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Multi-Fuzzy Subalgebras of BG-Algebra and Its Level Subalgebras

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Abstract

In this paper, we introduce the notion of multi-fuzzy subalgebra of BG-algebra and investigate some of its related properties. The purpose of this study is to implement the concepts of BG-algebra in multi-fuzzy sets. Also we define the concept of multi-level subsets and discuss the properties of subalgebras by their family of level subalgebras.

Keywords: BG-algebra, fuzzy subalgebra ,multi-fuzzy subalgebra , multi-level subalgebras.

1. INTRODUCTION

The notion of a fuzzy subset was initially introduced by Zadeh [7] in 1965, for representing uncertainity. In 2000, S.Sabu and T.V.Ramakrishnan [8,9] proposed the theory of multi-fuzzy sets in terms of multi-dimensional membership functions and investigated some properties of multilevel fuzziness. Theory of multi-fuzzy set is an extension of theory of fuzzy sets. Complete characterization of many real life problems can be done by multi-fuzzy membership functions of the objects involved in the problem.

Y.Imai and K.Iseki introduced two classes of abstract algebras: BCK algebras and BCI-algebras [1,2,3]. It is shown that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. J.Neggers and H.S.Kim [4] introduced a new notion, called a B-algebra. In 2005, C.B.Kim and H.S.Kim [5] introduced the notion of a BG-algebra which is a generalization of B-albebras. With these ideas, fuzzy subalgebras of BG-algebra were developed by S.S.Ahn and H.D.Lee [6]. In this paper, we apply the concept of BG-algebra in multi-fuzzy sets and define a new algebraic structure of multi-fuzzy subalgebra of BG-algebra. Also we classify the subalgebras by their family of level subalgebras in BG-algebra.

2. PRELIMINARIES

In this section we site the fundamental definitions that will be used in the sequel.

2.1 Definition : Let X be a non-empty set. A multi-fuzzy set A in X is defined as a set of ordered sequences:

$A = \{ (x , \mu_1(x), \mu_2(x),......,\mu_i(x),.....) : x \in X\}$, where $\mu_i : X \to [0 , 1]$ for all i.

Remarks:

(i). If the sequences of the membership functions have only k-terms (finite number of terms), k is called the dimension of A.

(ii). The set of all multi-fuzzy sets in X of dimension k is denoted by $M^kFS(X)$.

(iii). The multi-fuzzy membership function μ_A is a function from X to $[0,1]^k$ such that for all x in X , $\mu_A(x) = (\mu_1(x), \mu_2(x),......,\mu_k(x))$.

(iv). For the sake of simplicity, we denote the multi-fuzzy set $A = \{(x, \mu_1(x), \mu_2(x),......,\mu_k(x)): x \in X\}$ as $A = (\mu_1, \mu_2,......,\mu_k)$.

2.2 Definition: Let k be a positive integer and let A and B in $M^kFS(X)$, where $A = (\mu_1, \mu_2,......,\mu_k)$ and $B = (\nu_1, \nu_2 ,........ \nu_k)$, then we have the following relations and operations :

i. $A \subseteq B$ if and only if $\mu_i \leq \nu_i$, for all i=1,2,………..,k;

ii. $A = B$ if and only if $\mu_i = \nu_i$, for all i=1,2,………..,k;

iii. $A \cup B = (\mu_1 \cup \nu_1,.....\mu_k \cup \nu_k) = \{(x , \max(\mu_1(x),\nu_1(x)),...\max(\mu_k(x),\nu_k(x)): x \in X\}$

iv. $A \cap B = ((\mu_1 \cap \nu_1,.....\mu_k \cap \nu_k) = \{(x , \min(\mu_1(x),\nu_1(x)),....\min(\mu_k(x),\nu_k(x)) : x \in X\}$

2.3 Definition : A non-empty set X with a constant 0 and a binary operation "$*$" is called a BG-algebra if it satisfies the following axioms:

1. $x * x = 0$
2. $x * 0 = x$
3. $(x * y) * (0 * y) = x$, $\forall\, x, y \in X$.

2.4 Example: Let X = { 0, 1, 2 } be a set with the following table :

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then (X; $*$, 0) is a BG-algebra.

2.5 Definition : Let S be a non-empty subset of a BG-algebra X, then S is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

2.6 Definition : Let μ be a fuzzy set in BG-algebra . Then μ is called a fuzzy subalgebra of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}, \forall\, x, y \in X.$$

3. MULTI-FUZZY SUBALGEBRA

In this section, we define multi-fuzzy subalgebra of a BG-algebra and discuss some of its properties.

3.1 Definition : Let A be a multi-fuzzy set in a BG-algebra X . Then A is called a multi-fuzzy subalgebra of X if

$$A(x * y) \geq \min\{A(x), A(y)\}, \forall\, x, y \in X.$$

3.2 Example : Consider the BG-algebra X = { 0, 1, 2 } as in example 2.4.

Define a multi-fuzzy set $A : X \rightarrow I^2$ by

A(0) = A(1) = { 0.8 , 1 } and A(2) = { 0.3, 0.6 }

Then A is a multi-fuzzy subalgebra of X.

3.3 Theorem: If A is a multi-fuzzy subalgebra of X , then for all $x \in X$, $A(0) \geq A(x)$.

Proof :

For all $x \in X$, we have $x * x = 0$

$A(0) = A(x * x) \geq \min \{ A(x) , A(x) \} = A(x)$, which completes the proof.

3.4 Theorem: If $\{ A_i : i \in I \}$ is a family of multi-fuzzy subalgebras of a BG-algebra X, then $\cap A_i$ is a multi-fuzzy subalgebra in X.

Proof:

Let $A = \cap A_i$

For any x , y ∈ X, $A(x * y) = (\cap A_i)(x * y)$

$= \min \{ A_i (x * y) \} \geq \min\{ \min(A_i (x)) , A_i (y) \}$

$= \min \{ (\cap A_i)(x) , (\cap A_i)(y) \}$

$= \min \{ A(x) , A(y) \}$

Hence , $\cap A_i$ is a multi-fuzzy subalgebra in X.

Corrollary: Union of the family of multi-fuzzy subalgebras of a BG-algebra X need not be a multi-fuzzy subalgebra.

4. PROPERTIES OF MULTI-LEVEL SUBSETS OF A MULTI-FUZZY SUBALGEBRA

In this section, we introduce the concept of multi-level subsets of a multi-fuzzy subalgebra and discuss some of its properties.

4.1 Definition : Let A be a multi-fuzzy set of a BG-algebra X. For any $t = (t_1, t_2, \ldots t_k, \ldots)$ where $t_i \in [0,1]$, for all i , we define the multi-level subset of A is the set $U(A ; t) = \{ x \in X / A(x) \geq t \}$.

4.2 Theorem : A multi-fuzzy set A of a BG-algebra X is a multi-fuzzy subalgebra if and only if for every $t \in [0,1]$, U(A ; t) is either empty or a subalgebra of X.

Proof:

Suppose A is a multi-fuzzy subalgebra of X and $U(A ; t) \neq \emptyset$,

then for any x , y ∈ U(A ; t), we have $A(x * y) \geq \min \{ A(x) , A(y) \} \geq t$

Therefore , $x * y \in U(A ; t)$

Hence U(A ; t) is a subalgebra of X .

Conversely, take $t = \min \{ A(x) , A(y) \}$ for any x , y ∈ X.

Then by assumption , U(A ; t) is a subalgebra of X

This implies that $x * y \in$ U(A ; t) .

Therefore A($x * y$) $\geq$ t = min { A(x) , A(y) }

Hence A is a multi-fuzzy subalgebra of X.

4.3 Theorem : Let X be a BG-algebra and A be a multi-fuzzy set in X such that U(A ; t) is a subalgebra in X for all t = $(t_1, t_2, \ldots t_k, \ldots)$ where $t_i \in [0,1]$ for all i , t $\leq$ A(0). Then A is a multi-fuzzy subalgebra of X.

Proof:

Let x , y ∈ X and A(x) = r , A(y) = s where r = $(r_1, r_2, \ldots r_k, \ldots)$ and s = $(s_1, s_2, \ldots s_k, \ldots)$ for r_i , $s_i \in [0,1]$ for all i.

Suppose r < s

Then A(x) = r which implies x ∈ U(A ; r)

And A(y) = s > r which implies y ∈ U(A ; r)

Therefore x , y ∈ U(A ; r).

U(A ; r) is a subalgebra of X , $x * y \in$ U(A ; r)

Hence , A($x * y$) $\geq$ r = min{ r , s }

$\geq$ min { A(x) , A(y) }

That is , A($x * y$) $\geq$ min { A(x) , A(y) }.

Hence A is a multi-fuzzy subalgebra in X.

4.4 Definition : Let X be a BG-algebra and let A be multi-fuzzy subalgebra of X. The subalgebras U(A ; t) for t = $(t_1, t_2, \ldots t_k, \ldots)$ where each $t_i \in [0,1]$, for all i and t $\leq$ A(0) are called multi-level subalgebras of A.

4.5 Theorem : If two multi-level subalgebras U(A ; r) and U(A ; s) for r = $(r_1, r_2, \ldots r_k, \ldots)$ and s = $(s_1, s_2, \ldots s_k, \ldots)$ for r_i , $s_i \in [0,1]$ for all i and r , s $\leq$ A(0) with r < s of a multi-fuzzy subalgebra A are equal if and only if there is no x ∈ X such that r $\leq$ A(x) < s.

Proof:

Suppose that U(A ; r) = U(A ; s) for some r < s

If there exists a x ∈ X such that r $\leq$ A(x) < s , then U(A ; s) is a proper subset of U(A ; r) , That is , x $\in$ U(A ; r) since A(x) $\geq$ r but x $\notin$ U(A ; s) since A(x) < s , which contradicts to the assumption that, U(A ; r) = U(A ; s).

Hence there is no x in X such that r $\leq$ A(x) < s.

Conversely, suppose that there is no x in X such that r $\leq$ A(x) < s.

If x $\in$ U(A ; r) , then A(x) $\geq$ r and so A(x) $\geq$ s , since A(x) does not lie between r and s.

Thus x $\in$ U(A ; s) which gives , U(A ; r) $\subseteq$ U(A ; s).

Also U(A ; s) $\subseteq$ U(A ; r) , since r < s.

Hence U(A ; r) = U(A ; s).

4.6 Theorem : A multi-fuzzy subset A of X is a multi-fuzzy subalgebra of a BG-algebra X if and only if the multi-level subsets U(A ; t) , for t = $(t_1, t_2, \ldots t_k, \ldots)$ where t_i $\in$ [0,1], for all i and t $\leq$ A(0) , are subalgebras of X.

Proof : It is clear.

4.7 Theorem : Any subalgebra of a BG-algebra X can be realized as a multi-level subalgebra of some multi-fuzzy subalgebra of X.

Proof:

Let S be a subalgebra of X and A be a multi-fuzzy subset in X defind by

A(x) =

$$\begin{cases} 0 \quad if \ x \ \notin \ S \\ t \quad if \ x \ \in S \ , for \ t = (\ t_1, t_2, \ldots \ldots t_k, \ldots) where \ t_i \in [0,1] for \ all \ i \ and \ t \ \leq A(0) \end{cases}$$

We will prove that such defined A is a multi-fuzzy subalgebra of X.

Let x , y $\in$ X

Case (i): Suppose x , y $\in$ S .

Then x $*$ y $\in$ S

Hence A(x) = A(y) = A(x $*$ y) = t and A(x $*$ y) $\geq$ min { A(x) , A(y) }

Case (ii): Suppose x , y $\notin$ S . Then A(x) = A(y) = A(x $*$ y) = 0

and so A(x $*$ y) $\geq$ min { A(x) , A(y) }.

Case(iii) : If atmost one of x , y $\in$ S , then atleast one of A(x) and A(y) is equal to zero.

Therefore min { A(x) , A(y) } = 0

Hence A(x ∗ y) ≥ min { A(x) , A(y) }.

Thus in all cases, A is a multi-fuzzy subalgebra of X.

4.8 Theorem: Let X be a BG-algebra. Then given any chain of subalgebras

$$A_0 \subset A_1 \subset \ldots\ldots\subset A_r = X,$$

there exists a multi-fuzzy subalgebra A of X whose level subalgebras are exactly the subalgebras of this chain.

Proof:

Consider a set of numbers

$$t_0 > t_1 > \cdots \ldots\ldots > t_r$$

where each $t_i \in [0,1]$ and $t_i = (t_{i_1}, t_{i_2}, \ldots\ldots t_{i_k})$

Let A be a multi-fuzzy set defined by

$$A(A_0) = t_0 \text{ and } A(A_i - A_{i-1}) = t_i \text{ , } 0 < i \leq r .$$

We claim that A is a multi-fuzzy subalgebra of X. Let x , y ∈ X. Then we distinguish two cases as follows:

Case 1: Let x , y ∈ $A_i - A_{i-1}$. Then by definition of A, A(x) = t_i = A(y).

Since A_i is a subalgebra , it follows that x ∗ y ∈ A_i ,

and so either x ∗ y ∈ $A_i - A_{i-1}$ or x ∗ y ∈ A_{i-1}.

In any case we conclude that

$$A(x * y) \geq t_i = \min \{ A(x), A(y) \}.$$

Case 2: For $i > i_1$, let x ∈ $A_i - A_{i-1}$ and y ∈ $A_{i_1} - A_{i_1-1}$

Then A(x) = t_i and A(y) = t_{i_1}, and x ∗ y ∈ A_i since A is a subalgebra of X and $A_{i_1} \subset A_i$.

Hence A(x ∗ y) ≥ t_{i_1} = min { A(x), A(y) }

Thus A is a multi-fuzzy subalgebra of X. From the definition of A , it follows that Im(A) = { $t_0, t_1, \ldots\ldots t_r$ }. Hence the level subalgebras of A are given by the chain of subalgebras

$$U(A ; t_0) \subset U(A; t_1) \subset\ldots\ldots\ldots\subset U(A; t_r)$$

Now U(A; t_0) = { x ∈ X / A(x) ≥ t_0 } = A_0.

Finally we prove that U(A; t_i) = A_i for 0 < i ≤ r.

Clearly, $A_i \subset U(A; t_i)$.

If $x \in U(A; t_i)$, then $A(x) \geq t_i$ which implies that $x \notin A_j$ for some $j > i$.

Hence $A(x) \in \{t_0, t_1, \ldots \ldots t_i\}$ and so $x \in A_k$ for some $k \leq i$. As $A_k \subseteq A_i$,

it follows that $x \in A_i$.

Therefore we obtain $L(A; t_i) = A_i$. This completes the proof.

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