

# COMMON FIXED POINT THEOREMS FOR COMMUTING MAPPINGS OF TYPE (P\*) IN GENERALIZED INTUITIONISTIC FUZZY METRIC SPACES

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## Abstract

In this paper, we first formulate the definition of weakly commuting and weakly community mappings in generalized intuitionistic fuzzy metric spaces and prove the intuitionistic fuzzy version of Pant's theorem [9].

**Keywords:** R-weakly commuting mappings, Intuitionistic fuzzy metric spaces, R-weakly commuting mappings of type (P-1) and type (P-2).

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## 1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh [15] following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michlek [5 and George and Veeramani [4] modified the notion of fuzzy metric spaces with the help of continuous t-norms. As a generalization of fuzzy sets, Atanassove [2] introduced and studied the concept of intuitionistic fuzzy sets. Park [9] using the idea of intuitionistic fuzzy sets defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalized of fuzzy metric spaces,

George and Veeramani [4] showed that every metric induces an intuitionistic fuzzy metric, every fuzzy metric space in an intuitionistic fuzzy metric space and found a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete. Choudhary [3] introduced mutually contractive sequence of self maps and proved a fixed point theorem. Kramosil and Michlek [5] introduced the notion of Cauchy sequences in an intuitionistic fuzzy metric space and proved the well known fixed point theorem of Banach. Turkoglu et al [14] gave the generalization of Jungck's Common fixed point theorem to intuitionistic fuzzy metric spaces.

In 2006, Sedghi and Shobe [12] defined  $\mathcal{M}$ -fuzzy metric spaces and proved a common fixed point theorem for four weakly Compatible mappings in this spaces. In 2009, Mehra and Gugnani [6] defined the notion of an intuitionistic  $\mathcal{M}$ -fuzzy metric spaces due to Sedghi and Shobe and proved a common fixed point theorem for six mappings for property (E) in this newly defined space. Our result is an generalized intuitionistic fuzzy metric space in R-weakly commuting mappings. We first formulate the definition of weakly commuting and R-weakly community mappings in generalized intuitionistic fuzzy metric spaces and prove the intuitionistic fuzzy version of Pant's theorem[8].

## 2. PRELIMINARIES

### Definition 2.1:

A binary operation  $*$  :  $[0,1] \times [0,1]$  is a continuous t-norm if it satisfies the following condition.

- 1)  $*$  is associative and commutative,
- 2)  $*$  is continuous ,
- 3)  $a*1 = a$  for all  $a \in [0,1]$ ,
- 4)  $a*b < c*d$  whenever  $a \leq c$  and  $b \leq d$  for each  $a, b, c, d \in [0,1]$ .

Two typical example of a continuous t-norm are  $a * b = ab$  and  $a * b = \min \{a, b\}$

### Definition 2.2:

A binary operation  $\diamond$  :  $[0,1] \times [0,1] \rightarrow [0,1]$  is a continuous t-conorm if it satisfies the following conditions:

- 1)  $\diamond$  is associative and commutative,
- 2)  $\diamond$  is continuous,
- 3)  $a \diamond 0 = a$  for all  $a \in [0,1]$ ,
- 4)  $a \diamond b \leq c \diamond d$  whenever  $a \leq c$  and  $b \leq d$ , for each  $a, b, c, d \in [0, 1]$ .

Two typical examples of a continuous t-conorm are  $a \diamond b = \min \{1, a + b\}$  and  $a \diamond b = \max \{a, b\}$ .

**Definition 2.3:**

A 5-tuple  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is called generalized intuitionistic fuzzy metric space if  $X$  is an arbitrary (non-empty) set,  $*$  is a continuous t-norm,  $\diamond$  a continuous t-conorm and  $\mathcal{M}, \mathcal{N}$  are fuzzy sets on  $X^3 \times (0, \infty)$ , satisfying the following conditions: for each  $x, y, z, a \in X$  and  $t, s > 0$ .

- a)  $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \leq 1$ ,
- b)  $\mathcal{M}(x, y, z, t) > 0$ ,
- c)  $\mathcal{M}(x, y, z, t) = 1$  if and only if  $x = y = z$ ,
- d)  $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$ , where  $p$  is a permutation function,
- e)  $\mathcal{M}(x, y, z, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$ ,
- f)  $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous,
- g)  $\mathcal{N}(x, y, z, t) > 0$ ,
- h)  $\mathcal{N}(x, y, z, t) = 0$ , if and only if  $x = y = z$ ,
- i)  $\mathcal{N}(x, y, z, t) = \mathcal{N}(p\{x, y, z\}, t)$  where  $p$  is a permutation function,
- j)  $\mathcal{N}(x, y, z, a, t) \diamond \mathcal{N}(a, z, z, s) \geq \mathcal{N}(x, y, z, t + s)$ ,
- k)  $\mathcal{N}(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$  is continuous.

Then  $(\mathcal{M}, \mathcal{N})$  is called an generalized intuitionistic fuzzy metric on  $X$ .

**Example 2.4:**

$$\text{Let } X = \mathbb{R} \text{ and } \mathcal{M}(x, y, z, t) = \frac{t}{t + |x-y| + |y-z| + |z-x|},$$

$\mathcal{N}(x, y, z, t) = \frac{|x-y| + |y-z| + |z-x|}{t + |x-y| + |y-z| + |z-x|}$  for every  $x, y, z$  and  $t > 0$ , let  $A$  and  $B$  be defined as

$$Ax = 2x+1, Bx = x + 2, \text{ consider the sequence } x_n = \frac{1}{n} + 1, n = 1, 2, \dots$$

Thus we have  $\lim_{n \rightarrow \infty} \mathcal{M}(Ax_n, 3, 3, t) = \lim_{n \rightarrow \infty} \mathcal{M}(Bx_n, 3, 3, t) = 1$  and

$$\lim_{n \rightarrow \infty} \mathcal{N}(Ax_n, 3, 3, t) = \lim_{n \rightarrow \infty} \mathcal{N}(Bx_n, 3, 3, t) = 0, \text{ for every } t > 0.$$

Then  $A$  and  $B$  satisfying the property (E).

**Lemma 2.5:**

Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an generalized intuitionistic fuzzy metric space. Then  $\mathcal{M}(x, y, z, t)$  and  $\mathcal{N}(x, y, z, t)$  are non-decreasing with respect to  $t$ , for all  $x, y, z$  in  $X$ .

**Definition 2.6:**

Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an generalized intuitionistic fuzzy metric space. Then

- (i) a sequence  $\{x_n\}$  in  $X$  is said to be canchy sequence if for all  $t > 0$  and  $p > 0$ ,  
 $\lim_{n \rightarrow \infty} \mathcal{M}(x_{n+p}, x_n, x_n, t) = 1$  and  $\lim_{n \rightarrow \infty} \mathcal{N}(x_{n+p}, x_n, x_n, t) = 0$
- (ii) a sequence  $\{x_n\}$  in  $X$  is said to be convergent to a point  $x \in X$ , if for all  $t > 0$ ,  
 $\lim_{n \rightarrow \infty} \mathcal{M}(x_n, x, x, t) = 1$  and  $\lim_{n \rightarrow \infty} \mathcal{N}(x_n, x, x, t) = 0$
- (iii) An generalized intuitionistic fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is said to be complete if and only if every Cauchy sequence in  $X$  is convergent.

**Definition 2.7:**

Let  $A$  and  $S$  be maps from an generalized intuitionistic fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  into itself. The maps  $A$  and  $S$  are said to be weakly commuting if  $\mathcal{M}(ASz, SAz, SAz, t) \geq \mathcal{M}(Az, Sz, Sz, t)$  and  $\mathcal{N}(ASz, SAz, SAz, t) \leq \mathcal{N}(Az, Sz, Sz, t)$  for all  $z \in X$  and  $t > 0$ .

**Definition 2.8:**

Let  $A$  and  $S$  be maps from an generalized intuitionistic fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  into itself. The maps  $A$  and  $S$  are said to be compatible if for all  $t > 0$ ,  
 $\lim_{n \rightarrow \infty} \mathcal{M}(ASx_n, SAx_n, SAx_n, t) = 1$ , and  $\lim_{n \rightarrow \infty} \mathcal{N}(ASx_n, SAx_n, SAx_n, t) = 0$ , whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = z$  for some  $z \in X$ .

**Definition 2.9:**

A pair of self mappings  $(A, S)$  of a generalized intuitionistic fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is said to be point wise  $R$ -weakly commuting, if given  $x$  in  $X$ , there exist  $R > 0$  such that for all  $t > 0$ ,  $\mathcal{M}(ASx, SAx, SAx, t) \geq \mathcal{M}\left(Ax, Sx, Sx, \frac{t}{R}\right)$ , and  $\mathcal{N}(ASx, SAx, SAx, t) \leq \mathcal{N}\left(Ax, Sx, Sx, \frac{t}{R}\right)$  clearly, every pair of weakly commuting mappings is point wise  $R$ -weakly commuting with  $R=1$ .

**Definition 2.10:**

A pair of self mappings  $(A, S)$  of a generalized intuitionistic fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is said to be  $R$ -weakly commuting of type (P-1) if there exists some  $R > 0$  such that  $\mathcal{M}(SSx, ASx, ASx, t) \geq \mathcal{M}\left(Sx, Ax, Ax, \frac{t}{R}\right)$ , and  $\mathcal{N}(SSx, ASx, ASx, t) \leq \mathcal{N}\left(Sx, Ax, Ax, \frac{t}{R}\right)$  for all  $x \in X$  and  $t > 0$ .

**Definition 2.11:**

A pair of self mappings (A, S) of a generalized intuitionistic fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is said to be R-weakly commuting of type (P-2), if there exists some  $R > 0$ , such that  $\mathcal{M}(AAx, SAx, SAx, t) \geq \mathcal{M}\left(Ax, Sx, Sx, \frac{t}{R}\right)$ , and  $\mathcal{N}(AAx, SAx, SAx, t) \leq \mathcal{N}\left(Ax, Sx, Sx, \frac{t}{R}\right)$  for all  $x \in X$  and  $t > 0$ .

**Definition 2.12:**

Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be a generalized intuitionistic fuzzy metric space, A and S be self maps on X. A point  $x$  in X is called a coincidence point of A and S if and only if

$Ax = Sx$ . In this case,  $w = Ax = Sx$  is called a point of coincidence of A and S.

**Definition 2.13:**

A pair of self mappings (A, S) of a generalized intuitionistic fuzzy metric space

$(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is said to be weakly compatible if they commute at the coincidence points if

$Au = Su$  for some  $u$  in X, then  $ASu = SAu$ .

**Definition 2.14:**

A pair of self mappings (A, S) of a generalized intuitionistic fuzzy metric space

$(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  is said to be R-weakly commuting of type (P) if there exists some  $R > 0$  such that  $\mathcal{M}(AAx, SSx, SSx, t) \geq \mathcal{M}\left(Ax, Sx, Sx, \frac{t}{R}\right)$ , and  $\mathcal{N}(AAx, SSx, SSx, t) \leq \mathcal{N}\left(Ax, Sx, Sx, \frac{t}{R}\right)$  for all  $x \in X$  and  $t > 0$ .

**Lemma 2.15:**

Let  $\{x_n\}$  be a sequence in an generalized intuitionistic fuzzy metric space  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$ , if there exists a constant  $k \in (0, 1)$  such that

$\mathcal{M}(x_n, x_{n+1}, x_{n+1}, kt) \geq \mathcal{M}(x_{n-1}, x_n, x_n, t)$  and  $\mathcal{N}(x_n, x_{n+1}, x_{n+1}, kt) \leq \mathcal{N}(x_{n-1}, x_n, x_n, t)$  for all  $t > 0$ . Then  $\{x_n\}$  is Cauchy sequence in X.

**Lemma 2.16:**

Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be an generalized intuitionistic fuzzy metric space and for all

$x, y, z, \in X, t > 0$ , and if for a number  $k \in (0, 1)$ ,  $\mathcal{M}(x, y, z, kt) \geq \mathcal{M}(x, y, z, t)$  and  $\mathcal{N}(x, y, z, kt) \leq \mathcal{N}(x, y, z, t)$  then  $x = y = z$ .

### 3. FIXED POINT THEOREMS FOR COMMUTING MAPPINGS

#### Theorem 3.1:

Let  $(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be a complete generalized intuitionistic fuzzy metric space and

$f, g: X \rightarrow X$  be mappings satisfying the following conditions:

- (i)  $g(X) \subseteq f(X)$
- (ii)  $f$  is continuous
- (iii) There exists  $0 < k < 1$  such that for all  $x, y \in X$ ,  
 $\mathcal{M}(g(x), g(y), g(z), kt) \geq \mathcal{M}(f(x), f(y), g(z), t)$ ,  
 $\mathcal{N}(g(x), g(y), g(z), kt) \leq \mathcal{N}(f(x), f(y), f(z), t)$ .

Then  $f$  and  $g$  have a unique common fixed point in  $X$  provided  $f$  and  $g$  commute on  $X$ .

#### Proof:

Let  $x_0 \in X$  by (i) we can find  $x_1$  such that  $f(x_1) = g(x_0)$  by induction we can define a sequence  $\{x_n\}$  in  $X$  such that  $f(x_n) = g(x_{n-1})$  by induction again

$$\begin{aligned} \mathcal{M}(f(x_n), f(x_{n+1}), f(x_{n+1}), t) &= \mathcal{M}(g(x_{n-1}), g(x_n), g(x_n), t) \\ &\geq \mathcal{M}(f(x_{n-1}), f(x_n), f(x_n), t/k) \\ &\geq \dots \\ &\geq \mathcal{M}(f(x_0), f(x_1), f(x_1), t/k^n). \end{aligned}$$

$$\begin{aligned} \mathcal{N}(f(x_n), f(x_{n+1}), f(x_{n+1}), t) &= \mathcal{N}(g(x_{n-1}), g(x_n), g(x_n), t) \\ &\leq \mathcal{N}(f(x_{n-1}), f(x_n), f(x_n), t/k) \\ &\leq \dots \\ &\leq \mathcal{N}(f(x_0), f(x_1), f(x_1), t/k^n). \end{aligned}$$

Hence, for any positive integer  $p$ ,

$$\begin{aligned} \mathcal{M}(f(x_n), f(x_{n+p}), f(x_{n+p}), t) &\geq \mathcal{M}(f(x_n), f(x_{n+1}), f(x_{n+1}), t/k) * \underline{P \text{ times}} * \\ &\quad \mathcal{M}(f(x_{n+p-1}), f(x_{n+p}), f(x_{n+p}), t/k) \\ &\geq \mathcal{M}(f(x_0), f(x_1), f(x_1), t/pk^n) * \end{aligned}$$

$P \text{ times}$  \*

$$\mathcal{M}(f(x_0), f(x_1), f(x_1), t/pk^{n+p-1}).$$

$$\begin{aligned} \mathcal{N}(f(x_n), f(x_{n+p}), f(x_{n+p}), t) &\leq \mathcal{N}(f(x_n), f(x_{n+1}), f(x_{n+1}), t/k) \diamond \underline{P \text{ times}} \diamond \\ &\quad \mathcal{N}(f(x_{n+p-1}), f(x_{n+p}), f(x_{n+p}), t/k) \end{aligned}$$

$$\leq \mathcal{N}\left(f(x_0), f(x_1), f(x_1), t/pk^n\right) \diamond \underline{P \text{ times}} \diamond \mathcal{N}\left(f(x_0), f(x_1), f(x_1), t/pk^{n+P-1}\right).$$

By Definition, Since

$$\lim_{n \rightarrow \infty} \mathcal{M}\left(f(x_0), f(x_1), f(x_1), t/pk^n\right) = 1, \text{ and}$$

$$\lim_{n \rightarrow \infty} \mathcal{N}\left(f(x_0), f(x_1), f(x_1), t/pk^n\right) = 0.$$

It follows that

$$\lim_{n \rightarrow \infty} \mathcal{M}(f(x_n), f(x_{n+P}), f(x_{n+P}), t) \geq 1 * \dots * 1 \geq 1 \text{ and}$$

$$\lim_{n \rightarrow \infty} \mathcal{N}(f(x_n), f(x_{n+P}), f(x_{n+P}), t) \geq 0 \diamond \dots \diamond 0 \geq 0.$$

Thus  $\{f(x_n)\}$  is a Cauchy sequence and so, by the completeness of  $X$ ,  $\{f(x_n)\}$  converges to point the completeness of  $X$ ,  $\{f(x_n)\}$  converges to point  $y$  and  $g(x_{n-1}) = f(x_n)$  converges also to the point  $y$ . It can be shown from definition that the continuity of  $f$  implies the continuity of  $g$ . Therefore,  $\{g(f(x_n))\}$  converges to  $g(y)$ . However, since  $f$  and  $g$  commute on  $X$ ,  $g(f(x_n)) = f(g(x_n))$  and so  $f(g(x_n))$  converges to  $f(y)$ . Since the limits are unique,  $f(y) = g(y)$ , which implies  $f(f(y)) = f(g(y))$ .

$$\begin{aligned} \mathcal{M}(g(y), g(g(y)), g(g(y)), t) &\geq \mathcal{M}(f(y), f(g(y)), f(g(y)), t/k) \\ &\geq \mathcal{M}(g(y), g(g(y)), g(g(y)), t/k) \\ &\geq \dots \\ &\geq \mathcal{M}(g(y), g(g(y)), g(g(y)), t/k^n). \end{aligned}$$

$$\begin{aligned} \mathcal{N}(g(y), g(g(y)), g(g(y)), t) &\leq \mathcal{N}(f(y), f(g(y)), f(g(y)), t/k) \\ &\leq \mathcal{N}(g(y), g(g(y)), g(g(y)), t/k) \\ &\leq \dots \\ &\leq \mathcal{N}(g(y), g(g(y)), g(g(y)), t/k^n) \end{aligned}$$

It follows that  $g(y) = g(g(y))$ . Thus  $g(y) = g(g(y)) = f(g(y))$  which implies that  $g(y)$  is a common fixed point of  $f$  and  $g$ . If  $y$  and  $z$  are two common fixed points of  $f$  and  $g$  then

$$\begin{aligned} 1 &\geq \mathcal{M}(y, z, z, t) \\ &= \mathcal{M}(g(y), g(z), g(z), t) \geq \mathcal{M}(f(y), f(z), f(z), t/k) \\ &= \mathcal{M}\left(y, z, z, t/k\right) \geq \dots \geq \mathcal{M}\left(y, z, z, t/k^n\right) \\ 0 &\leq \mathcal{N}(y, z, z, t) \\ &= \mathcal{N}(g(y), g(z), g(z), t) \leq \mathcal{N}(f(y), f(z), f(z), t/k) \end{aligned}$$

$$= \mathcal{N}(y, z, z, t/k) \leq \dots \leq \mathcal{N}(y, z, z, t/k^n).$$

Thus by definition, since

$$\lim_{n \rightarrow \infty} \mathcal{M}(y, z, z, t/k^n) = 1 \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(y, z, z, t/k^n) = 0.$$

It follows that

$$1 \geq \mathcal{M}(y, z, z, t) \geq 1, 0 \leq \mathcal{M}(y, z, z, t) \leq 0,$$

Which implies  $y = z$  by definition.

**Theorem 3.2:**

$(X, \mathcal{M}, \mathcal{N}, *, \diamond)$  be a complete generalized intuitionistic fuzzy metric space and  $f, g$  be  $\mathbb{R}$ -weakly commuting self mappings of  $X$  satisfying the following conditions:

- (i)  $f(X) \subset g(X)$ ,
- (ii)  $f$  (or)  $g$  is continuous ,
- (iii) for all  $x, y, z \in X$  and  $0 < t < 1$ 

$$\mathcal{M}(fx, fy, fz, t) \geq \phi(\mathcal{M}(gx, gy, gz, t)),$$

$$\mathcal{N}(fx, fy, fz, t) \leq \psi(\mathcal{N}(gx, gy, gz, t)).$$

where  $\phi, \psi : [0,1] \rightarrow [0,1]$  are a continuous function such that  $\phi(t) > t$  and  $\psi(t) < t$ .

If the sequence  $\{x_n\}$  and  $\{y_n\}$  in  $X$  are such that for all  $x, y \in X$  and  $t > 0$ .

$$(iv) \quad \lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} y_n = y, \lim_{n \rightarrow \infty} z_n = z \text{ implies}$$

$$\lim_{n \rightarrow \infty} \mathcal{M}(x_n, y_n, z_n, t) = \mathcal{M}(x, y, z, t) \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(x_n, y_n, z_n, t) = \mathcal{N}(x, y, z, t).$$

Then  $f$  and  $g$  have a unique common fixed point in  $X$ .

**Proof :**

Let  $x_0$  be an arbitrary point in  $X$ . By (i) choose a point  $x_1$  in  $X$  such that  $fx_0 = gx_1$ .

In general, we choose  $x_{n+1}$  such that  $fx_n = gx_{n+1}$  for all  $n \geq 0$ . Then for all  $t > 0$ .

$$\begin{aligned} \mathcal{M}(fx_n, fx_{n+1}, fx_{n+1}, t) &\geq \phi(\mathcal{M}(gx_n, gx_{n+1}, gx_{n+1}, t)) \\ &= \phi(\mathcal{M}(fx_{n-1}, fx_n, fx_n, t)) \\ &> \mathcal{M}(fx_{n-1}, fx_n, fx_n, t) \end{aligned} \quad (3.2.1)$$

$$\begin{aligned} \mathcal{N}(fx_n, fx_{n+1}, fx_{n+1}, t) &\leq \psi(\mathcal{N}(gx_n, gx_{n+1}, gx_{n+1}, t)) \\ &= \psi(\mathcal{N}(fx_{n-1}, fx_n, fx_n, t)) \end{aligned} \quad (3.2.2)$$

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$$\mathcal{N}(fx_{n-1}, fx_n, fx_n, t).$$

Since  $\phi(t) > t$  and  $\psi(t) < t$  for all  $0 < t < 1$ . Thus  $\{\mathcal{M}(fx_n, fx_{n+1}, fx_{n+1}, t)\}$  is an increasing sequence of positive real numbers in  $[0, 1]$  and  $\{\mathcal{N}(fx_n, fx_{n+1}, fx_{n+1}, t)\}$  is a decreasing sequence of positive real numbers in  $[0, 1]$ . Therefore, they converge to the limits  $L \geq 1$  and

$L' \geq 0$ , respectively. Now we claim that  $L = 1$  and  $L' = 0$  for, Let  $L < 1$ .

Letting  $n \rightarrow \infty$  in (3.2.1), we have  $L \geq \phi(L) > L$ , which is a contradiction and so  $L = 1$ .

Similarly, let  $L' > 0$ . Letting  $n \rightarrow \infty$  in (3.2.2), we have  $L' \leq \psi(L') < L'$ , which is a contradiction and so  $L' = 0$ . Now, for any positive integer  $P$  and  $t > 0$ , we have

$$\begin{aligned} \mathcal{M}(fx_n, fx_{n+P}, fx_{n+P}, t) &\geq \mathcal{M}(fx_n, fx_{n+1}, fx_{n+1}, t/P) * \dots * \\ &\mathcal{M}(fx_{n+P-1}, fx_{n+P}, fx_{n+P}, t/P) \\ &\geq \mathcal{M}(fx_n, fx_{n+1}, fx_{n+1}, t/P) * \dots * \\ &\mathcal{M}(fx_n, fx_{n+1}, fx_{n+1}, t/P). \\ \mathcal{N}(fx_n, fx_{n+P}, fx_{n+P}, t) &\leq \mathcal{N}(fx_n, fx_{n+1}, fx_{n+1}, t/P) \diamond \dots \diamond \\ &\mathcal{N}(fx_{n+P-1}, fx_{n+P}, fx_{n+P}, t/P) \\ &\leq \mathcal{N}(fx_n, fx_{n+1}, fx_{n+1}, t/P) \diamond \dots \diamond \\ &\mathcal{N}(fx_n, fx_{n+1}, fx_{n+1}, t/P). \end{aligned}$$

Since we have

$$\lim_{n \rightarrow \infty} \mathcal{M}(fx_n, fx_{n+1}, fx_{n+1}, t/P) = 1, \text{ and } \lim_{n \rightarrow \infty} \mathcal{N}(fx_n, fx_{n+1}, fx_{n+1}, t/P) = 0$$

It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{M}(fx_n, fx_{n+P}, fx_{n+P}, t) &\geq 1 * \dots * 1 \geq 1, \\ \lim_{n \rightarrow \infty} \mathcal{N}(fx_n, fx_{n+P}, fx_{n+P}, t) &\leq 0 \diamond \dots \diamond 0 \leq 0. \end{aligned}$$

Thus by definition  $\{fx_n\}$  is a Cauchy sequence and by the completeness of  $X$ ,  $\{fx_n\}$  converges to a point  $z \in X$ . Also  $\{gx_n\}$  converges to the point  $z$ . Suppose that by (ii), the mapping  $f$  is continuous. Then  $\lim_{n \rightarrow \infty} ffx_n = fz$  and  $\lim_{n \rightarrow \infty} fgx_n = fz$ , further since  $f$  and  $g$  are  $R$ -weakly commuting, we have

$$\begin{aligned} \mathcal{M}(fgx_n, gfx_n, gfx_n, t) &\geq \mathcal{M}(fx_n, gx_n, gx_n, t/R) \text{ and} \\ \mathcal{N}(fgx_n, gfx_n, gfx_n, t) &\leq \mathcal{N}(fx_n, gx_n, gx_n, t/R) \end{aligned}$$

Letting  $n \rightarrow \infty$  in the inequality by (iv), we have  $\lim_{n \rightarrow \infty} gfx_n = fz$ .

Now, we prove that  $z = fz$  suppose  $z \neq fz$ . Then there exists  $t > 0$  such that

$$\begin{aligned} \mathcal{M}(z, fz, fz, t) &< 1 \text{ and } \mathcal{N}(z, fz, fz, t) > 0. \text{ By (iii), we have} \\ \mathcal{M}(fx_n, ffx_n, ffx_n, t) &\geq \phi(\mathcal{M}(gx_n, gfx_n, gfx_n, t)) \text{ and} \\ \mathcal{N}(fx_n, ffx_n, ffx_n, t) &\leq \psi(\mathcal{N}(gx_n, gfx_n, gfx_n, t)). \end{aligned}$$

Letting  $n \rightarrow \infty$  in the above inequalities,

we have  $\mathcal{M}(z, fz, fz, t) \geq \phi(\mathcal{M}(z, fz, fz, t)) > M(z, fz, fz, t)$ ,

$\mathcal{N}(z, fz, fz, t) \leq \psi(\mathcal{N}(z, fz, fz, t)) < N(z, fz, fz, t)$ , which is a contradiction. Therefore  $z = fz$ .

By (i) since we can find a point  $z_1 \in X$  such that  $z = fz = gz_1$ . Now it follows that

$$\mathcal{M}(fx_n, fz_1, fz_1, t) \geq \phi(\mathcal{M}(gfx_n, gz_1, gz_1, t)) \text{ and } \mathcal{N}(fx_n, fz_1, fz_1, t) \leq \psi(\mathcal{N}(gfx_n, gz_1, gz_1, t))$$

Letting  $n \rightarrow \infty$  in the above inequalities, we have

$$\mathcal{M}(fz, fz_1, fz_1, t) \geq \phi(\mathcal{M}(fz, gz_1, gz_1, t)) = 1 \text{ and } \mathcal{N}(fz, fz_1, fz_1, t) \leq \psi(\mathcal{N}(fz, gz_1, gz_1, t)) = 0$$

which implies that  $fz = fz_1$  since  $\phi(t) = 1$  and  $\psi(t) = 0$  for  $t = 1$ .

ie.,  $z = fz = fz_1 = gz_1$ . Also for any  $t > 0$ .

$$\begin{aligned} \mathcal{M}(fz, gz, gz, t) &= \mathcal{M}(fgz_1, gfz_1, gfz_1, t) \\ &\geq \mathcal{M}(fz_1, gz_1, gz_1, t/R) = 1, \end{aligned}$$

$$\begin{aligned} \mathcal{N}(fz, gz, gz, t) &= \mathcal{N}(fgz_1, gfz_1, gfz_1, t) \\ &\leq \mathcal{N}(fz_1, gz_1, gz_1, t/R) = 0. \end{aligned}$$

Which again imply that  $fz = gz$ . Therefore,  $z$  is a common fixed point of  $f$  and  $g$ .

Now, to prove the uniqueness of the common fixed point  $z$ , let  $y$  ( $y \neq z$ ) be another common fixed point of  $f$  and  $g$ . Then there exists  $t > 0$  such that  $\mathcal{M}(z, y, y, t) < 1$  and  $\mathcal{N}(z, y, y, t) > 0$

$$\begin{aligned} \mathcal{M}(z, y, y, t) &= \mathcal{M}(fz, fy, fy, t) \geq \phi(\mathcal{M}(gz, gy, gy, t)) \\ &= \phi(\mathcal{M}(z, y, y, t)) > M(z, y, y, t), \end{aligned}$$

$$\begin{aligned} \mathcal{N}(z, y, y, t) &= \mathcal{N}(fz, fy, fy, t) \leq \psi(\mathcal{N}(gz, gy, gy, t)) \\ &= \psi(\mathcal{N}(z, y, y, t)) < N(z, y, y, t). \end{aligned}$$

Which is a contradiction since  $\phi(t) > t$  and  $\psi(t) < t$  for any  $0 < t < 1$ .

Therefore,  $z = y$  ie,  $z$  is a unique common fixed point of  $f$  and  $g$ .

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