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A Method of Solution of a Class of Second-Order Ordinary Differential Equations

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Abstract

A second-order ordinary differential equation which is linear in the second-order derivative term can be converted into an infinite number of inequivalent total differential equations in three variables. If at least two of the latter equations are integrable, they yield two first integrals of the ordinary differential equation and, hence, the general solution.

Keywords: Ordinary differential equation, total differential equation, integrable, symmetry group.

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1. INTRODUCTION

A second-order ordinary differential equation with a two-parameter symmetry group can be solved by quadratures by transformations of the independent and the dependent variables [1]. In this paper, we describe an alternative method. Consider a second-order ordinary differential equation of the form

$$a(x, y, y') + b(x, y, y')y'' = 0, \tag{1}$$

where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$. Setting $z = y'$, equation (1) can be expressed as

$$a(x, y, z) + b(x, y, z)\frac{dz}{dx} = 0, \tag{2}$$

or as the total differential equation

$$a(x, y, z)dx + b(x, y, z)dz = 0. \tag{3}$$

If $a(x,y,z)$ consists of a sum of n terms, $n \geq 1$,

$$a(x,y,z) = \sum_{i=1}^{n} a_i(x,y,z),$$

then, with $\gamma_i + \mu_i = 1$, $1 \leq i \leq n$, employing $dx = \frac{1}{z}dy$, every term $a_i(x,y,z)dx$, $1 \leq i \leq n$, can be expressed as

$$\begin{aligned} a_i(x,y,z)dx &= (\gamma_i + \mu_i)a_i(x,y,z)dx \\ &= \gamma_i a_i(x,y,z)dx + \mu_i a_i(x,y,z)dx \\ &= \gamma_i a_i(x,y,z)dx + \mu_i a_i(x,y,z)\frac{1}{z}dy. \end{aligned}$$

Thus, equation (1) can be expressed as the total differential equation

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0. \tag{4}$$

With $\mathbf{F} = (P,Q,R)$, equation (4) is *integrable*, i.e., there exists an integrating factor which makes it exact, if and only if [2]

$$\mathbf{F} \cdot \mathrm{curl}(\mathbf{F}) = 0. \tag{5}$$

Every choice of γ_i (with $\mu_i = 1 - \gamma_i$) yields a different total differential equation of the form (4). If (at least) two of the total differential equations are integrable, their solutions yield two first integrals of equation (1), elimination of the first-order derivative terms from which then yields the general solution of equation (1). The integrability of equation (4) is required because, otherwise, its solution would involve an arbitrary relation $\varphi(x,y,z) = 0$ which, in general, would be inconsistent with $z = \frac{dy}{dx}$.

2. SYMMETRIES AND INTEGRATING FACTORS OF TOTAL DIFFERENTIAL EQUATIONS

Let $G = \{g_\varepsilon : \varepsilon \in I\}$, where $I \subset \mathbb{R}$ is an open interval containing 0, be a one-parameter local Lie group [3] acting on $\mathbb{R}^3$, $g_\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$,

$$g_\varepsilon \cdot (x,y,z) = (\tilde{x},\tilde{y},\tilde{z}).$$

If, expressing equation (4) in the new variables $(\tilde{x},\tilde{y},\tilde{z})$ results in an equation which is equivalent to

$$P(\tilde{x},\tilde{y},\tilde{z})d\tilde{x} + Q(\tilde{x},\tilde{y},\tilde{z})d\tilde{y} + R(\tilde{x},\tilde{y},\tilde{z})d\tilde{z} = 0,$$

i.e., equation (4) in the new variables, then equation (4) is *invariant* under G and the group G is a *symmetry group* of equation (4). A necessary condition that G be a symmetry group of equation (4) is that the components ξ, η and ζ of the infinitesimal generator

$$\mathbf{v} = \xi(x,y,z)\frac{\partial}{\partial x} + \eta(x,y,z)\frac{\partial}{\partial y} + \zeta(x,y,z)\frac{\partial}{\partial z}$$

of G satisfy the equations [4]

$$R(P_x\xi+P_y\eta+P_z\zeta+P\xi_x+Q\eta_x+R\zeta_x)-P(R_x\xi+R_y\eta+R_z\zeta+P\xi_z+Q\eta_z+R\zeta_z)=0 \tag{6}$$

and

$$R(Q_x\xi+Q_y\eta+Q_z\zeta+P\xi_y+Q\eta_y+R\zeta_y)-Q(R_x\xi+R_y\eta+R_z\zeta+P\xi_z+Q\eta_z+R\zeta_z)=0. \tag{7}$$

Equations (6) and (7) imply

$$Q(P_x\xi+P_y\eta+P_z\zeta+P\xi_x+Q\eta_x+R\zeta_x)-P(Q_x\xi+Q_y\eta+Q_z\zeta+P\xi_y+Q\eta_y+R\zeta_y)=0, \tag{8}$$

and any two of equations (6), (7) and (8) imply the third. Without regard to a symmetry group, any triple (ξ,η,ζ) which satisfies any two of equations (6), (7) and (8) yields an integrating factor

$$\mu = [P(x,y,z)\xi(x,y,z) + Q(x,y,z)\eta(x,y,z) + R(x,y,z)\zeta(x,y,z)]^{-1} \tag{9}$$

of an integrable equation (4) provided that

$$P(x,y,z)\xi(x,y,z) + Q(x,y,z)\eta(x,y,z) + R(x,y,z)\zeta(x,y,z) \not\equiv 0.$$

3. AN EXAMPLE

Consider the ordinary differential equation

$$2y\frac{dy}{dx} + 3x\left(\frac{dy}{dx}\right)^2 + 4xy\frac{d^2y}{dx^2} = 0. \tag{10}$$

Setting $z = \frac{dy}{dx}$ transforms equation (10) into

$$2yz + 3xz^2 + 4xy\frac{dz}{dx} = 0,$$

or the total differential equation

$$2yzdx + 3xz^2dx + 4xydz = 0. \tag{11}$$

Let $\gamma_i + \mu_i = 1$, $i = 1, 2$. Then equation (11) can be expressed as

$$2\gamma_1 yzdx + 2\mu_1 yzdx + 3\gamma_2 xz^2dx + 3\mu_2 xz^2dx + 4xydz = 0$$

or, employing $dx = \frac{1}{z}dy$,

$$2\gamma_1 yzdx + 2\mu_1 ydy + 3\gamma_2 xz^2 dx + 3\mu_2 xzdy + 4xydz = 0,$$

i.e.,

$$(2\gamma_1 yz + 3\gamma_2 xz^2)dx + (2\mu_1 y + 3\mu_2 xz)dy + 4xydz = 0. \tag{12}$$

The values, if any, of γ_i (with $\mu_i = 1 - \gamma_i$), $i = 1, 2$, will be determined in order that equation (12) be integrable. With

$$\mathbf{F} = (2\gamma_1 yz + 3\gamma_2 xz^2, 2\mu_1 y + 3\mu_2 xz, 4xy),$$

$$\text{curl}(\mathbf{F}) = ((4 - 3\mu_2)x, (2\gamma_1 - 4)y + 6\gamma_2 xz, (3\mu_2 - 2\gamma_1)z),$$

and the requirement that

$$\mathbf{F} \cdot \text{curl}(\mathbf{F}) = 0$$

results in the condition

$$xyz(12\mu_1\gamma_2) + x^2z^2(12\gamma_2 + 9\mu_2\gamma_2) + y^2(4\mu_1(\gamma_1 - 2)) = 0,$$

which gives

$$\mu_1\gamma_2 = 0,\ \gamma_2(4 + 3\mu_2) = 0 \text{ and } \mu_1(\gamma_1 - 2) = 0.$$

There are three solutions (γ_i, μ_i), $i = 1, 2$:

$$\gamma_1 = 1,\ \mu_1 = 0,\ \gamma_2 = 0,\ \mu_2 = 1, \tag{13}$$

$$\gamma_1 = 2,\ \mu_1 = -1,\ \gamma_2 = 0,\ \mu_2 = 1, \tag{14}$$

and

$$\gamma_1 = 1,\ \mu_1 = 0,\ \gamma_2 = \frac{7}{3},\ \mu_2 = -\frac{4}{3}. \tag{15}$$

The resulting integrable total differential equations, corresponding to the cases (13), (14) and (15), respectively, are

$$2yzdx + 3xzdy + 4xydz = 0, \tag{16}$$

$$4yzdx + (3xz - 2y)dy + 4xydz = 0 \tag{17}$$

and

$$(2yz + 7xz^2)dx - 4xzdy + 4xydz = 0. \tag{18}$$

Since any three first integrals of a second-order ordinary differential equation are related, only two of the equations (16), (17) and (18) need to be solved.

It is evident, by inspection, that equation (16) is invariant under the group defined by

$$(\tilde{x}, \tilde{y}, \tilde{z}) = (e^{\varepsilon} x, y, z),$$

with the infinitesimal generator $\mathbf{v} = x\frac{\partial}{\partial x}$. Then $(\xi, \eta, \zeta) = (x, 0, 0)$ satisfies equations (6) and (7),

$$(2yz, 3xz, 4xy) \cdot (x, 0, 0) = 2xyz$$

yields the integrating factor $\mu = \frac{1}{2xyz}$, and equation (16) becomes

$$\frac{2yz}{2xyz}dx + \frac{3xz}{2xyz}dy + \frac{4xy}{2xyz}dz = 0,$$

which is equivalent to

$$\frac{2}{x}dx + \frac{3}{y}dy + \frac{4}{z}dz = 0,$$

with the solution

$$2\ln|x| + 3\ln|y| + 4\ln|z| = c_1,$$

or

$$x^2 y^3 z^4 = k. \tag{19}$$

Equation (18) is invariant under the group defined by

$$(\tilde{x}, \tilde{y}, \tilde{z}) = (e^{\varepsilon} x, e^{2\varepsilon} y, e^{\varepsilon} z),$$

with the infinitesimal generator $\mathbf{v} = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Then $(\xi, \eta, \zeta) = (x, 2y, z)$ satisfies equations (6) and (7),

$$(2yz + 7xz^2, -4xz, 4xy) \cdot (x, 2y, z) = 7x^2z^2 - 2xyz$$

yields the integrating factor $\nu = \frac{1}{7x^2z^2 - 2xyz}$, and equation (18) becomes

$$\frac{2yz + 7xz^2}{7x^2z^2 - 2xyz}dx - \frac{4xz}{7x^2z^2 - 2xyz}dy + \frac{4xy}{7x^2z^2 - 2xyz}dz = 0,$$

which is equivalent to

$$\frac{2y + 7xz}{7x^2z - 2xy}dx - \frac{4x}{7x^2z - 2xy}dy + \frac{4y}{7xz^2 - 2yz}dz = 0,$$

with the solution

$$-\ln|x| + 2\ln|7xz - 2y| - 2\ln|z| = c_2,$$

or

$$\frac{(7xz - 2y)^2}{xz^2} = c. \tag{20}$$

Thus, with $z = \frac{dy}{dx}$, two first integrals of equation (10) are given by equations (19) and (20). In order to eliminate z, solve equation (20) for z^4 and, by equation (19), replace it by $\frac{k}{x^2y^3}$ to obtain the expression

$$(7\sqrt{x} \mp \sqrt{c})^4 k = 16y^7. \tag{21}$$

Certain algebraic manipulations are required in order to eliminate the sign ambiguity and the square roots. By equation (21),

$$\begin{aligned} 16y^7 &= k(49x + c \mp 14\sqrt{c}\sqrt{x})^2 \\ &= k(49^2x^2 + c^2 + 294cx \mp 28 \cdot 49\sqrt{c}\,x\sqrt{x} \mp 28c\sqrt{c}\sqrt{x}). \end{aligned}$$

Expressing the latter equation as

$$\pm 28k\sqrt{c}\sqrt{x}(49x + c) = 2401kx^2 + kc^2 + 294kcx - 16y^7$$

and squaring gives

$$784k^2cx(49x + c)^2 = (2401kx^2 + kc^2 + 294kcx - 16y^7)^2,$$

the general solution of equation (10).

4. THE SYMMETRY GROUPS

Let $G = \{g_\varepsilon : \varepsilon \in I\}$, where $I \subset \mathbb{R}$ is an open interval containing 0, be a one-parameter local Lie group, $g_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$, $g_\varepsilon \cdot (x, y) = (\tilde{x}, \tilde{y})$. Then G determines a group $\overline{G} = \{\overline{g}_\varepsilon : \varepsilon \in I\}$, $\overline{g}_\varepsilon \cdot (x, y, z) = (\tilde{x}, \tilde{y}, \tilde{z})$, where $(\tilde{x}, \tilde{y}) = g_\varepsilon \cdot (x, y)$ and $\tilde{z} = \frac{d\tilde{y}}{d\tilde{x}}$. If equation (1) is invariant under G, then the resulting integrable total differential equation (4) may or may not be invariant under the group $\overline{G}$. We present the results for the example in Section 3.

Equation (10) is invariant under the four one-parameter Lie groups G_1, G_2, G_3 and G_4, with infinitesimal generators $\mathbf{v}^1$, $\mathbf{v}^2$, $\mathbf{v}^3$ and $\mathbf{v}^4$, respectively, where

$$\begin{aligned} G_1 &= \{g_\varepsilon^1 : \varepsilon \in \mathbb{R}\},\ g_\varepsilon^1 \cdot (x, y) = (e^\varepsilon x, y),\ \mathbf{v}^1 = x\frac{\partial}{\partial x}, \\ G_2 &= \{g_\varepsilon^2 : \varepsilon \in \mathbb{R}\},\ g_\varepsilon^2 \cdot (x, y) = (x, e^\varepsilon y),\ \mathbf{v}^2 = y\frac{\partial}{\partial y}, \\ G_3 &= \{g_\varepsilon^3 : \varepsilon \in \mathbb{R}\},\ g_\varepsilon^3 \cdot (x, y) = ((x^{1/2} + \varepsilon/2)^2, y),\ \mathbf{v}^3 = x^{1/2}\frac{\partial}{\partial x}, \\ G_4 &= \{g_\varepsilon^4 : \varepsilon \in \mathbb{R}\},\ g_\varepsilon^4 \cdot (x, y) = (x, (y^{7/4} + 7\varepsilon/4)^{4/7}),\ \mathbf{v}^4 = y^{-3/4}\frac{\partial}{\partial y}. \end{aligned}$$

The groups $\overline{G}_1$, $\overline{G}_2$, $\overline{G}_3$ and $\overline{G}_4$, and their respective infinitesimal generators $\overline{\mathbf{v}}^1$, $\overline{\mathbf{v}}^2$, $\overline{\mathbf{v}}^3$ and $\overline{\mathbf{v}}^4$, are

$$\begin{aligned}
\overline{G}_1 &= \{\overline{g}^1_\varepsilon : \varepsilon \in \mathbb{R}\},\ \overline{g}^1_\varepsilon \cdot (x, y, z) = (e^\varepsilon x, y, e^{-\varepsilon} z),\ \overline{\mathbf{v}}^1 = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}, \\
\overline{G}_2 &= \{\overline{g}^2_\varepsilon : \varepsilon \in \mathbb{R}\},\ \overline{g}^2_\varepsilon \cdot (x, y, z) = (x, e^\varepsilon y, e^\varepsilon z),\ \overline{\mathbf{v}}^2 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \\
\overline{G}_3 &= \{\overline{g}^3_\varepsilon : \varepsilon \in \mathbb{R}\},\ \overline{g}^3_\varepsilon \cdot (x, y, z) = ((x^{1/2} + \varepsilon/2)^2, y, \frac{x^{1/2} z}{x^{1/2} + \varepsilon/2}), \\
&\quad \overline{\mathbf{v}}^3 = x^{1/2}\frac{\partial}{\partial x} - \frac{1}{2}x^{-1/2} z\frac{\partial}{\partial z}, \\
\overline{G}_4 &= \{\overline{g}^4_\varepsilon : \varepsilon \in \mathbb{R}\}, \\
&\quad \overline{g}^4_\varepsilon \cdot (x, y, z) = (x, (y^{7/4} + 7\varepsilon/4)^{4/7}, y^{3/4}(y^{7/4} + 7\varepsilon/4)^{-3/7} z), \\
&\quad \overline{\mathbf{v}}^4 = y^{-3/4}\frac{\partial}{\partial y} - \frac{3}{4}y^{-7/4} z\frac{\partial}{\partial z}.
\end{aligned}$$

All three of the total differential equations (16), (17) and (18) are invariant under the groups $\overline{G}_1$ and $\overline{G}_2$. Equation (18) is invariant under the group $\overline{G}_3$ but equations (16) and (17) are not. Equations (16) and (17) are invariant under the group $\overline{G}_4$ but equation (18) is not.

5. CONCLUSIONS

In this paper, it is shown how a second-order ordinary differential equation which is linear in the second-order derivative can be expressed as an infinite number of inequivalent total differential equations in three variables. If at least two of the latter equations are integrable, their solutions provide first integrals of the ordinary differential equation, and the elimination of the first-order derivatives yields the general solution. The solutions of the total differential equations are obtained by construction of integrating factors by means of the infinitesimal generators of the Lie groups which leave the total differential equations invariant. The relations, if any, between the symmetry groups of an ordinary differential equation and those of the resulting integrable total differential equations are investigated.

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