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The Lie Symmetry Solution of a First-Order Ordinary Differential Equation with a Quadratic Nonlinearity

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Abstract

The Lie symmetry method is employed to determine the general solution of a first-order ordinary differential equation with a quadratic nonlinearity. The infinitesimal generator of the symmetry group is determined and, by employing the group invariant as the new independent variable, the general solution of the equation is obtained.

Keywords: Ordinary differential equation, symmetry group, infinitesimal generator, one-parameter Lie group, group invariant.

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1. INTRODUCTION

The Lie symmetry method [1] will be applied to solve the equation

$$\frac{du}{dx} = \alpha + \beta x + \alpha^2 x^2 - 2\alpha x u + u^2,\ \alpha, \beta \in \mathbb{R},\ \beta \neq 0. \tag{1}$$

The particular choice of the parameters will ensure that the solution will be expressible in closed form. The special case with $\alpha = 0$ and $\beta = -1$ can be transformed by $u \to -u$ into

$$\frac{du}{dx} = x - u^2,$$

which has already been solved [2]. The infinitesimal generator of the one-parameter Lie group under which the general equation

$$\frac{du}{dx} = F(x, u) \tag{2}$$

is invariant is the vector field

$$\mathbf{v} = \xi(x, u)\frac{\partial}{\partial x} + \phi(x, u)\frac{\partial}{\partial u},$$

where (ξ, ϕ) is a solution of the underdetermined partial differential equation

$$\phi_x + (\phi_u - \xi_x)F - \xi_u F^2 = \xi F_x + \phi F_u. \tag{3}$$

Employing the new coordinates (y, w) given by $y = \eta(x, u)$, the *group invariant*, and $w = \zeta(x, u)$, where η and ζ are determined by the equations

$$\mathbf{v}(\eta) = \xi\frac{\partial \eta}{\partial x} + \phi\frac{\partial \eta}{\partial u} = 0 \text{ and } \mathbf{v}(\zeta) = \xi\frac{\partial \zeta}{\partial x} + \phi\frac{\partial \zeta}{\partial u} = 1, \tag{4}$$

Equation (2) can be transformed into

$$\frac{dw}{dy} = G(y)$$

and integrated, provided that Equations (4) can be solved.

2. THE INFINITESIMAL GENERATOR

With

$$F(x, u) = \alpha + \beta x + \alpha^2 x^2 - 2\alpha x u + u^2, \tag{5}$$

we seek a solution of Equation (3) in the form

$$\xi(x, u) = a(x) \text{ and } \phi(x, u) = b(x) + c(x)u. \tag{6}$$

Then Equation (3) is satisfied provided that

$$c(x) = -a'(x), \ \ b(x) = -\frac{1}{2}a''(x) + \alpha x a'(x) + \alpha a(x), \tag{7}$$

and

$$a'''(x) + 4\beta x a'(x) + 2\beta a(x) = 0. \tag{8}$$

Equation (8) has an ordinary point at $x_0 = 0$ and can be solved by infinite series. We shall determine the particular solution which satisfies the initial conditions

$$a(0) = 1, \ \ a'(0) = 0 \text{ and } a''(0) = 0, \tag{9}$$

the relevance of which will become clear shortly. Thus, substitution of

$$a(x) = \sum_{n=0}^{\infty} c_n x^n$$

into Equation (8) yields the coefficient recursion relation

$$c_{n+3} = \frac{-2\beta(2n+1)c_n}{(n+1)(n+2)(n+3)},\ n \geq 0,$$

and the solution

$$a(x) = 1 + \sum_{k=1}^{\infty} \frac{(-2\beta)^k \cdot 1 \cdot 7 \cdot 13 \cdots (6k-5)}{(3k)!} x^{3k}, \tag{10}$$

with infinite radius of convergence. Another condition satisfied by $a(x)$ will be required later and is obtained by multiplying Equation (8) by $a(x)$ and integrating by parts:

$$a(x)a'''(x) + 4\beta x a(x) a'(x) + 2\beta (a(x))^2 = 0$$

$$\Rightarrow\ a(x)a''(x) - \int a'(x)a''(x)dx + 2\beta x(a(x))^2 - \int 2\beta (a(x))^2 dx + \int 2\beta (a(x))^2 dx = k$$

$$\Rightarrow\ a(x)a''(x) - \frac{1}{2}(a'(x))^2 + 2\beta x (a(x))^2 = k,$$

and $k = 0$ due to the conditions (9). Hence,

$$a(x)a''(x) - \frac{1}{2}(a'(x))^2 + 2\beta x (a(x))^2 = 0. \tag{11}$$

By (6) and (7), the infinitesimal generator of a symmetry group of Equation (1) is the vector field

$$\mathbf{v} = a(x)\frac{\partial}{\partial x} + \left[-\frac{1}{2}a''(x) + \alpha x a'(x) + \alpha a(x) - a'(x)u\right]\frac{\partial}{\partial u}. \tag{12}$$

3. THE SOLUTION

Equations (4) require that

$$\mathbf{v}(\eta) = a(x)\frac{\partial \eta}{\partial x} + \left[-\frac{1}{2}a''(x) + \alpha x a'(x) + \alpha a(x) - a'(x)u\right]\frac{\partial \eta}{\partial u} = 0 \tag{13}$$

and

$$\mathbf{v}(\zeta) = a(x)\frac{\partial \zeta}{\partial x} + \left[-\frac{1}{2}a''(x) + \alpha x a'(x) + \alpha a(x) - a'(x)u\right]\frac{\partial \zeta}{\partial u} = 1, \tag{14}$$

and a solution is

$$\eta = a'(x) - 2\alpha x a(x) + 2a(x)u \text{ and } \zeta = f(x), \text{ where } f'(x) = \frac{1}{a(x)}. \tag{15}$$

Let

$$y = \eta(x,u) = a' - 2\alpha x a + 2au \text{ and } w = \zeta(x,u) = f(x). \tag{16}$$

Then $x = g(w)$ where g is an inverse function of f and $g(f(x)) = 1 \Rightarrow$

$$\frac{dg}{dw}\frac{dw}{dx} = 1 \quad \Rightarrow \quad \frac{dg}{dw} = \frac{1}{f'(x)} = a(x).$$

It follows that

$$\frac{dx}{dy} = \frac{dg}{dw}\frac{dw}{dy} = a(x)\frac{dw}{dy}$$

and, by (16),

$$u = \frac{y}{2a} - \frac{a'}{2a} + \alpha x.$$

Then

$$\begin{aligned}
\frac{du}{dy} &= \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{dx}{dy} \\
&= \frac{1}{2a} + \left(-\frac{a'}{2a^2}y - \frac{1}{2}\frac{aa'' - (a')^2}{a^2} + \alpha\right)\frac{dx}{dy} \\
&= \frac{1}{2a} + \left(\alpha - \frac{a'}{2a^2}y - \frac{a''}{2a} + \frac{(a')^2}{2a^2}\right)a\frac{dw}{dy}
\end{aligned}$$

and

$$\frac{du}{dx} = \frac{\frac{du}{dy}}{\frac{dx}{dy}} = \frac{1}{2a^2\frac{dw}{dy}} + \alpha - \frac{a'}{2a^2}y - \frac{a''}{2a} + \frac{(a')^2}{2a^2},$$

and Equation (1) transforms into

$$\frac{1}{2a^2\frac{dw}{dy}} + \alpha - \frac{a'}{2a^2}y - \frac{a''}{2a} + \frac{(a')^2}{2a^2}$$

$$= \alpha + \beta x + \alpha^2 x^2 - 2\alpha x\left(\frac{y}{2a} - \frac{(a')}{2a} + \alpha x\right) + \left(\frac{y}{2a} - \frac{(a')}{2a} + \alpha x\right)^2,$$

which simplifies to

$$\frac{dw}{dy} = \frac{1}{aa'' - \frac{1}{2}(a')^2 + 2\beta x a^2 + \frac{1}{2}y^2}$$

or, in view of Equation (11),

$$\frac{dw}{dy} = \frac{2}{y^2},$$

with the solution

$$w = -\frac{2}{y} + d, \tag{17}$$

where d is an arbitrary constant. By (16),

$$f(x) - d = -\frac{2}{a' - 2\alpha x a + 2au},$$

which gives

$$u = \alpha x - \frac{a'}{2a} - \frac{1}{a[f(x) - d]}$$

or, employing

$$f(x) - d = \int \frac{1}{a(x)}\, dx$$

by (15),

$$u = \alpha x - \frac{a'}{2a} - \frac{1}{a \int \frac{1}{a(x)} dx},$$

the solution of Equation (1) in terms of the original variables.

5. CONCLUSIONS

A solution of the partial differential equation (3) is determined in order to derive the infinitesimal generator of a Lie group under which Equation (1) is invariant. The coefficients of the vector field are determined by a third-order, linear ordinary differential equation which is solved by infinite series. The vector field reveals the suitable new coordinates which transform Equation (1) into one that can be solved by quadrature, thereby yielding the solution of Equation (1).

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