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ON FUZZY PRIME FILTERS OF A LATTICE

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Abstract

The merits and demerits of various definitions of fuzzy prime filters of a lattice are discussed. Characterization is given through level filters of that lattice. The relation between the definitions is discussed with examples.

Keywords: fuzzy filter, level filter, fuzzy prime filter

1. INTRODUCTION:

In the last few years considerable study has been made in the field of fuzzy filters of a lattice. There exist several fuzzifications of fuzzy prime filter. Therefore it is necessary to critically analyze and compare these definitions.

2. BACKGROUNDS:

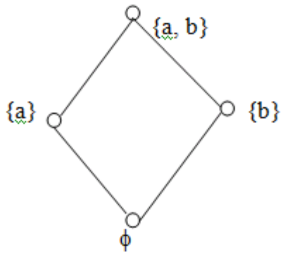
Here we discuss some basic definitions.

Definition 2.1:

A partial order set $(L, \leq)$ is said to form a **lattice** if for every $a, b \in L$, Sup $\{a, b\}$ and Inf $\{a, b\}$ exist in L. In that case, we write Sup $\{a, b\} = a \vee b$ and Inf $\{a, b\} = a \wedge b$.

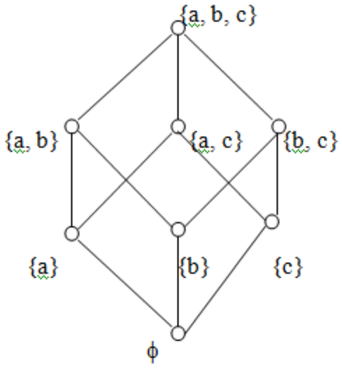
Example 2.2:

Consider the poset $\wp(A)$ where $A = \{a, b\}$. Then $(\wp(A), \vee, \wedge)$ is a lattice where $\vee$ is the union and $\wedge$ is the intersection.



Example 2.3:

Consider the poset $\wp(A)$ where A = {a, b, c}. Then ($\wp(A)$, $\vee$, $\wedge$) is a lattice where $\vee$ is the union and $\wedge$ is the intersection.



Definition 2.4:

Let (L, $\vee$, $\wedge$) be a lattice. A non – empty subset S of L is called a **sublattice** of L if for every, x, y $\in$ S, $x \vee y, x \wedge y \in S$.

Definition 2.5:

Let (L, $\vee$, $\wedge$) be a lattice. A non – empty subset S of L is called a **filter** of L if it satisfies the following conditions: (i) x, y $\in$ S $\Rightarrow x \wedge y \in$ S. (ii) x $\in$ S and r $\in$ L with r $\geq x \Rightarrow r \in$ S.

Definition 2.6:

A filter P in a lattice (L, $\vee$, $\wedge$) is called a **prime filter** of L if $x \vee y \in$ P, then either x $\in$ P or y $\in$ P.

Definition 2.7:

Let X be a non – empty set. A mapping μ: X → [0, 1] is called a **fuzzy subset** of X.

Definition 2.8:

Let μ be any fuzzy subset of a set X. Then, $\{t_i \in [0, 1] / i \in N \text{ and } \mu(x_i) = t_i\}$ is called the **image set of μ** and is denoted by Im μ.

Definition 2.9:

Let μ be any fuzzy subset of a set X and $t \in$ Im μ. Then the set $\mu_t = \{x \in X / \mu(x) \geq t\}$ is called the **level subset** of μ. Clearly, $\mu_t \subseteq \mu_s$ whenever $t > s$.

Definition 2.10:

A fuzzy subset μ of a lattice L is called a **fuzzy filter** of L if, for all $x, y \in L$ the following conditions are satisfied:

(i) $\mu(x \vee y) \geq \max\{\mu(x), \mu(y)\}$

(ii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$

Definition 2.11:

Let μ, θ be any two fuzzy filters of a lattice L.

Join $(\mu \vee \theta)$ of μ and θ is defined by,

$$(\mu \vee \theta)(x) = \underset{x = y \vee z}{Sup} \{\min\{\mu(y), \theta(z)\}\} \text{ where } x, y, z \in L.$$

Meet $(\mu \wedge \theta)$ of μ and θ is defined by,

$$(\mu \wedge \theta)(x) = \underset{x = y \wedge z}{Sup} \{\min\{\mu(y), \theta(z)\}\} \text{ where } x, y, z \in L.$$

3. FUZZY PRIME FILTER

Here are some Definitions related to fuzzy prime filter of a lattice.

Definition 3.1:

A non-constant fuzzy filter μ of a lattice L is called **prime fuzzy filter** of L, if $\rho \vee \eta \subseteq \mu$, for any two fuzzy filters ρ and η of L, then either $\rho \subseteq \mu$ or $\eta \subseteq \mu$.

Definition 3.2:

A fuzzy filter μ of a lattice L, is called **strongly fuzzy prime filter** of L, if for all $x, y \in L$, $\mu(x \vee y) = \mu(x)$ or $\mu(x \vee y) = \mu(y)$.

Definition 3.3:

A fuzzy filter μ of a L, is called **completely fuzzy prime filter** of L, if $\mu(a \vee b) = \mu(1)$ then either $\mu(a) = \mu(1)$ or $\mu(b) = \mu(1)$.

Definition 3.4:

A fuzzy filter μ of a lattice L, is called **fuzzy prime filter** of L, if the filter μ_t, where $t = \mu(1)$, is a prime filter of L.

Theorem 3.5:

Let μ be a non – constant prime fuzzy filter of a lattice L. Then Card Im μ = 2; that is μ is two – valued.

Proof:

Suppose Card Im $\mu \geq 3$. Let $\mu(1) = s$. There exists, $t, m \in \text{Im}\ \mu$ such that $t > m > s$. Let $k = \text{l.u.b.}\{\mu(x) / x \in L\}$. Then $t \leq k$ and $s < k$.

Consider the fuzzy subset σ of L such that $\sigma(x) = ½(t + m)$ for all $x \in L$. Then, σ is a fuzzy filter of L. Consider another fuzzy subset θ of L such that $\theta(x) = \begin{cases} s \text{ if } x \in \mu_m \\ k \text{ if } x \in L \sim \mu_m \end{cases}$, where $\mu_m = \{x \in L / \mu(x) \geq m\}$. Here μ_m is a filter of L. Then, θ is a fuzzy filter of L.

Next, we prove: $\sigma \vee \theta \subseteq \mu$. Let $x \in L$ be arbitrary.

Now, $(\sigma \vee \theta)(x) = \underset{x = r_1 \vee r_2}{Sup} \{\min\{\sigma(r_1), \theta(r_2)\}\}$ *where* $r_1, r_2 \in L$

$= \min\{\sigma(a), \theta(b)\}$ for some $a, b \in L$

Suppose $b \in \mu_m$. Then, $x \in \mu_m$, since $b \leq x$ and $\mu(x) \geq m$ and $\sigma(a) = ½(t + m)$ and $\theta(b) = s$. Then $(\sigma \vee \theta)(x) = \min\{½(t + m), s\} = s < m \leq \mu(x)$. Hence $(\sigma \vee \theta)(x) \leq \mu(x)$.

Suppose $b \notin \mu_m$. Then $\mu(b) < m$. Then $\mu(b) \leq k$. Here $k \leq \mu(x)$, since $\mu(b) \leq \mu(x)$ as $b \leq x$.

And $\sigma(a) = ½(t + m)$ and $\theta(b) = k$. Then $(\sigma \vee \theta)(x) = \min\{½(t + m), k\} = ½(t + m) < k \leq \mu(x)$. Hence $(\sigma \vee \theta)(x) \leq \mu(x)$.

Thus $\sigma \vee \theta \subseteq \mu$

Let $u \in L$ such that $\mu(u) = m$. Here $\sigma(u) = ½(t + m) > m = \mu(u)$. Hence $\sigma \not\subseteq \mu$.

Let $v \in L$ such that $\mu(v) = s$. Here $\mu(v) < m$ and $v \notin \mu_m$. Here $\theta(v) = k > s = \mu(v)$. Hence $\theta \not\subseteq \mu$

Thus neither $\sigma \subseteq \mu$ nor $\theta \subseteq \mu$. Hence μ is not a prime fuzzy filter of L. This contradiction proves that Card Im $\mu < 3$. Since μ is non – constant, Card Im $\mu \geq 2$. Hence, Card Im $\mu = 2$

Theorem 3.6:

Let μ be a non – constant prime fuzzy filter of a lattice L. Then $\mu(1) = 1$.

Proof:

Here, Card Im $\mu = 2$. Let Im $\mu = \{t, s\}$ and $t < s$. Then $\mu(1) = s$.

Suppose $s \neq 1$. Let $s < m \leq 1$. Then $t < s < m \leq 1$

Consider the fuzzy subset σ of L such that $\sigma(x) = ½(t + s)$ for all $x \in L$. Then, σ is a fuzzy filter of L.

Consider another fuzzy subset θ of L such that

$$\theta(x) = \begin{cases} m \text{ if } x \in \mu_s \\ t \text{ if } x \notin \mu_s \end{cases}, \text{ where } \mu_s = \{x \in L / \mu(x) \geq s\}.$$

Here μ_s is a filter of L. Then, θ is a fuzzy filter of L. Let $x \in L$ be arbitrary.

Now, $(\sigma \vee \theta)(x) = \underset{x = r_1 \vee r_2}{Sup} \{\min\{\sigma(r_1), \theta(r_2)\}\}$ *where* $r_1, r_2 \in L$

$= \min \{\sigma(a), \theta(b)\}$ for some $a, b \in L$

Suppose $b \in \mu_s$. Then, $x \in \mu_s$, since $b \leq x$ and $\mu(x) \geq s$. And $\sigma(a) = ½(t + s)$ and $\theta(b) = m$.

Then $(\sigma \vee \theta)(x) = \min \{½(t + s), m\} = ½ (t + s) < s \leq \mu(x)$. Thus $(\sigma \vee \theta)(x) \leq \mu(x)$.

Suppose $b \notin \mu_s$. Then $\mu(b) < s$. Here $\mu(b) \leq k$ and $k \leq \mu(x)$, since $\mu(b) \leq \mu(x)$ as $b \leq x$. And $\sigma(a) = ½(t + s)$ and $\theta(b) = t$. Then $(\sigma \vee \theta)(x) = \min \{½(t + s), t\} = t < k \leq \mu(x)$. Thus $(\sigma \vee \theta)(x) \leq \mu(x)$.

Hence $\sigma \vee \theta \subseteq \mu$.

Let $u \in L$ such that $\mu(u) = t$. Here $\sigma(u) = ½(t + m) > t = \mu(u) \Rightarrow \sigma \not\subseteq \mu$.

Let $v \in L$ such that $\mu(v) = s$. Here $\theta(v) = m > s = \mu(v) \Rightarrow \theta \not\subseteq \mu$.

Thus neither $\sigma \subseteq \mu$ nor $\theta \subseteq \mu$. Hence μ is not a prime fuzzy filter of L. This contradiction proves that $\mu(1) = 1$.

Theorem 3.7:

Let μ be any prime fuzzy filter of a lattice L. Then the filter μ_t of L, $t = \mu(1)$ is a prime filter of L.

Proof:

Consider the filter μ_t of L, $t = \mu(1)$. Let A and B be any two filters of L such that $A \vee B \subseteq \mu_t$.

Now we define the fuzzy subset ρ and η of L as follows:

$$\rho(x) = \begin{cases} \mu(1) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(x) = \begin{cases} \mu(1) & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

Then, ρ and η are fuzzy filters of L. Let $u \in L$ be arbitrary.

Suppose $(\rho \vee \eta)(u) = 0$, then $(\rho \vee \eta) \subseteq \mu$, since, $0 \leq \mu(u)$, for all $u \in L$.

Assume that $(\rho \vee \eta)(u) \neq 0$.

Then $\underset{u = y \vee z}{Sup}\{\min\{\rho(y), \eta(z)\}\} \neq 0$, since

$$(\rho \vee \eta)(u) = \underset{u = y \vee z}{Sup}\{\min\{\rho(y), \eta(z)\}\} \; where\; u, y, z \in L$$

$\Rightarrow \min\{\rho(u_1), \eta(u_2)\} > 0$ for some $u_1, u_2 \in L$ such that $u = u_1 \vee u_2$.

$\Rightarrow \rho(u_1) \neq 0$ and $\eta(u_2) \neq 0$. Here $\rho(u_1) = \mu(1)$ with $u_1 \in A$ And $\eta(u_2) = \mu(1)$ with $u_2 \in B$.

Now $u = u_1 \vee u_2 \in A \vee B$. Then $u \in \mu_t$, since $A \vee B \subseteq \mu_t$ and $\mu(u) \geq t$. Hence $\mu(u) \geq \mu(1)$, but $\mu(u) \leq \mu(1)$ for $u \in L$, since μ is a filter of L. Therefore $\mu(u) = \mu(1)$.

Again $\min\{\rho(u_1), \eta(u_2)\} = \min\{\mu(1), \mu(1)\} = \mu(1)$

Now, $(\rho \vee \eta)(u) = \underset{u = y \vee z}{Sup}\{\min\{\rho(y), \eta(z)\}\} \; where\; u, y, z \in L$

$\leq \min\{\rho(u_1), \eta(u_2)\}$, since all the other three cases are 0

$= \mu(1) \;\; = \mu(u)$

Thus $(\rho \vee \eta)(u) \leq \mu(u)$ for all $u \in L$. Hence $(\rho \vee \eta) \subseteq \mu$. Since μ is a prime fuzzy filter of L, either $\rho \subseteq \mu$ or $\eta \subseteq \mu$. Assume that $\rho \subseteq \mu$. If $A \not\subset \mu_t$, then there exists $r \in A$ such that $r \notin \mu_t$. Here $\mu(r) \neq \mu(1)$ and $\mu(r) < \mu(1)$, since $\mu(r) \leq \mu(1)$ for all $r \in L$, but $\rho(r) = \mu(1) > \mu(r)$. This is a contradiction to the assumption that $\rho \subseteq \mu$. Hence $A \subset \mu_t$. Similarly, if $\eta \subset \mu$, then $B \subset \mu_t$.

Theorem 3.8:

Let μ be a fuzzy subset of a lattice L such that Card Im $\mu = 2$, $\mu(1) = 1$, and the level subset $\mu_s = \{x \in L / \mu(x) = \mu(1)\}$ be a prime filter of L. Then μ is a prime fuzzy filter of L.

Proof:

Let μ be a fuzzy subset of a lattice L such that Card Im $\mu = 2$, $\mu(1) = 1$. Then Im $\mu = \{t, 1\}$, $t \in [0, 1)$. Consider $\mu_s = \{x \in L / \mu(x) = \mu(1) = 1\}$. Assume that μ_s is a prime filter of L. Let $x, y \in L$ be arbitrary.

Suppose $x \in \mu_s$ and $y \in \mu_s$. Then $\mu(x) = 1$ and $\mu(y) = 1$. Here, min $\{\mu(x), \mu(y)\} = 1$ and max $\{\mu(x), \mu(y)\} = 1$. Since μ_s is a filter of L, $x \vee y, x \wedge y \in \mu_s$. Then $\mu(x \vee y) = \mu(x \wedge y) = 1$.

Suppose $x \in \mu_s$ and $y \notin \mu_s$. Then $\mu(x) = 1$ and $\mu(y) = t$. Here, min $\{\mu(x), \mu(y)\} = t$ and max $\{\mu(x), \mu(y)\} = 1$. Since μ_s is a filter of L and $x \leq x \vee y$, we have $x \vee y \in \mu_s$. Then $\mu(x \vee y) = 1$. Here $\mu(x \wedge y) \geq t$.

Suppose $x \notin \mu_s$ and $y \notin \mu_s$. Then $\mu(x) = t$ and $\mu(y) = t$. Here, min $\{\mu(x), \mu(y)\} = t$ and max $\{\mu(x), \mu(y)\} = t$. Here $\mu(x \vee y) \geq t$; $\mu(x \wedge y) \geq t$.

Thus the two inequalities are satisfied in all the cases. Thus μ is a fuzzy filter of L.

Let σ and θ be any two fuzzy filters of L such that $\sigma \vee \theta \subseteq \mu$. Suppose that $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Then there exist $x, y \in L$ such that $\sigma(x) > \mu(x)$ and $\theta(y) > \mu(y)$. Here Im $\sigma \in [0, 1]$ and Im $\mu = \{t, 1\}$, but $\mu(x) < \sigma(x)$. Hence $\mu(x) = t$. Similarly, $\mu(y) = t$. Hence $x, y \notin \mu_s$. Then $x \vee y \notin \mu_s$. Therefore, $\mu(x \vee y) = t$.

Now, $(\sigma \vee \theta)(x \vee y) = \underset{x \vee y = u \vee v}{Sup} \{\min\{\sigma(u), \theta(v)\}\}$ *where* $u, v \in L$

$$\begin{aligned} &\geq \min \{\sigma(x), \theta(y)\} \\ &> t, \text{ since } \sigma(x) > \mu(x) = t,\ \theta(y) > \mu(y) = t \\ &= \mu(x \vee y) \end{aligned}$$

Thus $(\sigma \vee \theta)(x \vee y) > \mu(x \vee y)$. Hence $(\sigma \vee \theta) \supset \mu$. Therefore, $(\sigma \vee \theta) \not\subseteq \mu$. This is a contradiction to our assumption $(\sigma \vee \theta) \subseteq \mu$. Thus either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$. Hence μ is a prime fuzzy filter of L.

From the Theorems stated above, we characterize the prime fuzzy filter of a lattice by the following Theorem:

Theorem 3.9: [Characterization Theorem]

Let μ be a fuzzy subset of a lattice L. Then μ is prime fuzzy filter of L if and only Card Im $\mu = 2$, $\mu(1) = 1$, and the level subset $\mu_s = \{x \in L \,/\, \mu(x) = \mu(1)\}$ be a prime filter of L.

Proof:

Theorems: 3.5, 3.6, 3.7 and 3.8 complete the proof.

Corollary 3.10:

A filter P of L, $P \neq L$, is a prime filter of L iff $\yen_P$ is a prime fuzzy filter of L.

Proof:

Consider $\yen_P(x) = \begin{cases} 1 \; if \;\; x \in P \\ 0 \; otherwise \end{cases}$ is a fuzzy subset of L. Here Card Im $\yen_P = 2$. Since P is a filter then $1 \in P$ and hence $\yen_P(1) = 1$. Clearly the level subset $\{x \in L \,/ \yen_P(x) = \yen_P(1)\} = P$. Then the proof completes from the above Theorem.

Proposition 3.11:

Let μ be a non – constant prime fuzzy filter of a lattice L. Then Im $\mu = \{t, 1\}$, where $t \in [0, 1)$. Hence the chain of level filters of μ consists of $\mu_s \subseteq L$, where $\mu_s = \{x \in L \,/\, \mu(x) = \mu(1)\}$.

Proof:

By Theorem: 3.6, $\mu(1) = 1$. By Theorem: 3.5, Card Im $\mu = 2$. Then there must be another one element in Im μ other than 1. Let it be t. Then $t \in [0, 1)$. Thus Im $\mu = \{t,$

1}. Then there are only two level filters of μ. They are $\mu_s = \{x \in L / \mu(x) = \mu(1) = 1\}$ and $\mu_t = \{x \in L / \mu(x) \geq t\} = L$. Clearly, $\mu_s \subseteq L$. Hence the chain of level filters of μ consists of $\mu_s \subseteq L$.

Proposition 3.12:

Let μ be any non – constant prime fuzzy filter of a lattice L. Then for all x, y ∈ L, $\mu(x \vee y) = \max\{\mu(x), \mu(y)\}$.

Proof:

Here Card Im μ = 2 and μ(1) = 1. Therefore, Im μ = {t, 1}, where t < 1. Let x, y ∈ L be arbitrary. Then, $\mu(x \vee y) \geq \max\{\mu(x), \mu(y)\}$. Let A be the filter of L generated by x and B be the filter of L generated by y such that $A \vee B \subseteq \mu_k$, where $k = \mu(x \vee y)$. Now we define the fuzzy subset ρ and η of L as follows:

$$\rho(w) = \begin{cases} \mu(x \vee y) & \text{if } w \in A \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \eta(w) = \begin{cases} \mu(x \vee y) & \text{if } w \in B \\ 0 & \text{otherwise} \end{cases}$$

Then, ρ, η are fuzzy filters of L. If $(\rho \vee \eta)(w) = 0$, then nothing to prove. Assume $(\rho \vee \eta)(w) \neq 0$. Then $\underset{w = u \vee v}{Sup}\{\min\{\rho(u), \eta(v)\}\} \neq 0$, since

$(\rho \vee \eta)(w) = \underset{w = u \vee v}{Sup}\{\min\{\rho(u), \eta(v)\}\}$ *where* $w, u, v \in L$. Hence, $\min\{\rho(u), \eta(v)\} > 0$ for some u, v ∈ L. Then, ρ(u) ≠ 0 and η(v) ≠ 0. And $\rho(u) = \eta(v) = \mu(x \vee y)$. Therefore, u ∈ A, v ∈ B and $w = u \vee v \in A \vee B \subseteq \mu_k$. Thus, $w \in \mu_k$. Then $\mu(w) \geq k$ and $\mu(w) \geq \mu(x \vee y)$. Now $(\rho \vee \eta)(w) \leq \mu(x \vee y)$ for all w ∈ L, since all the other cases becomes zero and $(\rho \vee \eta)(x \vee y) \leq \mu(x \vee y)$. Hence $(\rho \vee \eta) \subseteq \mu$. Since μ is a prime fuzzy filter of L, either $\rho \subseteq \mu$ or $\eta \subseteq \mu$.

Assume that $\rho \subseteq \mu$. Here, $\rho(x) \leq \mu(x)$ and $\mu(x \vee y) \leq \mu(x)$, since x ∈ A and $\rho(x) = \mu(x \vee y)$.

Suppose $\mu(x \vee y) = 1$, then μ(x) = 1 and $\max\{\mu(x), \mu(y)\} = 1$. Thus $\mu(x \vee y) = \max\{\mu(x), \mu(y)\}$.

Suppose $\mu(x \vee y) < 1$. Then, $\mu(x \vee y) = t$, but $\mu(x) \leq \mu(x \vee y)$ and $\mu(y) \leq \mu(x \vee y)$, since $x \leq x \vee y$. Hence μ(x) = t, μ(y) = t and $\max\{\mu(x), \mu(y)\} = t$. Here, $\mu(x \vee y) = \max\{\mu(x), \mu(y)\}$.

Assume that $\eta \subseteq \mu$. Then, $\eta(y) \leq \mu(y)$ and $\mu(x \vee y) \leq \mu(y)$, since y ∈ B and $\eta(y) = \mu(x \vee y)$.

Suppose $\mu(x \vee y) = 1$, then $\mu(y) = 1$ and max $\{\mu(x), \mu(y)\} = 1$. Thus $\mu(x\vee y) =$ max $\{\mu(x), \mu(y)\}$.

Suppose $\mu(x \vee y) < 1$. Then, $\mu(x \vee y) = t$, but $\mu(x) \leq \mu(x \vee y)$ and $\mu(y) \leq \mu(x \vee y)$, since $x \leq x\vee y$.

Hence $\mu(x) = t$, $\mu(y) = t$ and max $\{\mu(x), \mu(y)\}= t$. Here $\mu(x \vee y) =$ max $\{\mu(x), \mu(y)\}$.

Thus $\mu(x \vee y) =$ max $\{\mu(x), \mu(y)\}$ in all the cases.

Proposition 3.13:

Let μ be a non – constant strongly prime fuzzy filter of a lattice L. Then $\mu(x \vee y) = \mu(x)$ or $\mu(x \vee y) = \mu(y)$, for all $x, y \in L$.

Proof:

Let $x, y \in L$ be arbitrary. Then $\mu(x \vee y) =$ max $\{\mu(x), \mu(y)\}$. Hence $\mu(x \vee y) = \mu(x)$ if max $\{\mu(x), \mu(y)\} = \mu(x)$ and $\mu(x \vee y) = \mu(y)$ if max $\{\mu(x), \mu(y)\} = \mu(y)$.

Remark 3.14:

The above Proposition shows that prime fuzzy filter implies strongly fuzzy prime filter.

Theorem 3.15:

Let μ be any non – constant strongly fuzzy prime filter of a lattice L. Then each level filter of L is a prime filter of L.

Proof:

Let μ be any fuzzy subset of a lattice L. Now Card Im $\mu = 2$ and $\mu(1) = 1$. Hence Im $\mu = \{t, 1\}$, $t \in [0, 1)$ and $t < 1$. Then the only level filters of L are μ_1 and μ_t. Here $\mu_1 \subseteq \mu_t$. By Theorem: 3.9, we have μ_1 is a prime filter of L. Clearly $\mu_t = L$. Let $a \vee b \in L$. Then $\mu(a \vee b) \geq t$. Suppose $a \notin L$ and $b \notin L$. Then $\mu(a) < t$ and $\mu(b) < t$. Hence max $\{\mu(a), \mu(b)\} < t$. This makes a contradiction to $\mu(a \vee b) =$ max $\{\mu(a), \mu(b)\}$ as in Proposition: 3.12 and this contradiction proves that either $a \in L$ and $b \in L$. Thus $\mu_t = L$ is a prime filter of L.

Remark 3.16:

The converse of the above theorem is not true in general.

Proof: By Example,

Consider the lattice in Example 2.2, $(\wp(A), \vee, \wedge)$ where $A = \{a, b\}$.

Here, $\mu = \{(\phi, 2/5), (\{a\}, 2/5), (\{b\}, 3/4), (\{a, b\}, 3/4)\}$,

$\eta = \{(\phi, 1/4), (\{a\}, 1/4), (\{b\}, 4/5), (\{a, b\}, 1)\}$ and

$\rho = \{(\phi, 2/3), (\{a\}, 3/4), (\{b\}, 2/3), (\{a, b\}, 3/4)\}$ are fuzzy filters of L.

Clearly, the two level filters $\mu_{3/4} = \{\{b\}, \{a, b\}\}$ and $\mu_{2/5} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ are prime filters of L.

Now, $(\eta \vee \rho)(\phi) = \underset{\phi = y \vee z}{Sup} \{\min\{\eta(y), \rho(z)\}\}$ *where* $\phi, y, z \in L$

$= \min \{\eta(\phi), \rho(\phi)\}$

$= \min \{1/4, 2/3\}$

$= 1/4$

$(\eta \vee \rho)(\{a\}) = \underset{\{a\} = y \vee z}{Sup} \{\min\{\eta(y), \rho(z)\}\}$ *where* $\{a\}, y, z \in L$

$= Sup \{ (\min \{\eta(\{a\}), \rho(\phi)), (\min \{\eta(\phi), \rho(\{a\})), (\min \{\eta(\{a\}), \rho(\{a\})) \}$

$= Sup \{ (\min \{1/4, 2/3\}), (\min \{1/4, 3/4\}), (\min \{1/4, 3/4\}) \}$

$= Sup \{ 1/4, 1/4, 1/4 \}$

$= 1/4$

$(\eta \vee \rho)(\{b\}) = \underset{\{b\} = y \vee z}{Sup} \{\min\{\eta(y), \rho(z)\}\}$ *where* $\{b\}, y, z \in L$

$= Sup \{ (\min \{\eta(\phi), \rho(\{b\})), (\min \{\eta(\{b\}), \rho(\phi)), (\min \{\eta(\{b\}), \rho(\{b\})) \}$

$= Sup \{ (\min \{1/4, 2/3\}), (\min \{4/5, 2/3\}), (\min \{4/5, 2/3\}) \}$

$= Sup \{ 1/4, 2/3, 2/3 \} = 2/3$

$(\eta \vee \rho)(\{a, b\}) = \underset{\{a, b\} = y \vee z}{Sup} \{\min\{\eta(y), \rho(z)\}\}$ *where* $\{a, b\}, y, z \in L$

$= Sup \{ (\min \{ \eta(\{a, b\}), \rho(\{a, b\}) \}), (\min \{\eta(\{a\}), \rho(b)\}),$

$(\min \{\eta(\{b\}), \rho(a)\}), (\min \{\eta(\{a, b\}), \rho(\phi)\}), (\min \{\eta(\phi), \rho(\{a, b\})\})$

$= Sup \{ (\min \{1, 3/4\}), (\min \{1/4, 2/3\}), (\min \{4/5, 3/4\}), (\min \{1, 2/3\}),$

$(\min \{1/4, 3/4\}) \}$

$= Sup \{3/4, 1/4, 3/4, 2/3, 1/4\} = 3/4$

Here, $(\eta \vee \rho)(\phi) = 1/4 \leq 2/5 = \mu(\phi)$ and $(\eta \vee \rho)(\{a\}) = 1/4 \leq 2/5 = \mu(\{a\})$ and $(\eta \vee \rho)(\{b\}) = 2/3 \leq 3/4 = \mu(\{b\})$ and $(\eta \vee \rho)(\{a, b\}) = 3/4 \leq 3/4 = \mu(\{a, b\})$. Hence

$(\eta \vee \rho) \subseteq \mu$, but neither $\eta \subseteq \mu$ nor $\rho \subseteq \mu$, since $\eta(\{b\}) = 4/5 > 3/4 = \mu(\{b\})$ and $\rho(\{a\}) = 3/4 > 2/5 = \mu(\{a\})$.

Remark 3.17:

The fuzzy subset μ in the above example is strongly fuzzy prime filter according to Definitions: 3.2, completely fuzzy prime filter according to Definitions: 3.3 and fuzzy prime filter according to Definitions: 3.4.

But μ is not a prime fuzzy filter according to Definitions 3.1. And this shows that Definition: 3.2, 3.3, 3.4 does not imply Definition: 3.1

Theorem 3.18:

Let μ be any non – constant fuzzy filter of a lattice L. Then μ is strongly fuzzy prime filter of L if and only if each level filter of L is a prime filter of L.

Proof:

Let μ be any non – constant fuzzy filter of a lattice L. Then, $\mu(0) \leq \mu(x) \leq \mu(1)$ for $x \in L$, where 1 is the greatest element and 0 is the least element of L. Let $t \in \text{Im}\ \mu$ be arbitrary. Then $t \leq \mu(1)$. Let $x, y \in L$ be arbitrary.

Assume that μ is a strongly fuzzy prime filter of L. Then $\mu(x \vee y) = \mu(x)$ or $\mu(x \vee y) = \mu(y))$. If $t = \mu(0)$, then $\mu_t = \{x \in L/\mu(x) \geq t\} = L$. Clearly, L is a prime filter of L. Thus μ_t, $t = \mu(0)$ is a prime filter of L. If $\mu(x \vee y) = t > \mu(0)$ for some $x, y \in L$, then $x \vee y \in \mu_t$. Then, either $\mu(x \vee y) = \mu(x)$ or $\mu(x \vee y) = \mu(y)$. Hence either $x \in \mu_t$ or $y \in \mu_t$. This proves that μ_t is a prime filter of L.

Conversely, assume each level filter of L is a prime filter of L. Suppose $\mu(x \vee y) = \mu(0)$. Then $\mu(x \vee y) = \mu(x) = \mu(y)$. Suppose $\mu(x \vee y) = t > \mu(0)$. Then $x \vee y \in \mu_t$. Here $x \in \mu_t$ or $y \in \mu_t$, since μ_t is a prime filter of L. Hence $\mu(x) = \mu(x \vee y)$ or $\mu(y) = \mu(x \vee y)$. Thus μ is strongly fuzzy prime filter of L.

Proposition 3.19:

Let μ be a strongly fuzzy prime filter of a lattice L. If $\mu(a \vee b) = \mu(1)$ for some $a, b \in L$, then either $\mu(a) = \mu(1)$ or $\mu(b) = \mu(1)$.

Proof:

Let $\mu(a \vee b) = \mu(1)$, for some $a, b \in L$. Then $\mu(1) = \mu(a \vee b) = \mu(a)$ or $\mu(1) = \mu(a \vee b) = \mu(b)$.

Hence $\mu(a) = \mu(1)$ or $\mu(b) = \mu(1)$.

Remark 3.20:

The above Proposition shows that strongly fuzzy prime filter implies completely fuzzy prime filter.

Remark 3.21:

But completely fuzzy prime filter does not imply strongly fuzzy prime filter.

Proof: By Example,

Consider the lattice in Example 2.3, ($\wp$(A), $\vee$, $\wedge$) where A = {a, b, c}.

Here, μ = { (ϕ, .2), ({a}, .3), ({b}, .6), ({c}, .4), ({a, b}, .6), ({a, c}, .5), ({b, c}, .6), ({a, b, c}, .6) } is fuzzy filter of L.

Then μ is a completely fuzzy prime filter of L according to Definition: 3.3, since $\mu(x \vee y) = \mu(\{a, b, c\})$ implies either $\mu(x) = \mu(\{a, b, c\})$ and $\mu(y) = \mu(\{a, b, c\})$.

Let x = {a} and y = {c}. Here $\mu(x \vee y) = \mu(\{a\} \vee \{c\}) = \mu(\{a, c\}) = .5$, but $\mu(x) = \mu(\{a\}) = .3 \neq .5 = \mu(\{a, c\}) = \mu(x \vee y)$ and $\mu(y) = \mu(\{c\}) = .4 \neq .5 = \mu(\{a, c\}) = \mu(x \vee y)$. Hence neither $\mu(x \vee y) = \mu(x)$ nor $\mu(x \vee y) = \mu(y)$. Thus μ is not a strongly fuzzy prime filter of L according to Definition 3.2.

Example 3.22:

Consider the lattice in Example 2.2, ($\wp$(A), $\vee$, $\wedge$) where A = {a, b}. Here, θ = {(ϕ, .3), ({a}, .3), ({b}, .6), ({a, b}, 1)}. Then θ is a fuzzy prime filter of L according to Definition 3.4, since the level filter $\theta_t = \{x \in L / \theta(x) = \theta(1)\} = \{a, b\}$ is a prime filter of L.

Here $\theta(\{a\}\vee\{b\}) = \theta(\{a, b\}) = 1$; $\theta(\{a\}) = .3$ and $\theta(\{b\}) = .6$. Hence neither $\theta(\{a\}\vee\{b\}) = \theta(\{a\})$ nor $\theta(\{a\}\vee\{b\}) = \theta(\{b\})$.Thus θ is not a strongly fuzzy prime filter of L according to Definition: 3.2 as well as not a completely fuzzy prime filter according to Definition: 3.3. Here Card Im $\theta \neq 1$. Hence θ is not a prime fuzzy filter of L according to Definition 3.1.

Remark 3.23:

The above Example shows that Definition 3.4 does not imply Definition 3.2 and Definition 3.4 does not imply Definition 3.3. Also the above Example shows that Definition 3.4 does not imply Definition 3.1.

4. CONCLUSION

We conclude the relation between the four definitions as follows:

$$\begin{tikzcd}
\begin{array}{c}\text{Prime fuzzy filter}\\\text{Definition: 3.1}\end{array} \arrow[r, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] & \begin{array}{c}\text{Strongly fuzzy prime}\\\text{filter}\\\text{Definition: 3.2}\end{array} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow, shift left] \arrow[d, Rightarrow, shift left] & \begin{array}{c}\text{Completely fuzzy}\\\text{prime filter}\\\text{Definition: 3.3}\end{array} \arrow[l, Rightarrow, shift left, "/" marking] \arrow[dl, Rightarrow, shift left] \\
 & \begin{array}{c}\text{Fuzzy prime filter}\\\text{Definition: 3.4}\end{array} \arrow[ul, Rightarrow, shift left, "/" marking] \arrow[u, Rightarrow, shift left, "/" marking] \arrow[ur, Rightarrow, shift left, "/" marking] &
\end{tikzcd}$$

DISCUSSION:

Now we discuss the Definitions with respect to the following properties:

(P1). A filter J of L is a prime filter of L, if and only if the characteristic function $¥_J$ is a fuzzy filter of L and is prime.

(P2). A fuzzy filter μ of L is a fuzzy prime filter of L, if and only if each level filter μ_t is a fuzzy filter of L and is prime.

(P3). Card Im $\mu > 2$.

All the four Definitions satisfy (P1).

Definition 3.1 satisfies (P2) in one direction only. That is, "If a fuzzy filter μ of L is a prime fuzzy filter of L, then each level filter μ_t is a prime filter of L". But it does not satisfy the converse.

Definition 3.4 satisfies (P2) in the reverse direction only. That is, "If each level filter μ_t is a prime filter of L, then the fuzzy filter μ of L is a fuzzy prime filter of L". But it does not satisfy in forward direction.

The rest of the two Definitions satisfy (P2) in both the directions.

Definition 3.1 does not satisfy (P3). The other three Definitions do not hold any condition on Card Im μ.

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