

## FIXED POINT THEOREMS IN INTUITIONISTIC FUZZY METRIC SPACES USING (CLR<sub>g</sub>) PROPERTY

M. Jeyaraman<sup>1</sup> & M. Barveen Banu<sup>2</sup>

*Post Graduate and Research Department of Mathematics,  
Raja Doraisingam Government Arts College,  
Sivagangai -630 561, India. <sup>1</sup>Email: jeya.math@gmail.com  
<sup>2</sup>Email: barveenbanu44@gmail.com*

### Abstract

In this paper provides definitions and describe the properties of Common Limit in Range of G (CLR<sub>g</sub>) compatible mappings, and prove common fixed point for four self-mappings that are compatible in an intuitionistic fuzzy metric space.

**Keywords:** Compatible Map, Intuitionistic Fuzzy Metric Spaces, Occasionally Weakly Compatible.

**Mathematics Subject classification:** 54H25, 47H10.

### 1. INTRODUCTION

Grabiec [2] demonstrated the Banach contraction theorem in the fuzzy metric spaces introduced by Kramosil and Michalek [3], Park and Kim [6] also proved common fixed point theorem in intuitionistic fuzzy metric space. Recently, Park et al. [7] defined an intuitionistic fuzzy metric space while Park et al. [8] proved a fixed point Banach theorem for the contractive mapping of a complete intuitionistic fuzzy metric space. In this paper also presents an example of common fixed theorem for a pair of occasionally weakly compatible mappings by using the (CLR<sub>g</sub>) property in intuitionistic fuzzy metric spaces.

### 2. PRELIMINARIES

In this section, we recall some definitions and known results in intuitionistic fuzzy metric spaces.

**Definition 2.1:**

A binary operation  $*$  :  $[0,1] \times [0,1] \rightarrow [0,1]$  is a continuous t-norm if it satisfies the following conditions:

- (i)  $*$  is associative and commutative,
- (ii)  $*$  is continuous,
- (iii)  $a * 1 = a$  for all  $a \in [0,1]$ ,
- (iv)  $a * b \leq c * d$  whenever  $a \leq c$  and  $b \leq d$ , for each  $a, b, c, d \in [0,1]$ .

**Definition 2.2:**

A binary operation  $\diamond$  :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a continuous t-conorm if  $\diamond$  satisfies the following conditions:

- (i)  $\diamond$  is commutative and associative,
- (ii)  $\diamond$  is continuous,
- (iii)  $a \diamond 0 = a$  for all  $a \in [0, 1]$ ,
- (iv)  $a \diamond b \leq c \diamond d$  whenever  $a \leq c$  and  $b \leq d$ , for each  $a, b, c, d \in [0, 1]$ .

**Definition 2.3[7]:**

A 5-tuple  $(X, M, N, *, \diamond)$  is said to be an intuitionistic fuzzy metric space (shortly IFM-space) if  $X$  is an arbitrary set,  $*$  is a continuous t-norm,  $\diamond$  is a continuous t-conorm and  $M, N$

are fuzzy sets on  $X^2 \times (0, \infty)$  satisfying the following conditions:

for all  $x, y, z \in X$  and  $s, t > 0$ .

- (IFM-1)  $M(x, y, t) + N(x, y, t) \leq 1$ ,
- (IFM-2)  $M(x, y, 0) = 0$ ,
- (IFM-3)  $M(x, y, t) = 1$  if and only if  $x = y$ ,
- (IFM-4)  $M(x, y, t) = M(y, x, t)$ ,
- (IFM-5)  $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ,
- (IFM-6)  $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is left continuous,
- (IFM-7)  $\lim_{n \rightarrow \infty} M(x, y, t) = 1$ ,
- (IFM-8)  $N(x, y, 0) = 1$ ,
- (IFM-9)  $N(x, y, t) = 0$  if and only if  $x = y$ ,
- (IFM-10)  $N(x, y, t) = N(y, x, t)$ ,
- (IFM-11)  $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ ,
- (IFM-12)  $N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is right continuous,

$$(IFM-13) \quad \lim_{n \rightarrow \infty} N(x, y, t) = 0.$$

Then  $(M, N)$  is called an intuitionistic fuzzy metric on  $X$ . The functions  $M(x, y, t)$  and

$N(x, y, t)$  denote the degree of nearness and degree of non-nearness between  $x$  and  $y$  with respect to  $t$ , respectively.

**Remark 2.4:**

Every fuzzy metric space  $(X, M, *)$  is an intuitionistic fuzzy metric space if  $X$  of the form  $(X, M, 1-M, *, \diamond)$  such that  $t$ -norm  $*$  and  $t$ -conorm  $\diamond$  are associated, that is,  $x \diamond y = 1 - ((1 - x) * (1 - y))$  for any  $x, y \in X$ .

**Example 2.5[ 8 ]:**

Let  $(X, d)$  be a metric space. Denote  $a * b = ab$  and  $a \diamond b = \min\{1, a + b\}$  for all  $a, b \in [0, 1]$  and let  $M_d$  and  $N_d$  be fuzzy sets on  $X^2 \times (0, \infty)$  defined as follows.

$$M_d(x, y, t) = \frac{t}{t + d(x,y)}, \quad N_d(x, y, t) = \frac{d(x,y)}{t + d(x,y)}$$

Then  $(M_d, N_d)$  is an intuitionistic fuzzy metric on  $X$ . We call this intuitionistic fuzzy metric induced by a metric the standard intuitionistic fuzzy metric.

**Remark 2.6:**

Note the above example holds even with the  $t$ -norm  $a * b = \min\{a, b\}$  and the  $t$ -conorm  $a \diamond b = \max\{a, b\}$  and hence  $(M_d, N_d)$  is an intuitionistic fuzzy metric with respect to any continuous  $t$ -norm and continuous  $t$ -conorm.

**Definition 2.7[10]:**

- (i) A sequence  $\{x_n\}$  in  $X$  is said to be a Cauchy sequence iff for each  $\epsilon > 0, t > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x_m, t) > 1 - \epsilon$  and  $N(x_n, x_m, t) < \epsilon$  for all  $n, m \in \mathbb{N}$ .
- (ii) A sequence  $\{x_n\}$  in  $X$  is said to be a converge to a point  $x$  in  $X$  if and only if for each  $\epsilon > 0, t > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $M(x_n, x, t) > 1 - \epsilon$  and  $N(x_n, x, t) < \epsilon$  for all  $n \geq n_0$ .
- (iii) A intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  is said to be complete if every Cauchy sequence in it converges to a point in it.

**Definition 2.8:**

Two maps  $A$  and  $B$  from a fuzzy metric space  $(X, M, *)$  into itself are said to be compatible if  $\lim_{n \rightarrow \infty} M(ABx_n, BAx_n, t) = 1$  for all  $t > 0$ , whenever  $\{x_n\}$  is a sequence such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = x$  for some  $x \in X$ .

**Definition 2.9:**

Two maps A and B from a intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  into itself are said to be compatible if  $\lim_{n \rightarrow \infty} M(ABx_n, BAx_n, t) = 1$  and  $\lim_{n \rightarrow \infty} N(ABx_n, BAx_n, t) = 0$  for all  $t > 0$ , whenever  $\{x_n\}$  is a sequence such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = x$  for some  $x \in X$ .

**Definition 2.10:**

Two maps A and B from a intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  into itself are said to be weak compatible if they commute at their coincidence points, that is,  $Ax = Bx$  implies  $ABx = BAx$ .

**Definition 2.11:**

Self mappings A and S of a intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  are said to be Occasionally Weakly Compatible (OWC) if and only if there is a point  $x$  in  $X$ , which is coincidence point of A and S at which A and S commute.

**Definition 2.12:**

A pair of self mappings A and S of a intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  is said to satisfy the (CLR<sub>g</sub>) property if there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = Bu, \text{ for some } u \in X.$$

**Proposition 2.13 :**

Let S and T be compatible self maps of a intuitionistic fuzzy metric space  $(X, M, N, *, \diamond)$  and  $Su = Tu$  for some  $u$  in  $X$  then  $STu = TSu = SSu = TTu$ .

**3. COMMON FIXED POINT FOR FOUR SELF MAPPINGS****Theorem 3.1:**

Let  $(X, M, N, *, \diamond)$  be a intuitionistic fuzzy metric space,  $*$  being continuous t-norm and  $\diamond$  being continuous t-conorm with  $a * b \geq ab$ , and  $a \diamond b \leq ab$ , for all  $a, b \in [0, 1]$ .

Let  $P, Q: X \times X \rightarrow X$  and  $R, S: X \times X \rightarrow X$  be four mappings satisfying following conditions:

(3.1.1) The pairs  $(P, R)$  and  $(Q, S)$  satisfy CLR<sub>g</sub> property

(3.1.2)  $M(P(x, y), Q(u, v), kt) \geq \Phi\{M(Rx, Su, t) * M(P(x, y), Rx, t) * M(Q(u, v), Su, t)\}$

$$\text{and } N(P(x, y), Q(u, v), kt) \leq \psi\{N(Rx, Su, t) \diamond N(P(x, y), Rx, t) \diamond N(Q(u, v), Su, t)\}$$

for all  $x, y, u, v \in X, k \in (0, 1)$  and  $\Phi, \psi : [0, 1] \rightarrow [0, 1]$ , Such that  $\Phi(t) > t$  and  $\psi(t) < t$

for  $0 < t < 1$ . Then (P, R) and (Q, S) have point of coincidence. Moreover if the pairs (P, R) and (Q, S) are occasionally weakly compatible, then there exists unique  $x$  in  $X$ , Such that  $P(x, x) = S(x) = Q(x, x) = R(x) = x$ .

**Proof:**

Since the pairs (P, R) and (Q, S) satisfy CLRg property, there exist sequences  $\{x_n\}, \{y_n\}, \{x_n'\}$  and  $\{y_n'\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} P(x_n, y_n) = \lim_{n \rightarrow \infty} R(x_n) = Ra$ ,

$$\lim_{n \rightarrow \infty} P(y_n, x_n) = \lim_{n \rightarrow \infty} R(y_n) = Rb \text{ and } \lim_{n \rightarrow \infty} Q(x_n', y_n') = \lim_{n \rightarrow \infty} S(x_n') = Sa',$$

$$\lim_{n \rightarrow \infty} Q(y_n', x_n') = \lim_{n \rightarrow \infty} S(y_n') = Sb', \text{ for some } a, b, a', b' \text{ in } X.$$

**Step 1:** We now show that the pairs (P, R) and (Q, S) have common coupled coincidence point. We first show that  $Ra = Sa'$ , using (2.13) we have,

$$M(P(x_n, y_n), Q(x_n, y_n), kt) \geq \Phi\{M(Rx_n, Sx_n, t) * M(P(x_n, y_n), Rx_n, t) * M(Q(x_n, y_n), Sx_n, t)\}$$

$$N(P(x_n, y_n), Q(x_n, y_n), kt) \leq \psi\{N(Rx_n, Sx_n, t) \diamond N(P(x_n, y_n), Rx_n, t) \diamond N(Q(x_n, y_n), Sx_n, t)\}$$

Taking  $n \rightarrow \infty$ , we get

$$\begin{aligned} M(Ra, Sa', kt) &\geq \Phi\{M(Ra, Sa', t) * 1 * 1\} \\ &\geq \Phi\{M(Ra, Sa', t)\} \geq M(Ra, Sa', t) \end{aligned}$$

$$M(Ra, Sa', kt) \geq M(Ra, Sa', t) \text{ and}$$

$$\begin{aligned} N(Ra, Sa', kt) &\leq \psi\{N(Ra, Sa', t) \diamond 0 \diamond 0\} \\ &\leq \psi\{N(Ra, Sa', t)\} \\ &\leq N(Ra, Sa', t) \end{aligned}$$

$$N(Ra, Sa', kt) \leq N(Ra, Sa', t)$$

Therefore  $Ra = Sa'$ .

Similarly, we can have  $Rb = Sb'$ , Also,

$$M(P(y_n, x_n), Q(x_n', y_n'), kt) \geq \Phi\{M(Ry_n, Sx_n', t) * M(P(y_n, x_n), Ry_n, t) * M(Q(x_n', y_n'), Sx_n', t)\}$$

$$M(Rb, Sa', kt) \geq M(Rb, Sa', t) \text{ and}$$

$$N(P(y_n, x_n), Q(x_n', y_n'), kt) \leq \psi\{N(Ry_n, Sx_n', t) \diamond N(P(y_n, x_n), Ry_n, t) \diamond N(Q(x_n', y_n'), Sx_n', t)\}$$

$$N(Rb, Sa', kt) \leq N(Rb, Sa', t).$$

Therefore  $Rb = Sa'$ .

Hence  $Rb = Sa' = Ra = Sa'$ . Now, for all  $t > 0$ , using condition (2.13). We have

$$M(P(x_n, y_n), Q(a', b'), kt) \geq \Phi\{M(Rx_n, Sa', t) * M(P(x_n, y_n), Rx_n, t) * M(Q(a', b'), Sa', t)\}$$

$$M(Ra, Q(a', b'), kt) \geq M(Ra, Q(a', b'), t) \text{ and}$$

$$N(P(x_n, y_n), Q(a', b'), kt) \leq \psi\{N(Rx_n, Sa', t) \diamond N(P(x_n, y_n), Rx_n, t) \diamond N(Q(a', b'), Sa', t)\}$$

$$N(Ra, Q(a', b'), kt) \leq N(Ra, Q(a', b'), t).$$

Therefore  $Ra = Q(a', b')$ .

Similarly, we can get that  $Rb = Q(b', a')$ .

In a similar fashion, we can have  $Sa' = P(a, b)$  and  $Sb' = P(b, a)$ .

Thus,  $Q(a', b') = Ra = Sa' = P(a, b)$  and  $Q(b', a') = Rb = Sb' = P(b, a)$ .

Thus the pairs  $(P, R)$  and  $(Q, S)$  have coincidence points.

Let  $Ra = P(a, b) = Q(a', b') = Sa' = x$  and  $Rb = P(b, a) = Q(b', a') = Sb' = y$ .

Since  $(P, R)$  and  $(Q, S)$  are occasionally weakly compatible, so

$$Rx = RP(a, b) = P(Ra, Rb) = P(x, y) \text{ and } Ry = RP(b, a) = P(Rb, Ra) = P(y, x).$$

$$Sx = SQ(a', b') = Q(Sa', Sb') = Q(x, y) \text{ and } Sy = SQ(b', a') = Q(Sb', Sa') = Q(y, x).$$

**Step 2:** We next show that  $x = y$  from (2.13),

$$M(x, y, kt) = M(P(a, b), Q(a', b'), kt)$$

$$\geq \Phi\{M(Ra, Sa', t) * M(P(a, b), Sa', t) * M(Q(a', b'), Sa', t)\} = 1 \text{ and}$$

$$N(x, y, kt) = N(P(a, b), Q(a', b'), kt)$$

$$\leq \psi\{N(Ra, Sa', t) \diamond N(P(a, b), Sa, t) \diamond N(Q(a', b'), Sa', t)\} = 0.$$

Thus,  $x = y$ .

**Step 3:** Now, we prove that  $Rx = Sx$ , using (2.13) again

$$\begin{aligned} M(Rx, Sx, kt) &= M(P(x, y), Q(y, x), kt) \\ &\geq \Phi\{M(Rx, Sy, t) * M(P(x, y), Rx, t) * M(Q(y, x), Sy, t)\} \\ &\geq \Phi\{M(Rx, Sx, t) * M(P(x, y), Rx, t) * M(Q(y, x), Sy, t)\} \end{aligned}$$

$$M(Rx, Sx, kt) \geq M(Rx, Sy, t) \text{ and}$$

$$\begin{aligned} N(Rx, Sx, kt) &= N(P(x, y), Q(y, x), kt) \\ &\leq \psi\{N(Rx, Sy, t) \diamond N(P(x, y), Rx, t) \diamond N(Q(y, x), Sy, t)\} \\ &\leq \psi\{N(Rx, Sx, t) \diamond N(P(x, y), Rx, t) \diamond N(Q(y, x), Sy, t)\} \end{aligned}$$

$$N(Rx, Sx, kt) \leq N(Rx, Sy, t)$$

Therefore  $Rx = Sx = Sy$ .

**Step 4:** We prove that  $Rx = x$ .

$$\begin{aligned} M(Rx, x, kt) &= M(Rx, y, kt) \\ &= M(P(x, y), Q(x, y), kt) \\ &\geq \Phi\{M(Rx, Sx, t) * M(P(x, y), Rx, t) * M(Q(x, y), Sx, t)\} \text{ and} \end{aligned}$$

$$\begin{aligned} N(Rx, x, kt) &= N(Rx, y, kt) \\ &= N(P(x, y), Q(x, y), kt) \\ &\leq \psi\{N(Rx, Sx, t) \diamond N(P(x, y), Rx, t) \diamond N(Q(x, y), Sx, t)\} \end{aligned}$$

Hence  $x = Rx = Sx = P(x, x) = Q(x, x)$ .

This shows that  $P, Q, R, S$  have a common fixed point and uniqueness of  $x$ .

**Example 3.2:**

Let  $X = [0, \infty)$  be the usual metric space. Define  $f, g: X \rightarrow X$  by  $fx = x + 3$  and  $gx = 4x$  for all  $x \in X$ . We consider the sequence  $\{x_n\} = \{1 + \frac{1}{n}\}$ .

Since  $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = 4 = g(1) \in X$ . Therefore  $f$  and  $g$  satisfy the (CLRg) property.

**Example 3.3 :**

The conclusion of above example remains true if the self mappings  $f$  and  $g$  is defined on  $X$  by  $f(x) = \frac{x}{5}$  and  $g(x) = \frac{2x}{4}$  for all  $x \in X$ . Let a sequence  $\{x_n\} = \{\frac{1}{n}\}$  in  $X$ .

Since  $\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = 0 = g(0) \in X$ . Therefore  $f$  and  $g$  satisfy the (CLRg) property.

**Example 3.4:**

Let  $(X, M, N, *, \diamond)$  be a intuitionistic fuzzy metric space,  $*$  being a continuous t-norm and  $\diamond$  being a continuous t-conorm with  $X = [0, \infty)$ . Define  $M(x, y, t) = \frac{t}{t+|x-y|}$  and

$$N(x, y, t) = \frac{|x-y|}{t+|x-y|} \text{ for all } x, y \text{ in } X \text{ and } t > 0.$$

Define mappings  $f: X \times X \rightarrow X$  and  $g: X \rightarrow X$  as follows.

$$f(x, y) = \{x + y, x \in [0,1], y \in X \text{ and } \frac{x+y}{2}, x \in (1,\infty), y \in X\}.$$

We consider the sequence  $x_n = \{\frac{1}{n}\}$  and  $y_n = \{1 + \frac{1}{n}\}$  then,

$$f(x_n, y_n) = f\left(\frac{1}{n} + 1 + \frac{1}{n}\right) = 1 + \frac{2}{n}, \quad f(y_n, x_n) = f\left(1 + \frac{1}{n}, \frac{1}{n}\right) = \frac{1}{2} + \frac{1}{n}.$$

$$g(x_n) = g\left(\frac{1}{n}\right) = 1 + \frac{1}{n}, \quad g(y_n) = g\left(1 + \frac{1}{n}\right) = \frac{1}{2} + \frac{1}{2n}.$$

$$\lim_{n \rightarrow \infty} M(f(x_n, y_n), g(x_n), t) \rightarrow 1 = g(0) \text{ and } \lim_{n \rightarrow \infty} M(f(y_n, x_n), g(y_n), t) \rightarrow 1 = g(0)$$

$$\text{and } \lim_{n \rightarrow \infty} N(f(x_n, y_n), g(x_n), t) \rightarrow 0 = g(1) \text{ and } \lim_{n \rightarrow \infty} N(f(y_n, x_n), g(y_n), t) \rightarrow 0 = g(1)$$

therefore, the maps  $f$  and  $g$  satisfy (CLRg) property but the maps are not continuous.

**REFERENCES**

- [1] Atanassov . (1986) ,K, Intuitionistic fuzzy sets, *Fuzzy Sets and System*, 20,87-96.
- [2] Grabiec .M, (1988) Fixed points in fuzzy metric spaces, *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385-389.
- [3] Kramosil.I and Michalek.J, (1975), Fuzzy metrics and statistical metric spaces, *Kybernetica*, vol. 11, no. 5, pp. 336-344.

- [4] Park .J.S, “Some common fixed point theorems using compatible maps in intuitionistic Fuzzy metric space, (2011), International Journal of Fuzzy Logic and Intelligent Systems, vol. 11, no. 2, pp. 108-112.
- [5] Park .J.S, (2011),Fixed point theorem for common property (E.A.) and weak compatible functions in intuitionistic fuzzy metric space, International Journal of Fuzzy Logic and Intelligent systems, vol. 11, no. 3, pp. 149-152.
- [6] Park . J.S and Kim. S.Y, (2008), Common fixed point theorem and example in intuitionistic fuzzy metric space, Journal of Korean Institute of Intelligent Systems, vol. 18, no. 4, pp. 524-529.
- [7] Park . J.S, Kwun .Y.C , and Park . J.H, (2005),A fixed point theorem in the intuitionistic fuzzy metric spaces, Far East Journal of Mathematical Sciences, vol. 16, no. 2, pp. 137-149.
- [8] Park . J.S, Kwun .Y.C , and Park . J.H, (2007),Fixed point theorems in intuitionistic fuzzy metric space (I),” JP Journal of Fixed Point Theory and Applications, vol. 2, no. 1, pp. 79-89.
- [9] Park . J.S, Kwun .Y.C , and Park . J.H, (2008),On some results for five mappings using compatibility of type ( $\alpha$ ) in intuitionistic fuzzy metric space, International Journal of Fuzzy Logic and Intelligent Systems, vol. 8, no. 4, pp. 299-305.
- [10] Park . J.S, Kwun .Y.C , (2007), Some fixed point theorems in the intuitionistic fuzzy metric spaces, Far East Journal of Mathematical Sciences, vol. 24, no. 2, pp. 227-239.
- [11] Park . J.S, Kwun .Y.C , and Park . J.H, (2008), Fixed points in M-fuzzy metric spaces, Fuzzy Optimization and Decision Making, Vol.7, no.4, pp.305-315.
- [12] Park . J.S, (2013),Some Common Fixed Points for Type ( $\beta$ ) compatible Maps in an intuitionistic Fuzzy Metric Space, International Journal of Fuzzy Logic and Intelligent Systems, vol. 13, no. 2, pp. 147 - 153.
- [13] Turkoglu .D, Alaca .C, Cho .Y.J , Yildiz .C,(2006),Common fixed point theorems in intuitionistic fuzzy metric spaces, J. Appl. Math.& Computing, 22(1-2),411–424.
- [14] Zadeh .L.A, (1965), Fuzzy sets, Inform. and Control,8, 338-353.

