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Uniqueness of Entire Functions with Differential Difference Polynomials of a Weakly Weighted and Relaxed Weighted Sharing

[1]**Harina P. Waghamore and** [2]**Roopa M.**

[1]*Department of Mathematics, Jnanabharathi Campus, Bangalore University,*
Bengaluru-560 056, INDIA. Email: harinapw@gmail.com

[1]*Department of Mathematics, Jnanabharathi Campus, Bangalore University,*
Bengaluru-560 056, INDIA. Email: mroopaprakash@gmail.com

Abstract

In this paper, we prove the uniqueness theorems concerning with differential difference polynomials for an entire functions by using the concept of weakly weighted sharing and relaxed weighted sharing. We obtain some recent results which improve and generalizes the earlier results of Roopa M. and Harina P. Waghamore[19].

Keywords : Weakly weighted sharing, relaxed weighted sharing, Differentail difference Polynomial, entire functions, Uniqueness.

1. INTRODUCTION AND MAIN RESULTS

In this research paper, we consider a meromorphic function which always meant a meromorphic function in the complex plane $\mathbb{C}$. Here authors are assumed that readers are known about basic notations of Nevanlinna theory and uses some of the notations like $m(r,f)$, $N(r,0;f)$, $\overline{N}(r,\infty;f)$, and $T(r,f)$ etc., (see [4], [6], [22]). Denote $S(r,f)$ any quantity which satisfies $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ outside of an exceptional set of finite linear measure and we are also denoting $S(r,f) = T(r,\alpha(z))$, where $\alpha(z)$ is a small function of f. Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. Set $E(a,f) = \{z \mid f - a = 0\}$, where a zero with multiplicity k is counted k times. If the zeros are counted only once, then we denote the set by $\overline{E}(a,f)$.

If $E(a,f)=E(a,g)$ where f and g are two non-constant meromorphic functions, then we say that f and g share a CM (counting multiplicity). If $\overline{E}(a,f)=\overline{E}(a,g)$ then we say that f and g share a IM (ignoring multiplicity). Denoting $E_{k)}(a,f)$ by the set of all a points of f with multiplicities not exceeding k, where an a points is counted according to its multiplicity. Also we denote $\overline{E}_{k)}(a,f)$ the set of distinct a points of f with multiplicities not exceeding k. We denote by $N_{k)}(r,a;f)$ the counting function of zeros of $f-a$ with multiplicity less than or equal to k, and by $\overline{N}_{k)}(r,a;f)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r,a;f)$ the counting function of zeros of $f-a$ with multiplicity greater than or equal to k, and by $\overline{N}_{(k}(r,a;f)$ the corresponding one for which multiplicity is not counted. Set

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}_{(2}(r,a;f)+\ldots+\overline{N}_{(k}(r,a;f).$$

Let us define $P(z)=a_mz^m+a_{m-1}z^{m-1}+\ldots+a_0$ be a non-zero polynomial of degree m, where $a_m(\neq 0)$, $a_{m-1},\ldots,a_0$ $(\neq 0)$ are complex constants and m is a positive integer. Throughout the paper, we denote by $\rho(f)=\lim\limits_{r\to\infty}\frac{logT(r,f)}{logr}$ the order of $f(z)$[[4],[6],[22]].

2. SOME DEFINITIONS

The following definitions are necessary to prove our main results.

Definition 1. *[10] Let $f(z)$ be a nonconstant meromorphic function. An expression of the form*

$$P[f]=\sum_{i=1}^{p}a_i(z)\prod_{j=0}^{q}f^j(z)^{l_{ij}},$$

where $a_i(z)\in S(f)$ for i=1,2,...,p and l_{ij} are nonnegative integers for $i=1,2,\ldots,p; j=0,1,2,\ldots,q$ and $d=\sum_{j=0}^{q}l_{ij}$ for each $i=1,2,\ldots,p$ is called homogeneous differential polynomial of degree d generated by $f(z)$.

Definition 2. *[8] Let $a\in\mathbb{C}\cup\{\infty\}$. Denote by $N_E(r,a;f,g)(\overline{N}_E(r,a;f,g))$ by the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with same multiplicities by $N_0(r,a;f,g)(\overline{N}_0(r,a;f,g))$ the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ IM. If*

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a;f,g)=S(r,f)+S(r,g)$$

then we say that f and g share the value a CM. If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a;f,g)=S(r,f)+S(r,g)$$

then we say that f and g share the value a IM.

Definition 3. *[7] Let f and g share the value a IM and k be a positive integer or infinity. Then $\overline{N}^{E}_{k)}(r, a; f, g)$ denotes the reduced counting function of those a points of f whose multiplicities are equal to the corresponding a points of g, and both of their multiplicities are not greater than k. $\overline{N}^{0}_{(k}(r, a; f, g)$ denotes the reduced counting function of those a points of f which are a points of g, and both of their multiplicities are not less than k.*

In 2006, authors S. H Lin and W. C Lin [7] introduced the following definitions of weakly weighted sharing which is a scaling between sharing IM and CM.

Definition 4. *[7] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If*

$$\overline{N}(r, a; f| \leq k) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, f).$$

$$\overline{N}(r, a; g| \leq k) - \overline{N}^{E}_{k)}(r, a; f, g) = S(r, g).$$

$$\overline{N}(r, a; f| \geq k + 1) - \overline{N}^{0}_{k+1)}(r, a; f, g) = S(r, f).$$

$$\overline{N}(r, a; g| \geq k + 1) - \overline{N}^{0}_{k+1)}(r, a; f, g) = S(r, g).$$

$$\overline{N}(r, a; f) - \overline{N}_0(r, a; f, g) = S(r, f).$$

$$\overline{N}(r, a; g) - \overline{N}_0(r, a; f, g) = S(r, g).$$

then we say that f and g share the value a weakly with weight k and we write f and g share "(a, k)".

In 2007, A. Banerjee and S. Mukherjee [5] introduced a new type of sharing known as relaxed weighted sharing, weaker than weakly weighted sharing and is defined as follows.

Definition 5. *[5] We denote by $\overline{N}(r, a; f| = p; g| = q)$ the reduced counting function of common a points of f and g with multiplicities p and q respectively.*

Definition 6. *[5] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. Suppose that f and g share the value a IM. If for $p \neq q$,*

$$\sum_{p,q \leq k} \overline{N}(r, a; f| = p; g| = q) = S(r),$$

then we say that f and g share the value a with weight k in a relaxed manner and in that case we write f and g share $(a, k)^*$.

In 2015, Pulak Sahoo [11] proved the following results.

Theorem 1. *[11] Let f and g be two transcendental entire functions of finite order, and α $(\not\equiv 0,\infty)$ be a small function of both f and g. Suppose that η is non-zero complex constant, n and $m(\geq 1)$ are integers such that $n \geq m+6$. If $f^n(f^m-1)f(z+\eta)$ and $g^n(g^m-1)g(z+\eta)$ share "$(\alpha(z),2)$", then $f \equiv tg$ where $t^m = 1$.*

Theorem 2. *[11] Let f and g be two transcendental entire functions of finite order, and $\alpha(\not\equiv 0,\infty)$ be a small function of both f and g. Suppose that η is non-zero complex constant, n and $m(\geq 1)$ are integers such that $n \geq 2m+8$. If $f^n(f^m-1)f(z+\eta)$ and $g^n(g^m-1)g(z+\eta)$ share $(\alpha(z),2)^*$, then $f \equiv tg$ where $t^m = 1$.*

Theorem 3. *[11] Let f and g be two transcendental entire functions of finite order, and α $(\not\equiv 0,\infty)$ be a small function of both f and g. Suppose that η is non-zero complex constant, n and $m(\geq 1)$ are integers such that $n \geq 4m+12$. If $\overline{E}_{2)}(\alpha(z), f^n(f^m-1)f(z+\eta)) = \overline{E}_{2)}(\alpha(z), g^n(g^m-1)g(z+\eta))$, then $f \equiv tg$ where $t^m = 1$.*

In 2018, Pulak Sahoo and Gurudas Biswas [13] proved the following result.

Theorem 4. *[13] Let f and g be two transcendental entire functions of finite order, and α $(\not\equiv 0,\infty)$ be a small function of both f and g. Suppose that η is non-zero complex constant, n and $m(\geq 1)$, $k(\geq 0)$ are integers such that $n \geq 2k+m+6$. If $(f^n(f^m-1)f(z+\eta))^{(k)}$ and $(g^n(g^m-1)g(z+\eta))^{(k)}$ share "$(\alpha(z),2)$", then $f \equiv tg$ where $t^m = 1$.*

Theorem 5. *[13] Let f and g be two transcendental entire functions of finite order, and α $(\not\equiv 0,\infty)$ be a small function of both f and g. Suppose that η is non-zero complex constant, n and $m(\geq 1)$, $k(\geq 0)$ are integers such that $n \geq 3k+2m+8$. If $(f^n(f^m-1)f(z+\eta))^{(k)}$ and $(g^n(g^m-1)g(z+\eta))^{(k)}$ share $(\alpha(z),2)^*$, then $f \equiv tg$ where $t^m = 1$.*

Theorem 6. *[13] Let f and g be two transcendental entire functions of finite order, and α $(\not\equiv 0,\infty)$ be a small function of both f and g. Suppose that η is non-zero complex constant, n and $m(\geq 1)$, $k(\geq 0)$ are integers such that $n \geq 5k+4m+12$. If $\overline{E}_{2)}(\alpha(z), (f^n(f^m-1)f(z+\eta))^{(k)}) = \overline{E}_{2)}(\alpha(z), (g^n(g^m-1)g(z+\eta))^{(k)})$, then $f \equiv tg$ where $t^m = 1$.*

In 2022, B. Saha, S. Pal and T. Biswas [19] proved the following results.

Theorem 7. *[19] Let f and g be two transcendental entire functions of finite order, c_j $(j = 1,\dots,d)$ be finite complex constants and and α $(\not\equiv 0)$ be a small function of both f and g with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are positive integers satisfying $n \geq max\{2k+m+\sigma+5, \sigma+2d+3\}$. If*

$\left(f^n(f^m-1)\prod_{j=1}^{d}f(z+c_j)^{\mu_j}\right)^{(k)}$ *and* $\left(g^n(g^m-1)\prod_{j=1}^{d}g(z+c_j)^{\mu_j}\right)^{(k)}$ *share* "$(\alpha,2)$" *then* $f\equiv tg$ *for some constant* t *such that* $t^{n+\sigma}=t^m=1$.

Theorem 8. *[19] Let* f *and* g *be two transcendental entire functions of finite order,* c_j $(j=1,\dots,d)$ *be finite complex constants and and* α $(\not\equiv 0)$ *be a small function of both* f *and* g *with finitely many zeros. Suppose that* $n(\geq 1)$, $m(\geq 1)$ *and* $k(\geq 0)$ *are integers satisfying* $n\geq max\{3k+2m+2\sigma+6,\sigma+2d+3\}$. *If* $\left(f^n(f^m-1)\prod_{j=1}^{d}f(z+c_j)^{\mu_j}\right)^{(k)}$ *and* $\left(g^n(g^m-1)\prod_{j=1}^{d}g(z+c_j)^{\mu_j}\right)^{(k)}$ *share* $(\alpha,2)^*$ *then conclusion of Theorem 7 holds.*

Theorem 9. *[19] Let* f *and* g *be two transcendental entire functions of finite order,* c_j $(j=1,\dots,d)$ *be finite complex constants and and* α $(\not\equiv 0)$ *be a small function of both* f *and* g *with finitely many zeros. Suppose that* $n(\geq 1)$, $m(\geq 1)$ *and* $k(\geq 0)$ *are integers satisfying* $n\geq max\{5k+4m+4\sigma+8,\sigma+2d+3\}$. *If* $\overline{E}_{2)}\left(\alpha(z),\left(f^n(f^m-1)\prod_{j=1}^{d}f(z+c_j)^{\mu_j}\right)^{(k)}\right)$ *and* $\overline{E}_{2)}\left(\alpha(z),\left(g^n(g^m-1)\prod_{j=1}^{d}g(z+c_j)^{\mu_j}\right)^{(k)}\right)$ *then conclusion of Theorem 7 holds.*

Recently Roopa M. and Harina P. Waghamore [[18]] have proved the results on uniqueness of differential difference polynomials of the form $\left(f^n(f^m-1)\prod_{j=1}^{d}f(z+c_j)^{\mu_j}\right)^{(k)}$ and $\left(g^n(g^m-1)\prod_{j=1}^{d}g(z+c_j)^{\mu_j}\right)^{(k)}$, where $f(z)$ is a transcendental entire function of finite order, n,m,j,d,k and $\mu_j(j=1,2,\dots,p)$ are non negative integers and $c_j(j=1,2,...,p)$ are distinct finite complex numbers. They have proved the following uniqueness results which extend and improve the recent results in this direction.

Theorem 10. *Let* f *and* g *be two transcendental entire functions of finite order,* c_j $(j=1,\dots,d)$ *be finite complex constants and and* α $(\not\equiv 0)$ *be a small function of both* f *and* g *with finitely many zeros. Suppose that* $n(\geq 1)$, $m(\geq 1)$ *and* $k(\geq 0)$ *are positive integers satisfying* $n\geq max\{2k+\Gamma_1+\sigma+5,\sigma+2d+3\}$. *If* $\left(f^n(f^m-1)\prod_{j=1}^{d}f(z+c_j)^{\mu_j}\right)^{(k)}$ *and* $\left(g^n(g^m-1)\prod_{j=1}^{d}g(z+c_j)^{\mu_j}\right)^{(k)}$ *share* "$(\alpha,2)$" *then* $f\equiv tg$ *for some constant* t *such that* $t^{n+\sigma}=t^m=1$.

Theorem 11. *Let f and g be two transcendental entire functions of finite order, c_j $(j = 1, \dots, d)$ be finite complex constants and and α $(\not\equiv 0)$ be a small function of both f and g with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq max\{3k+2\Gamma_1+2\sigma+6, \sigma+2d+3\}$. If $\left(f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{\mu_j}\right)^{(k)}$ and $\left(g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{\mu_j}\right)^{(k)}$ share $(\alpha, 2)^*$ then conclusion of Theorem 10 holds.*

Theorem 12. *Let f and g be two transcendental entire functions of finite order, c_j $(j = 1, \dots, d)$ be finite complex constants and and α $(\not\equiv 0)$ be a small function of both f and g with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq max\{5k + 5\Gamma_1 + 5\sigma + 6, \sigma + 2d + 3\}$. If $\overline{E}_{2)}\left(\alpha(z), \left(f^n(f^m - 1)\prod_{j=1}^{d} f(z + c_j)^{\mu_j}\right)^{(k)}\right)$ and $\overline{E}_{2)}\left(\alpha(z), \left(g^n(g^m - 1)\prod_{j=1}^{d} g(z + c_j)^{\mu_j}\right)^{(k)}\right)$ then conclusion of Theorem 10 holds.*

Now we prove the following results.

Theorem 13. *Let f and g be two transcendental entire functions of finite order, c_j $(j = 1, \dots, d)$ be finite complex constants and and α $(\not\equiv 0)$ be a small function of both f and g with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are positive integers satisfying $n \geq max\{2k + m + \sigma + \tau + 5, \sigma + \tau + 2\}$, where $\sigma = \sum_{i=1}^{p} v_i$, and $\tau = \sum_{j=1}^{q} \mu_j$. If $\left(f^n(f^m - 1)\prod_{i=1}^{p} f(z + c_i)^{v_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)}$ and $\left(g^n(g^m - 1)\prod_{i=1}^{p} g(z + c_i)^{v_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k)}$ share "$(\alpha, 2)$" then $f \equiv tg$ for some constant t such that $t^{n+\sigma+\tau} = t^m = 1$.*

Theorem 14. *Let f and g be two transcendental entire functions of finite order, c_j $(i = 1, \dots, p)$ be finite complex constants and and α $(\not\equiv 0)$ be a small function of both f and g with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq max\{3k + 2m + 2\sigma + 2\tau + 6, \sigma + \tau + 2\}$, where $\sigma = \sum_{i=1}^{p} v_i$, and $\tau = \sum_{j=1}^{q} \mu_j$. If $\left(f^n(f^m - 1)\prod_{i=1}^{p} f(z + c_i)^{v_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)}$ and $\left(g^n(g^m - 1)\prod_{i=1}^{p} g(z + c_i)^{v_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k)}$ share $(\alpha, 2)^*$ then conclusion of Theorem 13 holds.*

Theorem 15. *Let f and g be two transcendental entire functions of finite order, c_j $(j = 1, \ldots, d)$ be finite complex constants and and α $(\not\equiv 0)$ be a small function of both f and g with finitely many zeros. Suppose that $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq max\{5k + 4m + 4\sigma + 4\tau + 8, \sigma + \tau + 2\}$, where $\sigma = \sum_{i=1}^{p} v_i$, and $\tau = \sum_{j=1}^{q} \mu_j$. If $\overline{E}_{2)}\left(\alpha(z), \left(f^n(f^m - 1)\prod_{i=1}^{p} f(z + c_i)^{v_i}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)}\right)$ and $\overline{E}_{2)}\left(\alpha(z), \left(g^n(f^m - 1)\prod_{i=1}^{p} g(z + c_i)^{v_i}\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k)}\right)$ then conclusion of Theorem 13 holds.*

3. SOME LEMMAS

The following sequence of Lemmas will be helpful to prove our main results.
We denote H by the following function.

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$$

Lemma 1. *[1] Suppose f is a meromorphic function in the complex plane $\mathbb{C}$ and the polynomial is defined by $P(z) = a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0$, where $a_n(\not\equiv 0)$, $a_0,\ a_1,\ ,\ldots,\ a_{n-1}$ are small functions of f. Then*

$$T(r, P(f)) = mT(r, f) + S(r, f).$$

Lemma 2. *[3] Let $f(z)$ be a nonconstant meromorphic function and $P[f]$ be defined by (1). Then*

$$\begin{aligned} T(r, P) &\leq dT(r, f) + Q\overline{N}\left(r, \infty; f\right) + S(r, f) \\ &\leq Q\overline{N}\left(r, \infty; f\right) + dN\left(r, 0; f\right) + S(r, f). \end{aligned}$$

and

$$N\left(0r, 0; P[f]\right) \leq T(r, P[f]) - dT(r, f) + dN\left(r, 0; f\right) + S(r, f)$$

Lemma 3. *[5] Let H be defined as above. If F and G share "$(1, 2)$" and $H \not\equiv 0$, then*

$$\begin{aligned} T(r, F) \leq\ & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) \\ & + N_2(r, \infty; G) - \sum_{p=3}^{\infty} \overline{N}_{(p}\Big(r, \frac{G}{G'}\Big) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 4. *[5] Let H be defined as above. If* F *and* G *share* $(1,2)^*$ *and* $H \not\equiv 0$*, then*

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + \overline{N}(r,0;F) \\ + \overline{N}(r,\infty;G) - m(r,1;G) + S(r,F) + S(r,G).$$

Lemma 5. *[6] Let* F *and* G *be two non-constant entire functions and* $p \geq 2$ *be an integer. If* $\overline{E}_{p)}(1,F) = \overline{E}_{p)}(1,G)$ *and* $H \not\equiv 0$*, then*

$$T(r,F) \leq N_2(,0;F) + N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) + S(r,G).$$

Lemma 6. *[14] Let H be defined as above. If* F *and* G *share* $(1,2)^*$ *and* $H \equiv 0$ *and*

$$\varlimsup_{r\to\infty} \frac{\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G)}{T(r)} < 1, r \in I.$$

where $T(r) = max\{T(r,F), T(r,G)\}$ *and I is a set with linear measure, Then* $F \equiv G$ *or* $FG \equiv 1$*.*

Lemma 7. *[20] Let* f *be a non-constant meromorphic function, and let* s, k *be two positive integers. Then*

$$N_s\Big(r, \frac{1}{F}\Big) \leq T(r, f^{(k)}) - T(r,f) + N_{s+k}\Big(r, \frac{1}{f}\Big) + S(r,f).$$

$$N_s\Big(r, \frac{1}{F}\Big) \leq k\overline{N}(r,f) + N_{s+k}\Big(r, \frac{1}{f}\Big) + S(r,f).$$

Clearly, $\overline{N}\Big(r, \frac{1}{f^{(k)}}\Big) = N_1\Big(r, \frac{1}{f^{(k)}}\Big)$.

Lemma 8. *[13] Let* f *and* g *be two transcendental entire function of finite order and* $c_j (j = 1,\ 2, \ldots, s)$ *be finite complex constants. Let* $m(\geq 1)$ *and* $n(\geq 1)$ *be integers such that* $n \geq \sigma + \tau + 2$*. If*

$$\left(f^n(f^m-1)\prod_{i=1}^{p} f(z+c_i)^{v_i} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)} \equiv \left(g^n(g^m-1)\prod_{i=1}^{p} g(z+c_i)^{v_i} \prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k)}$$

then $f \equiv tg$ *for some constant t such that* $t^m = t^{n+\sigma+\tau} = 1$

Lemma 9. *[[19],[3]] Let* f *and* g *be two entire functions,* $n(\geq 1), m(\geq 1), k(\geq 0), p, q, i, j$ *be integers and let us define* $F = \left(f^n(f^m-1)\prod_{i=1}^{p} f(z+c_i)^{v_i} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)}$ *and* $G = \left(g^n(g^m-1)\prod_{i=1}^{p} g(z+c_i)^{v_i} \prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k)}$*. If there exists non-zero constants* c_1 *and* c_2 *such that* $\overline{N}(r,c_1;F) = \overline{N}(r,0;G)$ *and* $\overline{N}(r,c_2;G) = \overline{N}(r,0;F)$ *then* $n \leq 2k + m + \sigma + \tau + 2$.

4. PROOF OF MAIN RESULTS

Proof of Theorem 10.

Proof. Let $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$ where $F_1 = f^n(f^m - 1)\prod_{i=1}^{p} f(z+c_i)^{v_i} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}$ and $G_1 = g^n(g^m - 1)\prod_{i=1}^{p} g(z+c_i)^{v_i} \prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}$. Then F and G are transcendental meromorphic functions that share "$(1,2)$" except the zeros and poles of $\alpha(z)$. From Lemma 1, Lemma 2 and Lemma 7 we see that

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) \\ &\leq T(r,F_1^{(k)}) - T(r,F_1) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\leq T(r,F) - (n+m+\sigma+\tau)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned}$$

which gives

$$(n+m+\sigma+\tau)T(r,f) \leq T(r,F) - N_2(r,0;F) + N_{k+2}(r,0;F_1) + S(r,f). \tag{1}$$

Also, by Lemma 7, we obtain,

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{2}$$

Similarly,

$$N_2(r,0;G) \leq N_{k+2}(r,0;G_1) + S(r,g). \tag{3}$$

By using the inequalities (2) and (3) and Lemma 2 we get from (1)

$$\begin{aligned} (n+m+\sigma+\tau)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) \\ &\quad + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (k+2)\{\overline{N}(r,0;F_1) + \overline{N}(r,0;G_1)\} \\ &\quad + N(r,1;f^m) + N(r,1;g^m) + N\left(r,0;\prod_{i=1}^{p} f(z+c_i)^{v_i}\right) \\ &\quad + N\left(r,0;\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right) + N\left(r,0;\prod_{i=1}^{p} g(z+c_i)^{v_i}\right) \\ &\quad + N\left(r,0;\prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right) + S(r,f) + S(r,g) \\ &\leq (k+m+\sigma+\tau+1)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned}$$

Therefore,

$$(n+m+\sigma+\tau)T(r,f) \leq (k+m+\sigma+\tau+1)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \quad (4)$$

Similarly,

$$(n+m+\sigma+\tau)T(r,g) \leq (k+m+\sigma+\tau+1)\{T(r,g)+T(r,f)\}+S(r,g)+S(r,f). \quad (5)$$

Adding the inequalities (4) and (5), we get,

$$(n-2k-m-\sigma-\tau-4)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

which is obviously a contradiction as $n \geq 2k+m+\sigma+\tau+5$.
Consider the case when $H \equiv 0$. i.e.,

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1} \equiv 0$$

Integrating the above equation, we get,

$$\frac{1}{F-1} = \frac{P}{G-1} + Q \quad (6)$$

where $P \neq 0$ and Q are integrating constants. From the equation (6) it is clear that F and G share 1 CM and hence they share "$(1,2)$". Therefore $n \geq 2k+m+\sigma+\tau+5$. Upon considering the some of the cases separately, we obtain as follows.

Case 1. Suppose $Q \neq 0$ and $P = Q$ then from equation (6), we get,

$$\frac{1}{F-1} = \frac{QG}{G-1}. \quad (7)$$

If $B=-1$ then from equation (7), we get, $FG \equiv 1$.
i.e.,

$$\left(f^n(f^m-1)\prod_{i=1}^{p} f(z+c_i)^{v_i} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)} \left(g^n(g^m-1)\prod_{i=1}^{p} g(z+c_i)^{v_i} \prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k)} \equiv \psi^2$$

Since the number of zeros of $\varphi(z)$ is finite, it follows that f as well as g has finitely many zeros. We put $f(z) = h(z)e^{\beta(z)}$, where $h(z)$ is a nonzero polynomial and $\beta(z)$ is a nonconstant polynomial. Now replacing $\prod_{i=1}^{p} h(z+c_i)^{v_i}$ by $\mu(z)$ and $\sum_{i=1}^{p} v_i\beta(z+c_i)$

by $\gamma(z)$ we deduce that,

$$\begin{aligned}
&\left(f^n(f^m-1)\prod_{i=1}^{p} f(z+c_i)^{v_j}\prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)}\\
&\quad=\left(h^n(z)e^{n\beta(z)}\left(h^m(z)e^{m\beta(z)}-1\right)h(z+c)e^{\beta(z+c)}\right)^{(k)}\\
&\quad=\left(h^n(z)\mu(z)e^{n\beta(z)+\gamma(z)}\left(h^m(z)e^{m\beta(z)}-1\right)\right)^{(k)}\\
&\quad=\left(h^{n+m}(z)\mu(z)e^{(n+m)\beta(z)+\gamma(z)}-h^n(z)\mu(z)e^{n\beta(z)+\gamma(z)}\right)^{(k)}\\
&\quad=e^{(n+m)\beta(z)+\gamma(z)}P_1(\beta(z),\gamma(z),h(z),\mu(z),\ldots,\beta^{(k)}(z),\gamma^{(k)},\\
&\qquad\quad h^{(k)}(z),\mu^{(k)}(z))-e^{n\beta(z)+\gamma(z)}P_2(\beta(z),\gamma(z),h(z),\mu(z),\ldots,\\
&\qquad\quad \beta^{(k)}(z),\gamma^{(k)},h^{(k)}(z),\mu^{(k)}(z))\\
&\quad=e^{n\beta(z)+\gamma(z)}\left(P_1e^{m\beta(z)}-P_2\right).
\end{aligned}$$

Obviously $P_1e^{m\beta(z)}-P_2$ has infinite number of zeros, which contradicts with the fact that g is an entire function.
It can be easily verified from above that, $N(r,0;f)=S(r,f)$ and $N(r,1;f)=S(r,f)$. Thus

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3.$$

which is not possible.
If $B\neq-1$ from equation (7), we have, $\frac{1}{F}=\frac{QG}{(1+Q)G-1}$ and so $\overline{N}\Big(r,\frac{1}{1+Q};G\Big)=\overline{N}(r,0;F)$. Using Lemma 7 and Second main theorem of Nevanlinna, we get

$$\begin{aligned}
T(r,G)&\le\overline{N}(r,0;G)+\overline{N}\Big(r,\frac{1}{1+Q};G\Big)+\overline{N}(r,\infty;G)+S(r,G)\\
&\le\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+S(r,G)\\
&\le N_{k+1}(r,0;F_1)+T(r,G)+N_{k+1}(r,0;G_1)-(n+m+\sigma+\tau)T(r,g)+S(r,g).
\end{aligned}$$

Therefore,

$$(n+m+\sigma+\tau)T(r,g)\le(k+m+\sigma+\tau+1)\{T(r,f)+T(r,g)\}+S(r,g).\quad(8)$$

Likewise, we also get,

$$(n+m+\sigma+\tau)T(r,f)\le(k+m+\sigma+\tau+1)\{T(r,g)+T(r,f)\}+S(r,f).\quad(9)$$

From the inequalities (8) and (9) we obtain a contradiction as $n\ge 2k+m+\sigma+\tau+3$.

Case 2. Let $Q \neq 0$ and $P \neq Q$, then from equation (6) we get,

$$F = \frac{(Q+1)G - (Q-P+1)}{QG + (P-Q)}$$

and so $\overline{N}\Big(r, \frac{Q-P+1}{Q+1}; G\Big) = \overline{N}(r, 0; F)$. By providing the same argument as in case 1, we obviously get a contradiction.

Case 3. If $Q = 0$ and $P \neq 0$ then from equation (6) we get $F = \frac{G+P-1}{P}$ and $G = PF - (P-1)$. If $P \neq 1$, it follows that $\overline{N}\Big(r, \frac{P-1}{P}; F\Big) = \overline{N}(r, 0; G)$ and $N(r, 1-A; G) = N(r, 0; F)$. Now by using Lemma 9, it can be shown that $n \leq 2k + m + \sigma + \tau + 2$, a contradiction. Thus $P = 1$ and then $F \equiv G$ i.e.,

$$\left(f^n(f^m - 1)\prod_{i=1}^{p} f(z+c_i)^{\mu_j} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k)} \equiv \left(g^n(g^m - 1)\prod_{i=1}^{p} g(z+c_i)^{\mu_j} \prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k)}$$

Anti-Differentiate the above equation, we get,

$$\left(f^n(f^m - 1)\prod_{i=1}^{p} f(z+c_i)^{\mu_j} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right)^{(k-1)}$$

$$\equiv \left(g^n(g^m - 1)\prod_{i=1}^{p} g(z+c_i)^{\mu_j} \prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)^{(k-1)} + E_{k-1}.$$

where E_{k-1} is a constant. If $E_{k-1} \neq 0$, using Lemma 9 it follows that $n \leq 2k + m + \sigma + \tau + 2$, which is a contradiction. Hence $E_{k-1} = 0$. Repeating the above process k times we get

$$\left(f^n(f^m - 1)\prod_{i=1}^{p} f(z+c_i)^{v_i} \prod_{j=1}^{q} f^{(j)}(z)^{\mu_j}\right) \equiv \left(g^n(g^m - 1)\prod_{i=1}^{p} g(z+c_i)^{v_i} \prod_{j=1}^{q} g^{(j)}(z)^{\mu_j}\right)$$

which gives $f = tg$, where t is a constant satisfying $t^{n+m+\sigma} = 1$.

This completes the proof of Theorem 10.

□

Proof of Theorem 11.

Proof. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 10. Then F and G are transcendental meromorphic functions that share $(1, 2)^*$ except the zeros and poles of $\alpha(z)$. Let $H \not\equiv 0$. Then by using Lemma 3, Lemma 7 and Lemma 8, we get,

$$\begin{aligned}(n+m+\sigma+\tau)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G)\\ &\quad + \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g)\\ &\leq (2k+2m+2\sigma+2\tau+3)T(r,f)\\ &\quad + (k+m+\sigma+\tau+2)T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Therefore,

$$\begin{aligned}(n+m+\sigma+\tau)T(r,f) &\leq (2k+2m+2\sigma+2\tau+3)T(r,f)\\ &\quad + (k+m+\sigma+\tau+2)T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{10}$$

Likewise,

$$\begin{aligned}(n+m+\sigma+\tau)T(r,g) &\leq (2k+2m+2\sigma+2\tau+3)T(r,g)\\ &\quad +(k+m+\sigma+\tau+2)T(r,f) + S(r,f) + S(r,g).\end{aligned} \tag{11}$$

Adding the inequalities (10) and (11), we get,

$$\begin{aligned}(n+m+\sigma+\tau)\{T(r,f)+T(r,g)\} &\leq (3k+3m+3\sigma+2\tau+5)\\ &\quad + \{T(r,f)+T(r,g)\} + S(r,f) + S(r,g).\end{aligned}$$

which is a contradiction as $n \geq 3k+2m+2\sigma+2\tau+6$. Thus $H \equiv 0$. Proceeding similarly as done in Theorem 10 we get the proof of Theorem 11. □

Proof of Theorem 12.

Proof. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 10. Then F and G are transcendental meromorphic functions such that $\overline{E}_{2)}(1;F) = \overline{E}_{2)}(1;G)$ except the zeros and poles of $\alpha(z)$. Let $H \not\equiv 0$. Then by using Lemma 5, Lemma 7, we get,

$$\begin{aligned}(n+m+\sigma+\tau)T(r,f) &\leq N_2(r,0;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G)\\ &\quad + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g)\\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + 2N_{k+1}(r,0;F_1)\\ &\quad + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g)\\ &\leq (3k+3m+3\sigma+3\tau+4)T(r,f)\\ &\quad + (2k+2m+2\sigma+2\tau+3)T(r,g) + S(r,f) + S(r,g).\end{aligned}$$

Therefore,

$$\begin{aligned}(n+m+\sigma+\tau)T(r,f) &\leq (3k+3m+3\sigma+3\tau+4)T(r,f)\\ &\quad + (2k+2m+2\sigma+2\tau+3)T(r,g) + S(r,f) + S(r,g).\end{aligned} \tag{12}$$

Likewise,

$$\begin{aligned}(n+m+\sigma+\tau)T(r,g) &\leq (3k+3m+3\sigma+3\tau+4)T(r,g)\\ &\quad+(2k+2m+2\sigma+2\tau+3)T(r,f)S(r,f)+S(r,g).\end{aligned} \tag{13}$$

Adding the inequalities (12) and (13), we get,

$$\begin{aligned}(n+m+\sigma+\tau)\{T(r,f)+T(r,g)\} &\leq (5k+5m+5\sigma+5\tau+7)\{T(r,f)\\ &\quad+T(r,g)\}+S(r,f)+S(r,g).\end{aligned}$$

which is a contradiction as $n \geq 5k+4m+4\sigma+4\tau+8$. Thus $H \equiv 0$. Proceeding similarly as done in Theorem 10 we get the proof of Theorem 12. □

5. CONCLUSION

The main aim of this paper is to generalize and improve the some of the results of differential difference polynomial by using the concepts of weakly weighted sharing "$(\alpha, 2)$" and relaxed weighted sharing $(\alpha, 2)^*$. We have proved three theorems which extends the previous results of Roopa M. and Harina P. Waghamore.

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AVAILABILITY OF DATA AND MATERIALS

Data sharing will not be applicable to this article as no data sets were generated or analysed during the current study.

COMPETING INTEREST

The authors declare that they have no competing interest.

REFERENCES

[1] A. Banerjee and S. Mukherjee, Uniqueness of meromorphic functions concerning differential monomials sharing the same value, Bull. Math. Soc. Sci. Math. Roum. Nouv. Ser., 50 (2007), 191-206.

[2] S. S. Bhoosnurmath and S. R. Kabbur, Value distribution and uniqueness theorems for difference of entire and meromorphicfunctions, Int. J. Anal. Appl., 2 (2013), 124-136.

[3] G. Biswas, Weakly weighted and relaxed weighted sharing of differential differenc polynomials of entire functions, Analysis and Applications 9 (2021), 248-260.

[4] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[5] R. G. Halburd and R. J. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. Math., 31 (2006), 463-478.

[6] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin/Newyork, 1993.

[7] I. Lahiri and A. Sarkar, Uniqueness of a meromorphic function and its derivatives, J. inequal. pure appl. Math, 5(1), 20, 2004.

[8] I. Lahiri, Value distribution of certain differential polynomials, Int. J. Math. Math. Sci., 28 (2001), 83-91.

[9] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, Complex Var. Theory Appl., 46 (2001), 241-253.

[10] I. Lahiri, Indrajit and B. Pal, Uniqueness of meromorphic functions with their homogeneous and linear differential polynomials sharing a small function. Bulletin of the Korean Mathematical Society 54, no. 3 (2017): 825-838.

[11] X. Q. Lin and W. C. Lin, Uniqueness of entire functions sharing one value, Acta Math. Sci., Ser. B. Engl. Ed., 31 (2011), 1062-1076.

[12] S. H. Lin and W. C. Lin, Uniqueness of meromorphic functions concerning weakly weighted sharing, Kodai Math. J., 29 (2006), 269-280.

[13] K. Liu, X. G Qi, Meromorphic solutions of q-shift difference equations, Ann. Pol. Math. 101 (2011), 215-225.

[14] J. Li and K. Liu, Sharing values of q-difference differential polynomials, Adv. in Diff. Equations, 212 (2020), 1-12.

[15] C. Meng, Uniqueness of entire functions concerning difference polynomials, Math. Bohem., 139 (2014), no.1, 89-97.

[16] P. Sahoo, Uniqueness of entire functions related to difference polynomials, Commu. Math. Stat., 3 (2015), 227-238.

[17] P. Sahoo and G. Biswas, Some results on uniqueness of entire functions concerning difference polynomials, Tamkang Journal of Mathematics., 49 (2018), 85-97.

[18] B. Saha, S. Pal and T. Biswas, Some results on uniqueness of certain type of shift polynomials sharing a small function, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math., 29 (2022), no.1, 37-50.

[19] Roopa, M., and Harina P. Waghamore, Results Related to Value Distribution and Uniqueness of Entire Functions Concerning with Difference Polynomials, International Journal of Difference Equations (IJDE) 18, no. 2 (2023), 331-342.

[20] L. Yang, Value Distribution Theory, translated and revised from the 1982, Chinese Original, Springer, Berlin, 1993.

[21] H. X Yi, Meromorphic functions that share one or two values, Complex Variables Theory, Appl. 28 (2015) no. 1, 1-11.

[22] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer, Dordrecht,2003.

[23] J. L. Zhang and L. Z. Yang, Some results related to a conjecture of R. Brück, JIPAM. J. Inequal. Pure Appl. Math., 8 (2007), no.1, Article 18, 11 pp.