International Journal of Applied Mathematical Sciences
ISSN 0973-0176 Volume 9, Number 1 (2016), pp. 61-71

http://www.ripublication.com

COMMON FIXED POINT THEOREMS FOR SIX MAPPINGS ON GENERALIZED FUZZY METRIC SPACES

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Abstract

In this paper we extend the result of Turkoglu et.al[10] and prove a common fixed point theorems for compatible maps of type (α) on generalized fuzzy metric spaces.

Keywords: Metric Spaces, Compatible Mappings, $\mathcal{M}$- Fuzzy Metric Space, Weak Compatible of type (α)

2000 AMS Subject classification: 47H10, 54H25

1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [11]. Since then, many authors have tried to use this concept in topology and analysis and developed the theory of fuzzy sets and applications. Especially, Deng [3], Erceg [4], Kaleva and Seikkala [7], Kramosil and Michalek [8] have introduced the concept of fuzzy metric spaces in different ways. Grabiec [5] followed Kramosil and Michalek [8] and obtained the fuzzy version of Banach's fixed point theorem. Many authors have studied the fixed point theory in fuzzy metric spaces.

Jungck et al [6] introduced the concept of compatible maps of type (A) in metric spaces and proved common fixed point theorems in metric spaces. Cho [1] introduced the notion of compatible maps of type (α) in fuzzy metric spaces. In 2006, Sedghi and Shobe [9] introduced D* - metric space as a probable modification of the definition of D - metric introduced by Dhage [2], and prove some basic properties in D* - metric spaces. Sedghi and Shobe [9] introduced $\mathcal{M}$-fuzzy metric space which is based on D*-metric concept. In this paper we extend the result of Turkoglu et.al[10] and prove a common fixed point theorems for compatible maps of type (α) on generalized fuzzy metric spaces.

2. PRELIMINARIES

In this section we recall some definitions and known results in fuzzy metric spaces.

Definition 2.1:

A 3-tuple $(X, \mathcal{M},*)$ is called $\mathcal{M}$-fuzzy metric space if X is an arbitrary non-empty set

* is a continuous t -norm, and $\mathcal{M}$ is a fuzzy set on $X^3 x[0,\infty)$, satisfying the following conditions: for each x, y, z, a $\in$ X and t, s > 0

(FM-1) $\mathcal{M}(x,y,z,t)>0$

(FM-2) $\mathcal{M}(x,y,z,t) = 1$ iff $x = y = z$

(FM-3) $\mathcal{M}(x,y,z,t) = \mathcal{M}(p\{x,y,z\},t)$, where p is a permutation function

(FM-4) $\mathcal{M}(x,y,a,t) * \mathcal{M}(a,z,z,s) \leq \mathcal{M}(x,y,z,t+s)$

(FM-5) $\mathcal{M}(x,y,z,.) :(0,\infty) \rightarrow [0,1]$ is continuous

(FM-6) $\lim_{t\to\infty}\mathcal{M}(x,y,z,t) = 1$

Lemma 2.2:

Let $(X,\mathcal{M},*)$ be a $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non decreasing with respect to t, for all x, y, z in X.

Lemma 2.3:

Let $\{x_n\}$ be a sequence in a $\mathcal{M}$- fuzzy metric space $(X, \mathcal{M}, *)$ with the condition (FM-6). If there exists a number $k \in (0, 1)$ such that $\mathcal{M}(x_n, x_{n+1}, x_{n+1}, kt) \geq \mathcal{M}(x_{n-1}, x_n, x_n, t)$ for all $t > 0$ and

n =1, 2, 3 …, then $\{x_n\}$ is a Cauchy sequence.

Definition 2.4:

Let A and S be self mappings of a $\mathcal{M}$-fuzzy metric space (X, $\mathcal{M}$,*). Then the mappings are said to be compatible if $\lim_{n\to\infty} \mathcal{M}$ (ASx_n, SAx_n, SAx_n, t) = 1, for all t>0, whenever {x_n} is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z∈X.

Definition 2.5:

Let A and S be self mappings of a $\mathcal{M}$-fuzzy metric space (X, $\mathcal{M}$,*). Then the mappings are said to be compatible of type (α), if

$\lim_{n\to\infty} \mathcal{M}$ (ASx_n, SSx_n, SSx_n, t)= $\lim_{n\to\infty}\mathcal{M}$ (SAx_n, AAx_n, AAx_n, t) for all t>0, whenever {x_n} is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z∈X.

Proposition 2.6:

Let (X, $\mathcal{M}$, *) be a $\mathcal{M}$-fuzzy metric space with t * t ≥ t for all t ∈ [0, 1] and A, S be continuous maps from X into itself. Then A and S are compatible if and only if they are compatible of type (α).

Proposition 2.7:

Let (X, $\mathcal{M}$, *) be a $\mathcal{M}$-fuzzy metric space with t * t ≥ t for all t ∈ [0, 1] and A, S be continuous maps from X into itself. If A and B are compatible of type (α) and Az = Sz for some

z∈ X, then ASz = SSz = SAz = AAz.

Proposition 2.8:

Let (X, $\mathcal{M}$, *) be a $\mathcal{M}$-fuzzy metric space with t * t ≥ t for all t ∈ [0, 1] and A, S be compatible maps of type (α) from X into itself. Let {x_n} be a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$for some z ∈ X. Then we have the following:

(i) $\lim_{n\to\infty} Sx_n = Az$ if A is continuous at z,

(ii) ASz = SAz and Az = Sz if A and S are continuous at z.

Example 2.9:

Let X = [0, ∞) with the metric D* defined by D* (x, y, z) = |x-y| + |y-z| + |z-x| and for each t > 0, define $\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x,y,z)}$ for all x, y, z ∈ X. Clearly (X, $\mathcal{M}$, *) is a fuzzy metric space, where * is defined by a * b = ab. Define A, S: X →X by

$Ax = \begin{cases} x^2, & 0 \le x < 1 \\ 2, & x \ge 1 \end{cases}$, $Sx = \begin{cases} 2 - x^2, & 0 \le x < 1 \\ 2, & x \ge 1 \end{cases}$

Clearly A and S are discontinuous at x = 1. Consider the sequence $\{x_n\}$ in X defined by $x_n = 1 - \frac{1}{n}$

Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1 \in X$. Also $ASx_n \to 2$, $SSx_n \to 2$ as $n \to \infty$ and

$SAx_n = 1 - \frac{4}{n^2} - \frac{1}{n^4} + \frac{4}{n} + \frac{2}{n^3} - \frac{2}{n^2}$, $AAx_n = 1 + \frac{4}{n^2} + \frac{1}{n^4} - \frac{4}{n} - \frac{4}{n^3} + \frac{2}{n^2}$.

Then $\lim_{n\to\infty} \mathcal{M}(ASx_n, SAx_n, SAx_n, t) \neq 1$ but, $\lim_{n\to\infty} \mathcal{M}(ASx_n, SSx_n, SSx_n, t) = 1$

and $\lim_{n\to\infty} \mathcal{M}(SAx_n, AAx_n, AAx_n, t) = 1$, as $n \to \infty$.

Thus A and S are compatible of type (α) but they are not compatible.

Example 2.10 :

Let X = [0, ∞) with the metric D* defined by D* (x, y, z) = |x-y| +|y-z| + |z-x| and for each t > 0, define $\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x,y,z)}$, for all x, y, z ∈ X.

Clearly (X, $\mathcal{M}$, *) is a fuzzy metric space, where * is defined by a * b = ab.

Define A, S: X →X by $Ax = \begin{cases} 1 + x, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$, $Sx = \begin{cases} 1 - x, & 0 \le x < 1 \\ 2x, & x \ge 1 \end{cases}$

Clearly A and S are discontinuous at x = 1. Consider the sequence $\{x_n\}$ in X defined by $x_n = \frac{1}{n}$. Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1 \in X$.

Further, $ASx_n = 2 - \frac{1}{n}$, $SAx_n = 2 + \frac{2}{n}$. $AAx_n = 1 + \frac{1}{n}$, $SSx_n = \frac{1}{n}$

Therefore $\lim_{n\to\infty} \mathcal{M}(ASx_n, SAx_n, SAx_n, t) = 1$, $\lim_{n\to\infty} \mathcal{M}(ASx_n, SSx_n, SSx_n, t) \neq 1$

and $\lim_{n\to\infty} \mathcal{M}(SAx_n, AAx_n, AAx_n, t) \neq 1$, as $n \to \infty$.

Thus A and S are compatible but they are not compatible of type (α).

3. COMPATIBLE OF TYPE (α).

Theorem 3.1:

Let (X, $\mathcal{M}$, *) be a complete generalized fuzzy metric space with t * t ≥ t for all t ∈ [0, 1] and let A, B, P, Q, S and T be maps from X into itself such that

(i) P (ST)(X) ⊆ AB (ST)(X), Q(AB)(X) ⊆ AB (ST)(X),

(ii) There exists a constant k ∈ [0, 1) such that

$$\mathcal{M}^2(Px, Qy, Qy, kt) * [\mathcal{M}(ABx, Px, Px, kt)\mathcal{M}(STy, Qy, Qy, kt)] * \mathcal{M}^2(STy, Qy, Qy, kt) \ge [p\mathcal{M}(ABx, Px, Px, t) + q\mathcal{M}(ABx, STy, STy, t)]\mathcal{M}(ABx, Qy, Qy, 2kt)$$

for all x, y $\in$ X and t > 0, where 0 < p, q < 1 such that p + q = 1.

(iii) A, B, S and T are continuous .

(iv) AB = BA, ST = TS, PB = BP, TQ = QT, AB (ST) = ST(AB),

(v) The pairs (P, AB) and (Q, ST) are compatible of type (α).

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof :

Let $x_0 \in$ X be arbitrary. By (i) we can construct a sequence $\{x_n\}$ in X as follows.

$$P(ST)x_{2n} = AB(ST)x_{2n+1},\ Q(AB)x_{2n+1} = AB(ST)x_{2n+2},\ n = 0, 1, 2, 3, \ldots..$$

Let $z_n = AB(ST)x_n$, then by (ii),

$$\{\mathcal{M}^2(P(ST)x_{2n}, Q(AB)x_{2n+1}, Q(AB)x_{2n+1}, kt) *$$

$$\begin{bmatrix} \mathcal{M}(AB(ST)x_{2n}, P(ST)x_{2n}, P(ST)x_{2n}, kt) \\ \mathcal{M}(ST(AB)x_{2n+1}, Q(AB)x_{2n+1}, Q(AB)x_{2n+1}, kt) \end{bmatrix}$$

$$* \mathcal{M}^2\ (ST(AB)x_{2n+1}, Q(AB)x_{2n+1}, Q(AB)x_{2n+1}, kt)\}$$

$$\geq \begin{bmatrix} p\mathcal{M}(AB(ST)x_{2n}, P(ST)x_{2n}, P(ST)x_{2n}, t) + \\ q\mathcal{M}(AB(ST)x_{2n}, ST(AB)x_{2n+1}, ST(AB)x_{2n+1}, t \end{bmatrix}$$

$$\mathcal{M}(AB(ST)x_{2n}, Q(AB)x_{2n+1}, Q(AB)x_{2n+1}, 2kt) \quad \text{and}$$

$$\{\ \mathcal{M}^2(AB(ST)x_{2n+1}, AB(ST)x_{2n+2}, AB(ST)x_{2n+2}, kt) *$$

$$\begin{bmatrix} \mathcal{M}(z_{2n}, AB(ST)x_{2n+1}, AB(ST)x_{2n+1}, kt) \\ \mathcal{M}(z_{2n+1}, AB(ST)x_{2n+2}, AB(ST)x_{2n+2}, kt) \end{bmatrix} *$$

$$\mathcal{M}^2(z_{2n+1}, AB(ST)x_{2n+2}, AB(ST)x_{2n+2}, kt)\}$$

$$\geq \begin{bmatrix} p\mathcal{M}(z_{2n}, AB(ST)x_{2n+1}, AB(ST)x_{2n+1}, t) + \\ q\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, t) \end{bmatrix}$$

$$\mathcal{M}(z_{2n}, AB(ST)x_{2n+2}, AB(ST)x_{2n+2}, 2kt)\ , \quad \text{then}$$

$$\{\mathcal{M}^2(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt) *$$

$$[\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, kt)\mathcal{M}(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt)] * \mathcal{M}^2(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt)\}$$

$$\geq \begin{bmatrix} p\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, t) + \\ q\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, t) \end{bmatrix} \mathcal{M}\ (z_{2n}, z_{2n+2}, z_{2n+2}, 2kt)$$

so

$$\{\mathcal{M}^2(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt) *$$
$$[\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, kt)\mathcal{M}(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt)]\}$$

$$\geq [(p+q)\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, t)]\mathcal{M}\ (z_{2n}, z_{2n+2}, z_{2n+2}, 2kt)$$

and

$\{ \mathcal{M}^2(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt)[\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, kt) * \mathcal{M}(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt)]\}$

$$\geq (p+q)\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, t)\mathcal{M}\ (z_{2n}, z_{2n+2}, z_{2n+2}, 2kt)$$

$\{\mathcal{M}(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt)\mathcal{M}(z_{2n}, z_{2n+2}, z_{2n+2}, 2kt)\}$

$$\geq (p+q)\mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, t)\mathcal{M}(z_{2n}, z_{2n+2}, z_{2n+2}, 2kt)$$

Then it follows that

$\mathcal{M}(z_{2n+1}, z_{2n+2}, z_{2n+2}, kt) \geq \mathcal{M}(z_{2n}, z_{2n+1}, z_{2n+1}, t)$,for $0 < k < 1$ and for all $t > 0$.

Similarly, we also have

$\mathcal{M}(z_{2n+2}, z_{2n+3}, z_{2n+3}, kt) \geq \mathcal{M}(z_{2n+1}, z_{2n+2}, z_{2n+2}, t)$,for $0 < k < 1$ and for all $t > 0$.

In general,

$\mathcal{M}(z_{m+1}, z_{m+2}, z_{m+2}, kt) \geq \mathcal{M}(z_m, z_{m+1}, z_{m+1}, t)$, $2n+1 = m = 1, 2, 3, \ldots$

for $0 < k < 1$ and for all $t > 0$.

Therefore by Lemma (2.3), $\{z_n\}$ is a Cauchy sequence in X. Since (X, $\mathcal{M}$, *) is complete,$\{z_n\}$ converges to a point $x \in X$ and since $\{P(ST)\, x_{2n}\}$ and $\{Q(AB)\, x_{2n+1}\}$ are subsequences of $\{z_n\}$, $P(ST)x_{2n} \to z$ and $Q\,(AB)\, x_{2n+1} \to z$ as $n \to \infty$.

Let $y_n = STx_n$ and $w_n = ABx_n$, for n = 1, 2, 3 …, then we have

$Py_{2n} \to z$, $ABy_{2n} \to z$, $STw_{2n+1} \to z$ and $Qw_{2n+1} \to z$ as $n \to \infty$.

Since the pairs (P, AB) and (Q, ST) are compatible of type (α), we have as $n \to \infty$

$\mathcal{M}(P(AB)y_{2n}, AB(AB)y_{2n}, AB(AB)y_{2n}, t) \to 1$,

$\mathcal{M}(P(AB)y_{2n},\ PPy_{2n},\ PPy_{2n},\ t) \to 1$,

$\mathcal{M}\big((ST)Qw_{2n+1},\ QQy_{2n+1},\ QQy_{2n+1},\ t\big) \to 1$,

$\mathcal{M}(Q(ST)w_{2n+1},\ ST(ST)w_{2n+1},\ ST(ST)w_{2n+1},\ t) \to 1$,

Moreover, by the continuity of A, B, S and T and Proposition (2.8), we have

$Q(ST)w_{2n+1} \to STz$, $ST(ST)w_{2n+1} \to STz$,

$P(AB)y_{2n} \to ABy_{2n}$, $AB(AB)y_{2n} \to ABy_{2n}$, as $n \to \infty$.

Now taking $x = y_{2n}$ and $y = STw_{2n+1}$ in (ii),

We have

$\{\mathcal{M}^2(Py_{2n},\ Q(ST)w_{2n+1}, Q(ST)w_{2n+1}, kt) *$

$$\begin{bmatrix} \mathcal{M}(ABy_{2n}, Py_{2n}, Py_{2n}, kt) \\ \mathcal{M}(ST(ST)w_{2n+1}, Q(ST)w_{2n+1}, Q(ST)w_{2n+1}, kt) \end{bmatrix}$$

$* \mathcal{M}^2(ST(ST)w_{2n+1},\ \ Q(ST)w_{2n+1}, Q(ST)w_{2n+1}, kt)\}$

$$\geq \begin{bmatrix} p\mathcal{M}(ABy_{2n}, Py_{2n}, Py_{2n}, \quad t) + \\ q\mathcal{M}\ (ABy_{2n}, ST(ST)w_{2n+1}, ST(ST)w_{2n+1}, t) \end{bmatrix}$$

$\mathcal{M}(ABy_{2n}, Q(ST)w_{2n+1}, Q(ST)w_{2n+1}, 2kt)$.

This implies as n $\rightarrow \infty$

$\{\mathcal{M}^2(z, STz, STz, kt) * [\mathcal{M}(z, z, z, kt)\mathcal{M}(STz, STz, STz, kt)] *$

$\mathcal{M}^2\ (STz, STz, STz, kt)\ \} \geq [p\mathcal{M}(z, z, z, t) +$
$q\mathcal{M}(z, STz, STz, t)]\mathcal{M}\ (z, STz, STz, 2kt)$.

Then it follows that $\mathcal{M}^2(z, STz, STz, kt) \geq [p + q\mathcal{M}(z, STz, STz, t)]\mathcal{M}(z, STz, 2kt)$

and since $\mathcal{M}(x, y, z, .)$ is non-decreasing for all x, y $\in$ X, we have

$\{\mathcal{M}(z, STz, STz, 2kt)\mathcal{M}(z, STz, STz, t)\} \geq [p +$
$q\mathcal{M}(z, STz, STz, t)]\mathcal{M}(z, STz, STz, 2kt)$

Thus, $\mathcal{M}(z, STz, STz, t) \geq p + q\mathcal{M}\ (z, STz, STz, t)$

$\Longrightarrow \mathcal{M}(z, STz, STz, t) \geq \frac{p}{1-q} = 1$, for all t > 0.

So z = STz. Similarly z = ABz.

Now taking x = y_{2n} and y =z in (ii), we have

$\{\mathcal{M}^2(Py_{2n}, Qz, Qz, kt) * \begin{bmatrix} \mathcal{M}(ABy_{2n}, Py_{2n}, Py_{2n}, kt) \\ \mathcal{M}(STz, Qz, Qz, kt) \end{bmatrix} * \mathcal{M}^2\ (STz, Qz, Qz, kt)\}$

$\geq [p\mathcal{M}(ABy_{2n}, Py_{2n}, Py_{2n}, t) +\ q\mathcal{M}(ABy_{2n}, STz, STz, t)]\mathcal{M}(ABy_{2n}, Qz, Qz, 2kt)$.

This implies as n $\rightarrow \infty$

$\{\mathcal{M}^2(z, Qz, Qz, kt) * \mathcal{M}(z, Qz, Qz, kt)\} \geq (p + q)\mathcal{M}\ (z, Qz, Qz, 2kt)$.

So,$\mathcal{M}(z, Qz, Qz, kt)[\mathcal{M}(z, Qz, Qz, kt) * 1] \geq \mathcal{M}(z, Qz, Qz, 2kt)$

and since $\mathcal{M}(x, y, y, .)$ is non decreasing for all x, y $\in$ X, we have

$\{\ \mathcal{M}(z, Qz, Qz, 2kt)\mathcal{M}(z, Qz, Qz, t)\} \geq \mathcal{M}\ (z, Qz, Qz, 2kt)$.

Then it follows that $\mathcal{M}(z,\ \ Qz, Qz, t) =\ 1$, for all t > 0. So z = Qz.

Similarly, we have z = Pz. Now we show Bz = z and Tz = z.

Taking x = Bz and y = z in (ii), we get $\{\mathcal{M}^2(P(Bz), Qz, Qz, kt) * [\mathcal{M}(AB(Bz), P(Bz), P(Bz), kt)\mathcal{M}(STz, Qz, Qz, kt)] * \mathcal{M}^2(STz, Qz, Qz, kt)\} \geq \begin{bmatrix} p\mathcal{M}(AB(Bz), P(Bz), P(Bz), t) + \\ q\mathcal{M}(AB(Bz), STz, STz, t) \end{bmatrix} \mathcal{M}\ (AB(Bz), Qz, Qz, 2kt)$,

which gives

$$\{\mathcal{M}^2(Bz, z, z, kt) * [\mathcal{M}(Bz, Bz, Bz, kt)\mathcal{M}\ (z, z, z, kt)] * \mathcal{M}^2\ (z, z, z, kt)\}$$

$$\geq [p\mathcal{M}(Bz, Bz, Bz, t) +\ q\mathcal{M}(Bz, z, z, t)]\mathcal{M}(Bz, z, z, 2kt$$

$$\Rightarrow \mathcal{M}^2(Bz, z, z, kt) \geq [p + q\mathcal{M}(Bz, z, z, t)\mathcal{M}(Bz, z, z, 2kt)]$$

and since $\mathcal{M}(x, y, y, .)$ is non-decreasing for all x, y ∈ X, we have

$$\{\mathcal{M}(Bz, z, z, 2kt)\mathcal{M}(Bz, z, z, t)\} \geq [p + q\mathcal{M}\ (Bz, z, z, t)]\mathcal{M}(Bz, z, z, 2kt)$$

Thus

$$\mathcal{M}(Bz, z, z, t) \geq p + q\mathcal{M}\ (Bz, z, z, t)$$

$$\Rightarrow \mathcal{M}(Bz, z, z, t) \geq \frac{p}{1-q} = 1 \text{ for all } t > 0.$$

So Bz = z. Similarly, we have Tz = z.

Since z = ABz, therefore Az = z and since Tz = z therefore Sz = z.

By combining the above results, we have Az = Bz = Sz = Tz = Pz = Qz = z,

that is, z is the common fixed point of A, B, P, Q, S and T.

Uniqueness:

Let v ≠ z be another fixed point of A, B, P, Q, S and T. Then using (ii),

$$\{\mathcal{M}^2(Pz, Q\mathrm{v}, Q\mathrm{v}, kt) * [\mathcal{M}(ABz, Pz, Pz, kt)\mathcal{M}(ST\mathrm{v}, Q\mathrm{v}, Q\mathrm{v}, kt)] * \mathcal{M}^2(ST\mathrm{v}, Q\mathrm{v}, Q\mathrm{v}, kt)\}$$

$$\geq [p\mathcal{M}(ABz, Pz, Pz, t) +\ q\mathcal{M}(ABz, ST\mathrm{v}, ST\mathrm{v}, t)]\mathcal{M}(ABz, Q\mathrm{v}, Q\mathrm{v}, 2kt)$$

$$\Rightarrow\ \{\mathcal{M}^2(z, \mathrm{v}, \mathrm{v}, kt) * [\mathcal{M}(z, z, z, kt)\mathcal{M}(\mathrm{v}, \mathrm{v}, \mathrm{v}, kt)] * \mathcal{M}^2(\mathrm{v}, \mathrm{v}, \mathrm{v}, kt)\}$$

$$\geq [p\mathcal{M}(z, z, z, t) +\ q\mathcal{M}(z, \mathrm{v}, \mathrm{v}, t)]\mathcal{M}(z, \mathrm{v}, \mathrm{v}, 2kt)$$

$\Rightarrow \mathcal{M}^2(z, \mathrm{v}, \mathrm{v}, kt) \geq [p + q\mathcal{M}(z, \mathrm{v}, \mathrm{v}, t)]\mathcal{M}(z, \mathrm{v}, \mathrm{v}, 2kt)$ and since $\mathcal{M}(x, y, y, .)$ is non-decreasing for all x, y ∈ X, we have

$$\mathcal{M}(z, \mathrm{v}, \mathrm{v}, kt)\mathcal{M}(z, \mathrm{v}, \mathrm{v}, 2kt) \geq [p + q\mathcal{M}(z, \mathrm{v}, \mathrm{v}, kt)]\mathcal{M}(z, \mathrm{v}, \mathrm{v}, 2kt).$$

Thus it follows that $\mathcal{M}(z, \mathrm{v}, \mathrm{v}, t) \geq \frac{p}{1-q} = 1$, for all t > 0.

So, v = z. Hence A, B, P, Q, S and T have a unique fixed point.

Corollary 3.2:

Let (X, $\mathcal{M}$, *) be a complete generalized fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and let A, P, S and Q be maps from X into itself such that

(i) $PS(X) \subseteq AS(X)$, $QA(X) \subseteq AS(X)$.

(ii) There exists a constant $k \in (0, 1)$ Such that

$$\{\mathcal{M}^2(Px,Qy,Qy,kt) * [\mathcal{M}(Ax,Px,Px,kt)\mathcal{M}(Sy,Qy,Qy,kt)] * \mathcal{M}^2(Sy,Qy,Qy,kt)\}$$
$$\geq \begin{bmatrix} p\mathcal{M}(Ax,Px,Px,t) + \\ q\mathcal{M}(Ax,Sy,Sy,t) \end{bmatrix} \mathcal{M}(Ax,Qy,Qy,2kt)$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$,

(iii) A and S are continuous,

(iv) AS = SA

(v) The pairs (P, A) and (Q, S) are compatible of type (α).

Then A, P, S and Q have a unique common fixed point in X.

If we put A = S, B = T and P = Q in the theorem (3.1), we have the following.

Corollary 3.3:

Let (X, $\mathcal{M}$, *) be a complete generalized fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and let A, B and P be maps from X into itself such that

(i) $P(AB)(X) \subseteq AB$,

(ii) There exists a constant $k \in (0, 1)$ such that

$$\{\mathcal{M}^2(Px,Py,Py,kt) * [\mathcal{M}(ABx,Px,Px,kt)\mathcal{M}(ABy,Py,Py,kt)] * \mathcal{M}^2(ABy,Py,Py,kt)\}$$
$$\geq \begin{bmatrix} p\mathcal{M}(ABx,Px,Px,t) + \\ q\mathcal{M}(ABx,ABy,ABy,t) \end{bmatrix} \mathcal{M}(ABx,Py,Py,2kt)$$

for all $x, y \in X$ and $t > 0$ where $0 < p , q < 1$ such that $p+q = 1$,

(iii) A and B are continuous,

(iv) AB = BA, PB =BP,

(v) The pairs (P, AB) is compatible of type (α).

Then A, B and P have a unique common fixed point in X.

Example 3.4:

Let x = [-1, 1] with the metric D* defined by $D^*(x, y, z) = |x-y| + |y-z| + |z-x|$ and for each $t > 0$, define $\mathcal{M}(x,y,z,t) = \frac{t}{t + D^*(x, y, z)}$, for all $x, y, z \in X$.

Clearly (X, $\mathcal{M}$, *) is a complete generalized fuzzy metric space, where * is defined by $a * b = ab$.

Let A, B, P, Q, S and T be maps from X into itself defined as

$Ax = \frac{x}{2}$, $Bx = \frac{x}{8}$, $Sx = \frac{x}{3}$, $Tx = \frac{x}{5}$, $Px = \frac{x}{16}$, $Qx = \frac{x}{15}$.

Then $P(ST)(X) = \left[\frac{-1}{240}\ ,\ \frac{1}{240}\right] \subseteq AB(ST)(X) = \left[\frac{-1}{240}\ ,\ \frac{1}{240}\right]$ and

$$Q(AB)(X) = \left[\frac{-1}{240}\ ,\frac{1}{240}\right] \subseteq AB(ST)(X) = \left[\frac{-1}{240}\ ,\frac{1}{240}\right]$$

Thus (i) is satisfied. Also (iii) and (iv) are satisfied.

Now, define a sequence $\{x_n\}$ in X such that $x_n = \frac{n}{n+1}$.

Then $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = \frac{1}{16}$,

$\lim_{n\to\infty} \mathcal{M}(P(AB)x_n,\ AB(AB)x_n,\ AB(AB)x_n,\ t) = 1$ and

$\lim_{n\to\infty} \mathcal{M}\big((AB)Px_n, PPx_n, PPx_n, t\big) = 1$

Thus the pair (P, AB) is compatible of type (α). Similarly, $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} STx_n = \frac{1}{15}$,

$\lim_{n\to\infty} \mathcal{M}(Q(ST)x_n\ , ST(ST)x_n,\ ST(ST)x_n, t) = 1$ and

$\lim_{n\to\infty} \mathcal{M}\big((ST)Qx_n, QQx_n, QQx_n, t\big) = 1$

Therefore the pair (Q, ST) is also compatible of type (α). For $p = \frac{7}{8}$, $q = \frac{1}{8}$, $k = \frac{1}{4}$.

We can see that the condition (ii) is satisfied , Hence all the conditions of our main theorem (3.1)are satisfied and the unique common fixed point is x = 0.

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