Advances in Dynamical Systems and Applications.
ISSN 0973-5321, Volume 16, Number 2, (2021) pp. 1393-1413

https://www.ripublication.com/adsa.htm

Axisymmetric bending of FGM circular plate with parabolically- varying thickness resting on a non-uniform two-parameter partial foundation by DQM

B.M. Abdelbaki*, M.E. Sayed-Ahmed, A.M.A. Al-Kaisy*****

**Mathematics and Physics Department, Faculty of Engineering, Fayoum University, Fayoum, Egypt.*

***Mathematics and Physics Department, Faculty of Engineering, Fayoum University, Fayoum, Egypt.*

****Mathematics and Physics Department, Faculty of Engineering, Fayoum University, Fayoum, Egypt.*

**Corresponding author.*

Abstract

In this paper, the axisymmetric bending of a functionally graded material (FGM) circular plate with a non-uniform thickness fully/partially resting on a non-uniform two-parameter elastic foundation. The governing equation has been developed on the basis of the classical plate theory (CPT). To this end, the differential quadrature method (DQM) is applied to obtain the solution by discretizing the differential equations of bending with different boundary conditions. The material properties are considered to vary in the transverse direction following a power-law relation of volume fraction index of the constituents. The variation of plate thickness and subgrade modulus is assumed to be parabolic in the radial direction. A parametric study incorporating evaluation of various parameters (e.g., volume fraction index, thickness variation, subgrade modulus variation, surface-foundation contact ratio and different boundary conditions) is performed on the static analysis of the circular plate. It can be concluded that by the appropriate choice of geometrical parameters with restrains type, an optimum design can be achieved to provide uniform stresses under the plate in addition to minimize the transverse displacement. Furthermore, the obtained results are validated and compared with those found in literature, where excellent agreement is observed.

Keywords: Axisymmetric bending, Circular plates, Functionally graded material (FGM), Variable thickness, Elastic foundation, Differential quadrature method (DQM)

1 INTRODUCTION

Recently, circular plates supported by an elastic foundation have had extensive application in various engineering fields. They can be employed in a wide range of structural components such as foundation of liquid storage tanks, foundation of wind turbines, raft foundation for buildings, base of silos, machines and deck plates utilized in launch vehicles in addition to chimneys and heat exchangers in shape of tubes. Since achieving high strength and stiffness structural components is very important in modern industries, a new class of composite material called functionally graded material (FGM), an inhomogeneous composite usually made from a mixture of ceramic and metal by gradually varying material properties through the thickness direction. This type of material was employed in the industrial field for the first time by the Japanese [1]. FGMs can be found in manufacturing of aerospace structures, fusion reactors, pressure vessels, solar panels and heat exchange panels. Consequently, the following lines are addressing examples of the considerable research work conducted to demonstrate the static and dynamic analysis of FGM plate.

Reddy and Wang [2] demonstrated the axisymmetric bending analysis of FG circular and annular plate by employing the first-order shear deformation Mindlin plate theory. The solution of different terms such as deflection of the plate, forces and moment resultant was carried out based on Kirchhoff plate theory, whereas Mindlin solution is governed when the Kirchhoff solution is known. Ma and Wang [3] studied the axisymmetric analysis of bending and buckling of FGM circular plates through incorporating third-order and classical plate theory relationships. In addition, they proved that the first order shear theory is adequate to include the effects of shear deformation on the axisymmetric bending. Following the unconstrained three order shear deformation plate theory (UTST), Saidi and Rasouli [4] investigated the axisymmetric analysis of bending and buckling of FG circular plate in which the shear-free condition is released at the upper and lower faces of the plate as plates in flow fields. Numerical results of the deflection, resultant moments and critical buckling were presented and compared to corresponding results based on the classical plate theory. Yun and Rongqiao [5] presented an analytical solution to the axisymmetric bending of FG circular plate subjected to transverse load for different boundary conditions on the basis of three-dimensional theory adopting the direct displacement approach. The transverse load was expanded in the Fourier-Bessel series and the plate response was obtained by the superposition principle. Gupta et al. [6] studied the axisymmetric free vibration of nonhomogeneous circular plate with nonlinear variable thickness based on the classical plate theory using the differential quadrature method. Vullo and Vivio [7] developed an analytical approach for estimating the relations between elastic stresses and strains in non-uniform thickness rotating disks with a fictitious density distribution under the influence of thermal load. Naei and Masoumi [8] demonstrated the buckling analysis of FGM circular plate with variable thickness for different boundary conditions employing energy method on the basis of Love-Kirchhoff plate theory with Sander’s non-linear strain-displacement equation for thin plates. The influence of the variable thickness was investigated on the critical buckling of the plate using the finite-element method.

A considerable number of engineering applications deal with circular plates supported by elastic foundation. Winkler model is the simplest model to idealize the behavior of the foundation underneath the plate. The elastic foundation can be represented by a number of separate linear springs with constant subgrade modulus k_w [9]. Abbasi and Farhatnia [10] analyzed the bending analysis of circular plate resting on Winkler foundation based on the classical plate theory using differential transformation method. The study included the effects of Winkler foundation on the central deflection of the plate in addition to radial and tangential stresses. A refined extension of the Winkler model so called Pasternak foundation model that includes the interaction between the plate and the foundation in addition to consider the influence of in-plane shear [11] has been utilized. Arefi and Allam [12] studied the effects of the Winkler-Pasternak foundation on the nonlinear analysis of FG circular plate with piezoelectric layers. Shariyat and Alipour [13] presented a semi-analytical solution for free vibration of two-directional FGM circular plate resting on a two-parameter elastic foundation, illustrating the influence of various parameters such as elastic foundation on the natural frequency and modal stress of the plate.

In many cases, adopting the elastic foundation with a constant modulus has led to inaccurate results. For instance, in foundation mat subjected to uniformly distributed load, the foundation modulus at the edges has higher values than at the center of the mat as the distortion of the plate is not taken into consideration. Consequently, integrating the variation of foundation modulus is recommended in particular cases [14]. However, few studies have considered variable elastic foundation in the bending analysis of plates. Foyouzat and Mofid [15] investigated the static bending analysis of axisymmetric thin circular and annular plate resting on variable Winkler foundation under different boundary conditions. In addition, they provided a further extension of the addressed problem to a two-parameter elastic foundation. Rad and Shariyat [16] carried out the solution of static bending analysis of circular and annular plates supported by variable Winkler-Pasternak foundation using the exact three dimensions theory of elasticity by considering different sets of the foundation subgrade modulus. Furthermore, practical applications can be simulated as plates partially resting on elastic foundation such as plates that are used to cover openings or cavities in structures. Alinaghizadeh et al. [17] highlighted the utilization of plates partially resting on a two-parameter elastic foundation by investigating the bending behavior of two-directional FG circular/annular sector plates.

Many numerical methods have been employed to analyze the dynamic and static behavior of circular plate. Differential quadrature method (DQM) was one among the numerical methods that grasp the attention of the researchers in various engineering fields. Liew and Han [18] adopted the DQM to study the static analysis of rectangular plates on Winkler foundation for a combination of boundary conditions. Hossenin and Akhavan [19] investigated the buckling and dynamic behavior of sectorial plate supported by Pasternak foundation under in-plane compressive loads using DQM. The results revealed the stability and accuracy of the DQM in comparison with other numerical methods. Arshid et al. [20] employed the DQM to analyze the free vibration of circular plate made from porous material incorporated with piezoelectric actuators. Farhatnia and Saadat [21] studied the axisymmetric bending analysis of FGM sandwich plate supported by Winkler foundation via DQM. The numerical results of the transverse deflection in addition to the radial and tangential stress were compared with the results obtained by the finite-element method to demonstrate the efficiency of the proposed approach.

As mentioned above in the literature, no research work has been published yet on plates partially resting on a non-uniform two-parameter elastic foundation using the DQM as previous studies are limited to circular plates fully resting on foundation. In the present work, an axisymmetric bending solution of a FGM circular plate with varying thickness along the radial direction and subjected to transverse loading at the upper face of the plate with restrained edge is carried out using the differential quadrature method. The mechanical properties of the material vary across the thickness direction corresponding to a power-law relation in terms of volume fraction of constituents. A parametric study is carried out to investigate the influence of different parameters on the bending of the plate such as variable thickness, volume fraction index, existence of elastic foundation, variation of the subgrade modulus and the partially contact between the lower face of the plate and the elastic foundation. The governed equation is developed based on the classical plate theory and the solution was obtained by the differential quadrature method. Numerical results of the transverse displacement and radial stress are compared with other numerical methods from well-known literature, demonstrating the efficiency of the adopted approach.

2 MATHEMATICAL FORMULATION

2.1 Axisymmetric bending problem

Consider a FGM circular plate with outer radius b and initial thickness h_0 at the center of the plate, the plate is fully or partially resting on elastic foundation with variable Winkler's foundation modulus $k_w(r)$ represented by linear spring and Pasternak's foundation modulus $k_p(r)$ represented by shear layer under the plate as shown in

Fig. 1. The plate is assumed to be in continuous contact with the foundation except for the distance a and subjected to a uniformly distributed transverse load denoted by q_0. The cylindrical polar coordinate system (r,θ,z) is adopted in the analysis. The physical middle plane of the circular plate is located at $z=0$ and the top and bottom surfaces are $z=h/2$ and $z=-h/2$, respectively.

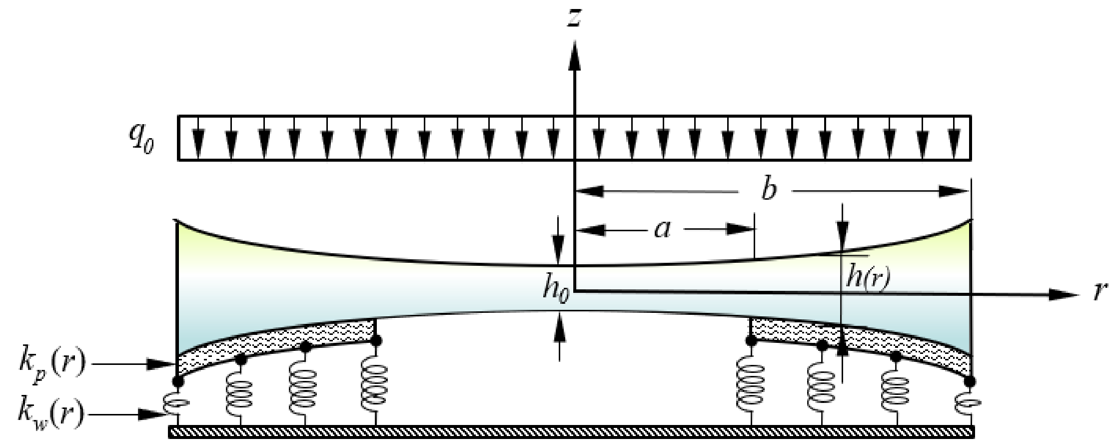


Fig. 1. FGM circular plate with variable thickness partially supported by a two-parameter elastic foundation with variable modulus.

The governing differential equation of circular plate with variable thickness resting on elastic foundation can be obtained by utilizing the classical plate theory (CPT) as [21], and by adding the term of Pasternak foundation obtained from [13]. Therefore, the equation can be written as:

$$r^3 D(r)\frac{d^4 w}{dr^4}+2r^2\left(D(r)+r\frac{dD(r)}{dr}\right)\frac{d^3 w}{dr^3}+r\left(-D(r)+(2+\nu)r\frac{dD(r)}{dr}+r^2\frac{d^2 D(r)}{dr^2}\right)\frac{d^2 w}{dr^2}$$
$$+\left(D(r)-r\frac{dD(r)}{dr}+r^2\nu\frac{d^2 D(r)}{dr^2}\right)\frac{dw}{dr}-k_p r^2\frac{d}{dr}\left(r\frac{dw}{dr}\right)+k_w r^3 w=q_0 r^3 \tag{1}$$

where w is the out-plane deflection, D is the flexural rigidity of the plate and ν is Poisson's ratio assumed to be constant.

Young's modulus $E(r,z)$ is smoothly varying from metal to ceramic according to the power-law distribution through the thickness of the plate as [10]:

$$E(r,z)=(E_c-E_m)\left(\frac{z}{h(r)}+\frac{1}{2}\right)^g+E_m \tag{2}$$

where the subscripts m and c refer to the metallic and ceramic constituents, respectively, and g is the volume fraction index.

Due to non-homogeneity of material properties, the physical neutral surface and geometric middle surface do not coincide. As a result, we have to select a proper reference plane where the stress and strain are zero. This plane is located at distance

(z_0) from the middle surface of the plate, which can be obtained from [22]:

$$z_0 = \frac{\int_{-h(r)/2}^{h(r)/2} zE(r,z)dz}{\int_{-h(r)/2}^{h(r)/2} E(r,z)dz}. \tag{3}$$

Therefore, the flexural rigidity of the plate is determined as follows:

$$D = \int_{-h(r)/2}^{h(r)/2} \frac{\left(z - z_0\right)^2 E(r,z)}{1-\nu^2} dz. \tag{4}$$

Furthermore, following the classical plate theory, one can deduce relations describing the radial and tangential stress using appropriate derivatives of the deflection function based on the following equations:

$$\begin{aligned} \sigma_r &= -\frac{E(r,z)z}{1-\nu^2}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) \\ \sigma_\theta &= -\frac{E(r,z)z}{1-\nu^2}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) \end{aligned} \tag{5}$$

2.2 Thickness and foundation modulus profiles

The thickness of the circular plate is assumed to vary continuously along the radial direction according to prescribed function as [21]:

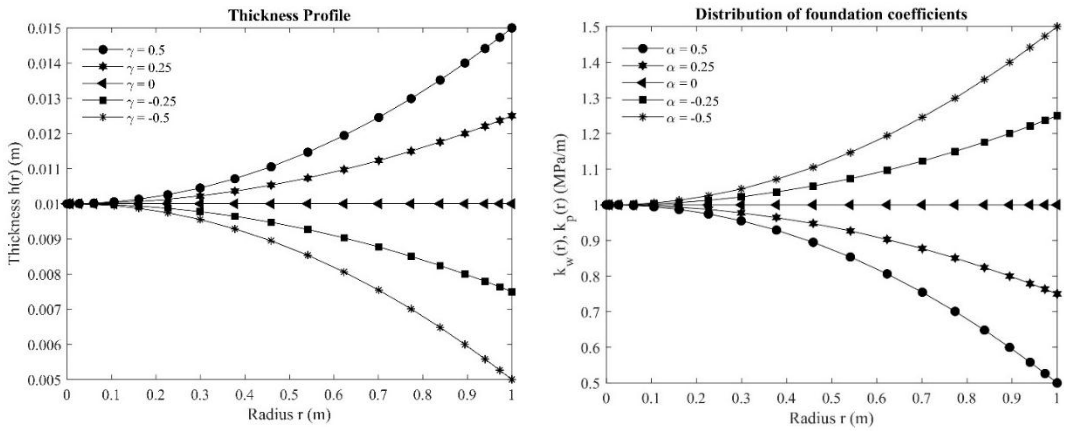
$$h(r) = h_0\left(1 + \gamma\left(\frac{r}{b}\right)^2\right). \tag{6}$$

in which γ is a geometrical parameter which controls the degree of curvature of the thickness profile.

Similarly, Pasternak and Winkler coefficients are varying parabolically along the radial direction as [16]:

$$k_w(r) = k_{w_o}\left(1 - \alpha\left(\frac{r}{b}\right)^2\right), \quad k_p(r) = k_{p_o}\left(1 - \alpha\left(\frac{r}{b}\right)^2\right) \tag{7}$$

in which γ and α are parameters controlling the curvature degree for thickness and foundation coefficients profiles, respectively. For better understanding of Eqs. (6) and (7), the variation of thickness profile and the distribution of the coefficients of the elastic foundation are shown in **Fig. 2**.



a) b)

Fig. 2. Thickness profile and the distribution of foundation coefficients along the radial direction.

2.3 Boundary and regularity conditions

Two different types of boundary conditions are adopted in the present work as follows:

- Clamped boundary condition:

$$w\big|_{r=b} = 0, \qquad \left.\frac{dw}{dr}\right|_{r=b} = 0. \tag{8}$$

- Simply-supported boundary condition:

$$w\big|_{r=b} = 0, \qquad M_r\big|_{r=b} = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) = 0 \tag{9}$$

In order to ensure that the solution is avoided from singularity, regularity condition must be fulfilled as below:

- Regularity condition at the center of the plate:

$$\left.\frac{dw}{dr}\right|_{r=0} = 0. \tag{10}$$

3 DQ Method

The differential quadrature method (DQM) approximates any derivative of a function with respect to a spatial variable at any grid point as a weighted linear summation of their values at all the grid points selected in the solution domain. According to the differential quadrature rule, the *n*th-order derivative of $f(r)$ can be discretely expressed at a grid point r_i, as [19]:

$$\left.\frac{d^n f(r)}{dr^n}\right|_{x=x_i} = \sum_{j=1}^{N} c_{ij}^{(n)} f\left(r_j\right), \qquad i = 1, 2, ..., N, \tag{11}$$

in which $c_{ij}^{(n)}$ are the weighting coefficients associated with *n*th-order derivative of $f(x)$, and N is the number of discrete grid points in the solution domain. The weighting coefficients in Eq. (11) are given as follows:

$$c_{ij}^{(1)} = \frac{M^{(1)}(r_i)}{(r_i - r_j)M^{(1)}(r_j)}, \qquad i, j = 1, 2, ..., N, \quad j \neq i, \tag{12}$$

where

$$M^{(1)}(r_i) = \prod_{j=1, j\neq i}^{N} (r_i - r_j) \tag{13}$$

and

$$c_{ij}^{(n)} = n\left(c_{ii}^{(n-1)} c_{ij}^{(1)} - \frac{c_{ij}^{(n-1)}}{r_i - r_j}\right), \qquad i, j = 1, 2, ..., N, \quad j \neq i, \text{ and } n = 2, 3, ..., \tag{14}$$

$$c_{ii}^{(n)} = -\sum_{j=1, j\neq i}^{N} c_{ij}^{(n)}, \qquad i, j = 1, 2, ..., N. \tag{15}$$

3.1 Numerical discretization

Several options can be employed to choose grid points location and the easiest method to select grid points in computational domain is uniform grid mesh, where all the grid points are equally spaced. However, in order to achieve more accurate results and reasonably convergence, non-uniform grid points are chosen. Shu and Richards [23] proposed a refined technique called Chebyshev-Gauss-Lobatto for selection of grid points. The grid points using this technique are given by:

$$r_i = \frac{b}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right), \qquad i = 1, 2, ..., N. \tag{16}$$

3.2 DQM applied to the governing equations

The discretized form of Eq. (1) can be achieved easily by substituting relation (12) into Eq. (1) as:

$$\begin{aligned} & r_i^3 D(r_i)\sum_{j=1}^{N} c_{ij}^{(4)} w_j + 2r_i^2\left(D(r_i) + r_i\frac{dD(r_i)}{dr_i}\right)\sum_{j=1}^{N} c_{ij}^{(3)} w_j - k_p(r_i) r_i^3 \sum_{j=1}^{N} c_{ij}^{(2)} w_j \\ & + r_i\left(-D(r_i) + (2+\nu) r_i \frac{dD(r_i)}{dr_i} + r_i^2\frac{d^2D(r_i)}{dr_i^2}\right)\sum_{j=1}^{N} c_{ij}^{(2)} w_j - k_p(r_i) r_i^2 \sum_{j=1}^{N} c_{ij}^{(1)} w_j \\ & + \left(D(r_i) - r_i\frac{dD(r_i)}{dr_i} + r_i^2 \nu \frac{d^2D(r_i)}{dr_i^2}\right)\sum_{j=1}^{N} c_{ij}^{(1)} w_j + k_w(r_i) r_i^3 w_i = q_0 r_i^3. \qquad i = 2,3,\ldots,N-2. \end{aligned} \tag{17}$$

Following the same procedure, the equations of radial and tangential stress can be transformed into the DQ formulation as [21]:

$$\begin{aligned} \sigma_r\big|_{r=r_i} &= -\frac{E(r_i,z)z}{1-\nu^2}\left(\sum_{j=1}^{N} c_{ij}^{(2)} w_j + \frac{\nu}{r_i}\sum_{j=1}^{N} c_{ij}^{(1)} w_j\right) \\ \sigma_\theta\big|_{r=r_i} &= -\frac{E(r_i,z)z}{1-\nu^2}\left(\frac{1}{r_i}\sum_{j=1}^{N} c_{ij}^{(1)} w_j + \nu\sum_{j=1}^{N} c_{ij}^{(2)} w_j\right). \end{aligned} \tag{18}$$

3.3 DQM applied to boundary and regularity conditions

During the implementation of DQM to the boundary conditions, a problem arises. This problem appears because of the existence of two boundary conditions at the same point. In order to overcome this problem, Bert [24] proposed δ-point technique as an appropriate way when applying DQM to the boundary conditions of beams and plates problems. In this technique, only one boundary condition is applied at the boundary grid point and the derivative condition is discretized at δ-point, where δ represents a very small distance. Then, boundary and regularity conditions can be discretized as follows:

- Clamped boundary conditions:

$$w_N = 0, \qquad \sum_{j=1}^{N} C_{N-1}^{(1)} w_j = 0. \tag{19}$$

- Simply supported boundary condition:

$$w_N = 0, \qquad \sum_{j=1}^{N} C_{(N-1)j}^{(2)} w_j + \nu\left(\frac{1}{r_i}\sum_{j=1}^{N} C_{(N-1)j}^{(1)} w_j\right) = 0. \tag{20}$$

- Regularity condition at the center of the plate:

$$\sum_{j=1}^{N} C_{1j}^{(1)} w_j = 0. \tag{21}$$

4 SOLUTION TECHNIQUE

In order to achieve the solution, the discretized forms of governing equation and boundary/regularity conditions result in a set of linear equations. This set can be expressed in a matrix form as follows:

$$\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N-1} & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N-1} & A_{1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{N-1,1} & A_{N-1,2} & \cdots & A_{N-1,N-1} & A_{N-1,N} \\ A_{N,1} & A_{N,2} & \cdots & A_{N-1,N-1} & A_{N,N} \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-1} \\ w_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix}, \tag{22}$$

Obviously, Eq. (22) includes all the functional values in the whole analysis domain. Nevertheless, it is difficult to couple Eq. (19) or (20) and Eq. (21) to obtain the solution of w_{N-1} and w_N. Hence, to overcome this problem, the governing equation and the boundary/regularity conditions must be solved simultaneously. The matrix of Eq. (22) can be reconstructed as:

$$[A_{BB}]\{w_B\} + [A_{BI}]\{w_I\} = \{0\}, \quad [A_{IB}]\{w_B\} + [A_{II}]\{w_I\} = \{q_0\}, \tag{23}$$

where *B* refers to boundary points and *I* refers to internal points. It is noted that the sizes of matrices $[A_{II}]$, $[A_{BB}]$ and $[A_{BI}]$ are $(N-3)\times(N-3)$, (3×3) and $3\times(N-3)$. The functional values of the internal points were separated from the boundary points. The functional values at the internal points and the boundary points are denoted by $\{w_I\}$ and $\{w_B\}$, respectively.

The functional values at the boundary points can be obtained as:

$$\{w_B\} = -[A_{BB}]^{-1}[A_{BI}]\{w_I\}. \tag{24}$$

By substituting $\{w_B\}$ from Eq. (24) into the second part of Eq. (23), we get:

$$\left([A_{II}] - [A_{IB}][A_{BB}]^{-1}[A_{BI}]\right)\{w_I\} = \{q_0\}. \tag{25}$$

The solving process was carried out by developing a MATLAB program to solve the set of the linear equation for the computational domain. The results are presented and discussed in the next section.

5 Numerical results

The numerical results presented in this section employ a value of $h_0 = 0.01\,\text{m}$, $b = 0.6$ m, $q_0 = 10^5\,\text{kPa}$. The mechanical properties of the materials employed in the study are taken from Ref. **[25]** as Young's modulus of elasticity of metal and ceramic are $E_m = 70\,\text{GPa}, E_c = 380\,\text{GPa}$, respectively. However, the value of Poisson's ratio is considered to be constant and taken as $\nu = 0.3$. Furthermore, the following non-dimensional quantities are adopted for presenting the results:

$$\sum_r = \sigma_r / q_0,\ \sum_\theta = \sigma_\theta / q_0,\ W = w / b,\ R = r / b,\ \xi = z / h$$

5.1 Comparison with results of past literature

In order to demonstrate the accuracy of the presented method, numerical results for the non-dimensional transversal displacement shown in **Table. 1** are compared with those reported in Ref. [10] for a clamped and simply supported circular plate. The plate is assumed to be fully rested on an elastic foundation in purpose of validation and subjected to uniform loading. The plate is assumed to be homogenous with uniform thickness and rested on one parameter elastic foundation (Winkler model). So as to reach the convergence and the stability of the proposed method, 20 grid points are adequate to obtain acceptable results for both clamped and simply supported conditions. The numerical results obtained by the differential quadrature method show excellent agreement with the differential transformation method (DTM) which reveals the accuracy and the efficiency of the adopted approach.

Table. 1. Comparison of results for uniform thickness, $(\gamma = 0)$ rested on Winkler foundation $(k_p = \alpha = a = 0)$.

k_w		Clamped edge				Simply supported edge			
		$g=0$	$g=1$	$g=10$	$g=100$	$g=0$	$g=1$	$g=10$	$g=100$
0	Present	0.0097	0.0195	0.0324	0.0470	0.0395	0.0793	0.1320	0.1915
	Ref. [10]	0.0097	0.0195	0.0324	0.0470	0.0395	0.0793	0.1320	0.1915
5	Present	0.0082	0.0141	0.0198	0.0243	0.0221	0.0306	0.0358	0.0386
	Ref. [10]	0.0082	0.0141	0.0198	0.0243	0.0221	0.0306	0.0358	0.0386
10	Present	0.0070	0.0110	0.0141	0.0161	0.0153	0.0186	0.0201	0.0206
	Ref. [10]	0.0070	0.0110	0.0141	0.0161	0.0153	0.0186	0.0201	0.0206
50	Present	0.0033	0.0038	0.0039	0.0038	0.0041	0.0041	0.0039	0.0037
	Ref. [10]	0.0033	0.0038	0.0039	0.0038	0.0041	0.0041	0.0039	0.0037

In comparison to another study as shown in **Table. 2,** it was carried out to guarantee the accuracy of the presented results. In order to validate the obtained results, tapered thickness profile was studied with different values of γ for homogenous metal material. The results were compared to the results reported in Ref. [21] for two numerical methods DQM and Finite Element method (FEM) using ABAQUS software. The results match those obtained by FEM with error percentage less than 0.02.

Table. 2. Validation of results for tapered section profile for homogeneous metal material, $(g=\infty, k_w=k_p=0)$.

Edge condition		Non-dimensional transversal displacement at the center of the plate (W)			
		$\gamma=-0.5$	$\gamma=-0.3$	$\gamma=0.3$	$\gamma=0.5$
Clamped	Present (DQM)	0.1688	0.0987	0.0319	0.0240
	Ref.[21] (DQM)	0.1688	0.0987	0.0319	0.0240
	Ref.[21] (FEM)	0.1683	0.0985	0.0316	0.0237
Simply supported	Present (DQM)	0.5046	0.3446	0.1438	0.1133
	Ref.[21] (DQM)	0.5044	0.3445	0.1438	0.1132
	Ref.[21] (FEM)	0.5040	0.3441	0.1434	0.1129

5.2 Parametric studies on bending analysis

To assess the influence of geometrical and foundation parameters on the absolute value of central displacement of the FGM circular plate, effects of different values of *k*-coefficients with different values of volume fraction index are shown in **Table. 3** for clamped and simply supported edges denoted by C and SS, respectively. In case of clamped edge, it can be observed that the central displacement decreases with increasing values of *k*-coefficients for negative values of geometrical parameter γ (concavity goes upward). In contrast, central displacement increases with increasing values of *k*-coefficients for positive values of geometrical parameter γ (concavity goes downward). In case of simply supported edge, a significant decrease of the value of central displacement with increasing values of *k*-coefficients is observed for negative values of γ. However, the decrease of the value of central displacement is weak with increasing values of *k*-coefficients. Moreover, for higher values of *k*-coefficients, a significant decrease of the value of central displacement during increasing values of *k*-coefficients when $g=1$ is observed. For better understanding of the effects of volume fraction index *g*, see **Fig. 3**.

In case of circular plate partially rested on an elastic foundation, numerical results for the non-dimensional central displacement are provided in **Table. 4**. For clamped edge, the central displacement increases for positive values of α (max values of the two-parameters foundation at the center of the plate). Furthermore, this increase is more noticeable when $a/b=0.5$. In contrast, for simply supported edge, the central displacement decreases for positive values of α and the decrease is significant when $a/b=0.2$.

The influences of the two-parameter elastic foundation are studied in **Fig. 4** for various combinations of foundation coefficients. In case of a clamped edge, unexpectedly, the values of the transverse displacement increase with increasing values of *k*-coefficients. This happens because the plate was partially rested on the elastic foundation. In addition, a sudden increase of transverse displacement takes place when $(k_w, k_p) = (10, 5)$. On the other hand, for simply supported edge, the values of the transverse displacement decrease with increasing values of *k*-coefficients.

In order to investigate the effects of foundation coefficients variation parameter α in the radial direction on the transverse displacement of the plate, a plate with a negative value of geometric parameter γ (concavity goes downward) is considered in **Fig. 5**. The influence of α on the transverse displacement appears more clearly when higher values of γ are used. As a result, it is recommended for a foundation with a variable stiffness specification to pick out a plate with cross section with positive values of γ taking into account that this variation will not significantly affect the transverse displacement in addition to minimizing the overall cross section of the plate. Furthermore, the bending behavior is almost the same in both clamped and simply supported edges. This happens because the flexural stiffness of the plate at the edge is minima in comparison with its value at the center of the plate.

Table. 3. Variation of the non-dimensional transversal displacement at the center of the plate of FGM circular plate fully rested on non-uniform two-parameter elastic foundation for $\alpha = 0.5$.

(k_w, k_p)	γ	Non-dimensional transversal displacement at the center of the plate (W)							
		$g = 0$		$g = 1$		$g = 10$		Metal	
		C	SS	C	SS	C	SS	C	SS
(0,0)	-0.5	0.0241	0.0642	0.0483	0.1288	0.0803	0.2143	0.1306	0.3485
	-0.3	0.0159	0.0521	0.0319	0.1046	0.0531	0.1740	0.0863	0.2830
	0	0.0097	0.0395	0.0195	0.0793	0.0324	0.1320	0.0526	0.2146
	0.3	0.0065	0.0307	0.0131	0.0616	0.0218	0.1024	0.0355	0.1666
	0.5	0.0052	0.0261	0.0105	0.0525	0.0174	0.0873	0.0283	0.1419
(5,1)	-0.5	0.0184	0.0234	0.0258	0.0278	0.0291	0.0297	0.0309	0.0310
	-0.3	0.0151	0.0247	0.0245	0.0299	0.0298	0.0317	0.0321	0.0325
	0	0.0105	0.0253	0.0203	0.0331	0.0287	0.0354	0.0337	0.0354
	0.3	0.0073	0.0244	0.0155	0.0355	0.0249	0.0394	0.0335	0.0392
	0.5	0.0059	0.0230	0.0127	0.0361	0.0218	0.0420	0.0321	0.0421
(10,5)	-0.5	0.0127	0.0130	0.0145	0.0145	0.0155	0.0153	0.0165	0.0158
	-0.3	0.0136	0.0144	0.0153	0.0154	0.0159	0.0159	0.0162	0.0162
	0	0.0142	0.0174	0.0172	0.0173	0.0172	0.0171	0.0170	0.0169
	0.3	0.0103	0.0221	0.0195	0.0202	0.0191	0.0186	0.0180	0.0178
	0.5	0.0113	0.0266	0.0209	0.0229	0.0208	0.0200	0.0190	0.0184
(20,10)	-0.5	0.0072	0.0073	0.0079	0.0077	0.0085	0.0080	0.0092	0.0081
	-0.3	0.0077	0.0077	0.0080	0.0080	0.0082	0.0081	0.0084	0.0082
	0	0.0086	0.0087	0.0085	0.0085	0.0084	0.0084	0.0084	0.0084
	0.3	0.0097	0.0101	0.0093	0.0091	0.0089	0.0088	0.0086	0.0086
	0.5	0.0105	0.0115	0.0100	0.0096	0.0092	0.0090	0.0088	0.0087

Table. 4. Variation of the non-dimensional transversal displacement at the center of the plate of FGM circular plate partially rested on non-uniform two-parameter elastic foundation for $(\gamma = 0.5, g = 1, k_w = 10, k_p = 1)$.

α	Non-dimensional transversal displacement at the center of the plate (W)									
	$a/b = 0.1$		$a/b = 0.2$		$a/b = 0.3$		$a/b = 0.4$		$a/b = 0.5$	
	C	SS	C	SS	C	SS	C	SS	C	SS
-0.5	0.0098	0.0218	0.0106	0.0251	0.0114	0.0302	0.0120	0.0372	0.0121	0.0451
-0.25	0.0099	0.0214	0.0107	0.0246	0.0116	0.0297	0.0122	0.0367	0.0123	0.0450
0	0.0100	0.0209	0.0108	0.0241	0.0117	0.0292	0.0124	0.0362	0.0125	0.0448
0.25	0.0101	0.0205	0.0109	0.0236	0.0119	0.0287	0.0126	0.0357	0.0127	0.0445
0.5	0.0102	0.0201	0.0110	0.0232	0.0121	0.0282	0.0128	0.0352	0.0130	0.0443

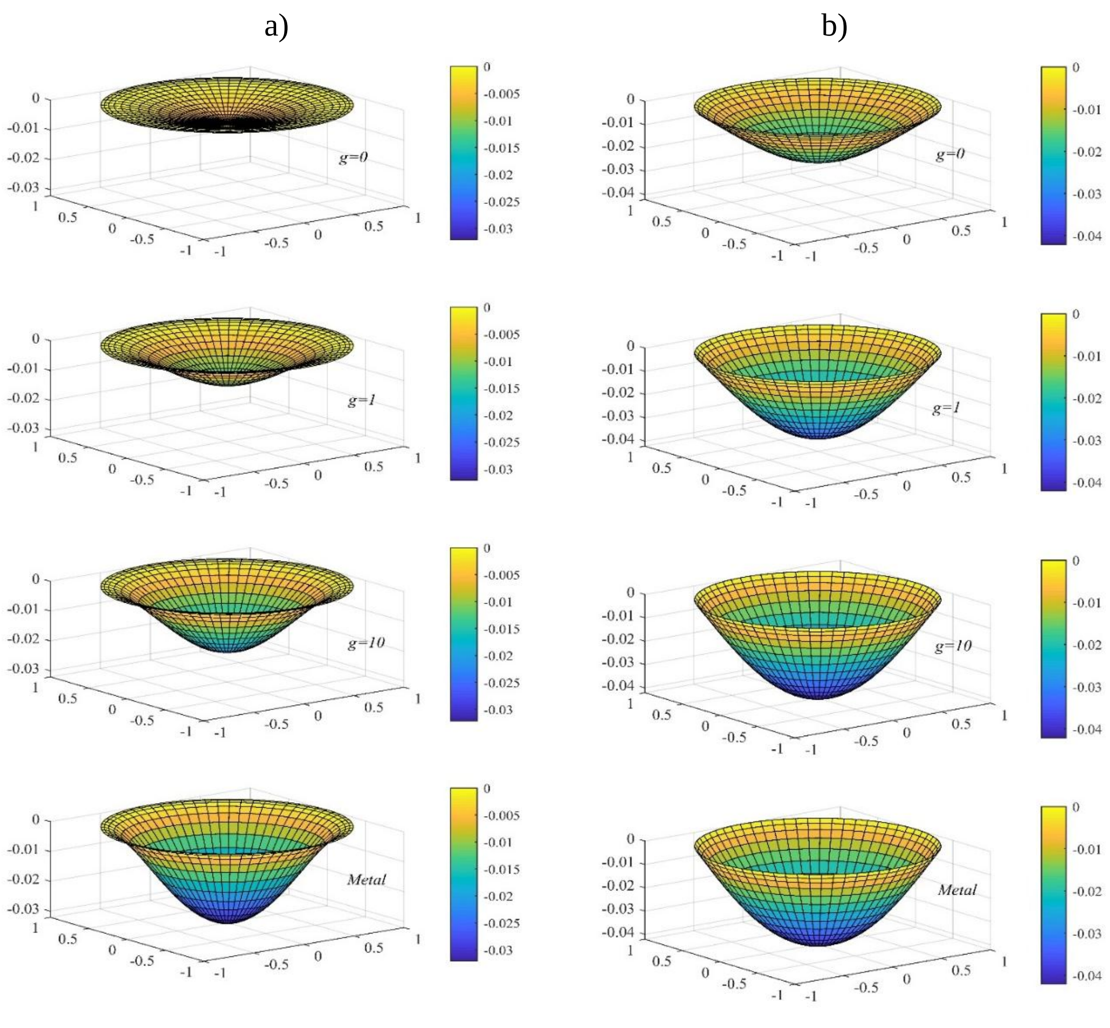


Fig. 3. Effects of volume fraction index (g) on the non-dimensional displacement of the circular plate for: a) clamped edge, b) simply supported edge; $(\gamma = 0.5\ , \alpha = 0.5, k_w = 5, k_p = 1, a/b = 0)$.

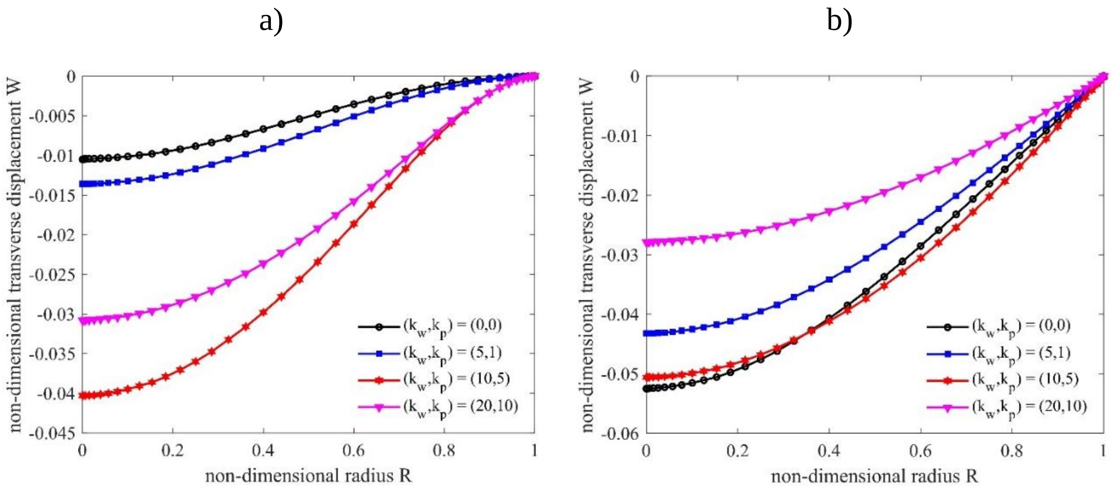


Fig. 4. Effects of a two-parameter elastic foundation on the transverse displacement of the plate for: a) clamped edge, b) simply supported edge; $(\gamma = 0.5, g = 1, \alpha = 0.5, a/b = 0.2)$.

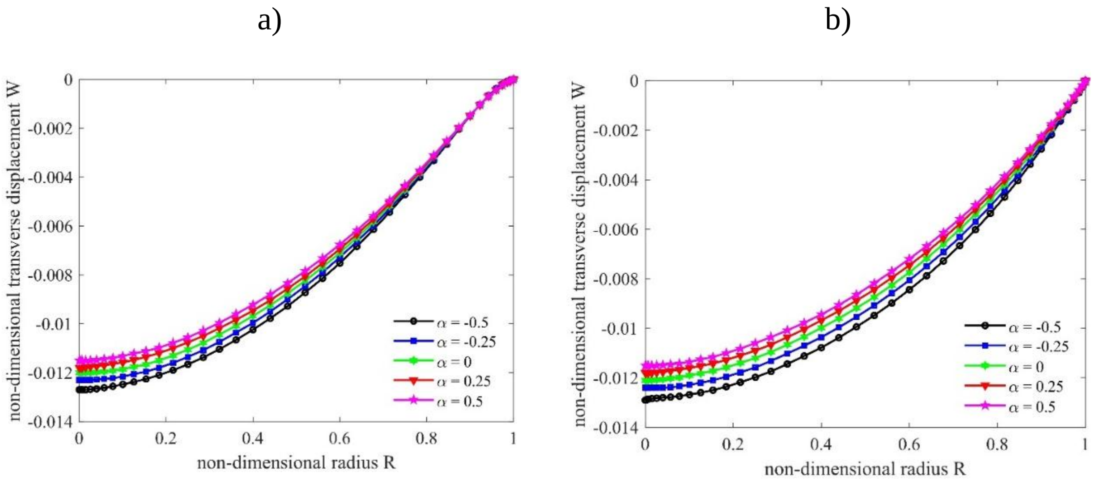


Fig. 5. Effects of the variation in the two-parameter elastic foundation coefficients on the transverse displacement of the plate for: a) clamped edge, b) simply supported edge; $(\gamma = -0.5, g = 1, k_w = 20, k_p = 2, a/b = 0.2)$.

5.3 Parametric studies on stress analysis

In this subsection, a parametric study has been conducted to investigate the influence of the ratio of contacting surface with the two-parameter elastic foundation, gradient index materials and geometric parameters on the radial stress of the FGM circular

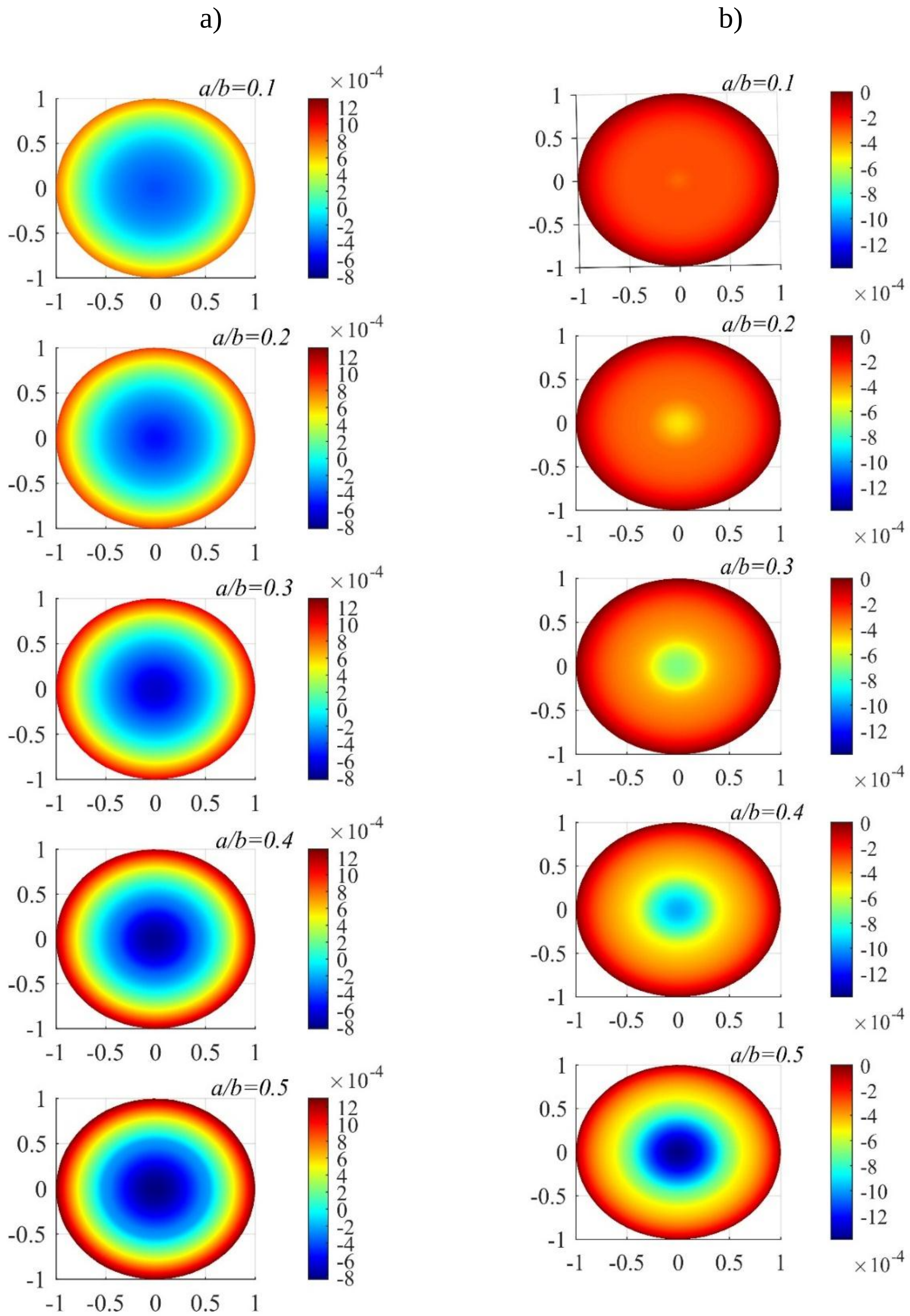


Fig. 6. Distribution of non-dimensional radial stress of FGM circular plate partially rested on elastic foundation plate for: a) clamped edge, b) simply supported edge;
$(\gamma = 0.5, g = 1, k_w = 20, k_p = 2, \alpha = 0.5, \xi = 0.5)$.

plate across the thickness of the elastic foundation (a/b). It is observed that the radial stress increases uniformly with increasing ratio a/b at both edge and center of the plate in case of clamped edge. This happens as a result of the absence of elastic foundation. On the other hand, for a simply supported edge, a sudden increase in the radial stress occurs for ratio a/b greater than 0.4.

a) b)

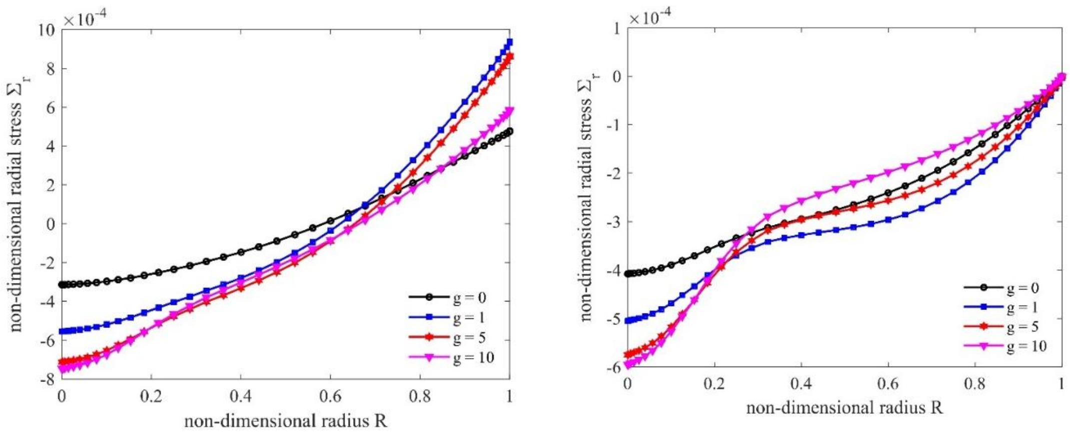


Fig. 7. Distribution of non-dimensional radial stress of FGM circular plate partially rested on elastic foundation plate for: a) clamped edge, b) simply supported edge; $(\gamma = 0.5, k_w = 20, k_p = 2, \alpha = 0.5, a/b = 0.2, \xi = 0.5)$.

a) b)

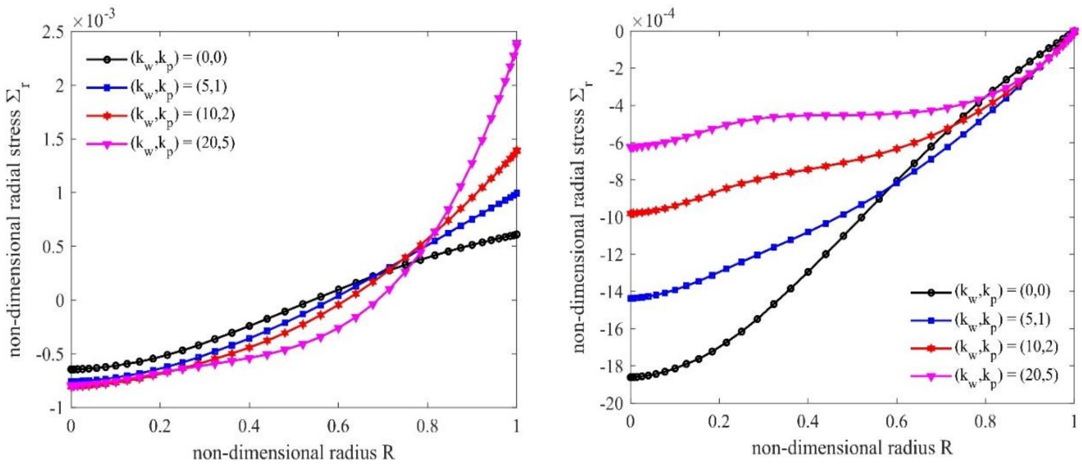
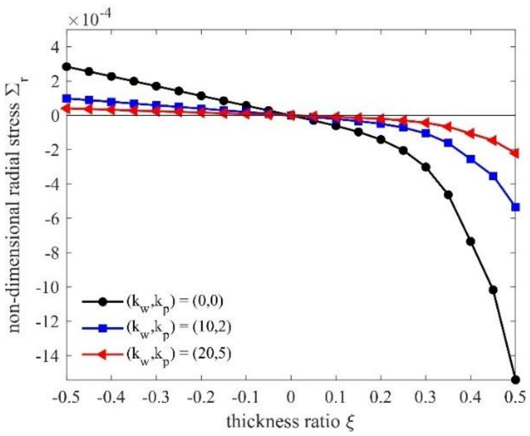


Fig. 8. Distribution of non-dimensional radial stress of FGM circular plate partially rested on elastic foundation plate for: a) clamped edge, b) simply supported edge; $(\gamma = 0.5, g = 1, \alpha = 0.5, a/b = 0.2, \xi = 0.5)$.

Fig. 7 shows the variation of non-dimensional radial stress at the top surface of the FGM circular for different values of volume fraction index g ($g = 0,1,5,10$). In case of clamped edge, it can be seen that maximum stresses at the edge and center of the plate occur when $g = 1$ and $g = 10$, respectively. For a simply supported edge, a uniform increase is noticed in both of stress at the center of the plate and in the rate of change of stress during increasing the value of g.

The compound effect of four different sets of two-parameter foundation coefficients (k_w, k_p) is plotted in **Fig. 8**. It is obvious that the influence of increasing the values of (k_w, k_p) on the radial stress is more obvious at the edge of the plate while it is barely noticeable at the center of the plate in case of clamped edge. On the contrary, increasing the value of (k_w, k_p) meets a uniform decrease of the radial stress. In **Fig. 9**, distribution of the radial stress through the thickness direction is plotted to demonstrate the effect of two-parameters elastic foundation on the variation of the radial stress corresponding to sections at R = 0 and R = 1 for FGM circular plate with clamped edge. As shown in **Fig. 9**, the degradation of the radial stress at the center of the plate (R = 0) is remarkable for higher sets of (k_w, k_p) while the values of radial stress go up by increasing values of (k_w, k_p). However, for an elastic foundation with higher shear coefficient (k_p), the values of the radial stress are slightly decreased by increasing values of (k_w, k_p).

a) b)



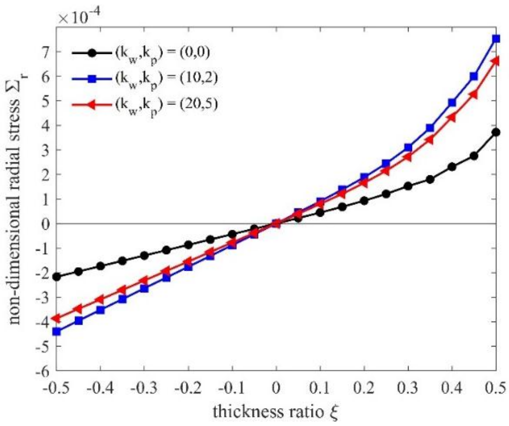


Fig. 9. Distribution of the radial stress through the thickness of the plate at: a) center of the plate, b) plate edge; $(\gamma = 0.5, g = 10, \alpha = 0.5, a/b = 0)$.

CONCLUSION

In the present paper, the differential quadrature method (DQM) was adopted to study the bending and stress analyses of axisymmetric functionally graded material circular plate with variable thickness partially resting on a non-uniform two-parameter elastic foundation subjected to uniform transverse pressure. The validity of the present work has been investigated for static analysis of the simply supported and clamped plate under various parameters. The obtained numerical results show excellent agreement with the results tabulated in the literature. Based on this study results, the main conclusions can be summarized as follows.

- The profile of the circular plate can affect the values of the central displacement. As for clamped edge it is better to choose a profile with concavity going upward $(+\gamma)$, while for a simply supported edge, it is better to choose a profile with concavity going downward $(-\gamma)$.
- The values of transverse displacement increase by increasing volume fraction index g toward metal material.
- For a variable elastic foundation, the central displacement of clamped circular plate increases for positive values of α where the values of the foundation coefficients (k_w, k_p) are maximum, while the central displacement decreases in case of a simply supported plate.
- A gradual stress distribution occurs by varying the contact surface ratio a/b for clamped edge while a sudden increase in the stress for a simply supported plate takes place. Thus, for plates partially resting on elastic foundation, it is recommended to choose plates with a clamped edge.
- The gradient of the material properties significantly influences the radial stress for both clamped and simply supported edges. In addition, the effect of the ratio a/b appears clearly in case of a simply supported edge. As a result, by proper choice of the gradient of the material, a uniform stress distribution may be achieved without the need to change the thickness profile.
- By increasing the values of foundation coefficients (k_w, k_p), the radial stress increases at the edge of the plate for clamped edge and decreases at the center of the plate for simply supported edge.

REFERENCES

[1] M. Yamanouchi, M. Koizumi, T. Hirai, and I. Shiota, "Proceedings of the first international symposium on functionally gradient materials," *Sendai, Japan,* 1990.

[2] J. Reddy, C. Wang, and S. Kitipornchai, "Axisymmetric bending of functionally graded circular and annular plates," *European Journal of Mechanics-A/Solids,* vol. 18, no. 2, pp. 185-199, 1999.

[3] L. Ma and T. Wang, "Relationships between axisymmetric bending and buckling solutions of FGM circular plates based on third-order plate theory and classical plate theory," *International Journal of Solids and Structures,* vol. 41, no. 1, pp. 85-101, 2004.

[4] A. Saidi, A. Rasouli, and S. Sahraee, "Axisymmetric bending and buckling analysis of thick functionally graded circular plates using unconstrained third-order shear deformation plate theory," *Composite Structures,* vol. 89, no. 1, pp. 110-119, 2009.

[5] W. Yun, X. Rongqiao, and D. Haojiang, "Three-dimensional solution of axisymmetric bending of functionally graded circular plates," *Composite Structures,* vol. 92, no. 7, pp. 1683-1693, 2010.

[6] U. Gupta, R. Lal, and S. Sharma, "Vibration analysis of non-homogeneous circular plate of nonlinear thickness variation by differential quadrature method," *Journal of sound and vibration,* vol. 298, no. 4-5, pp. 892-906, 2006.

[7] V. Vullo and F. Vivio, "Elastic stress analysis of non-linear variable thickness rotating disks subjected to thermal load and having variable density along the radius," *International Journal of Solids and Structures,* vol. 45, no. 20, pp. 5337-5355, 2008.

[8] M. Naei, A. Masoumi, and A. Shamekhi, "Buckling analysis of circular functionally graded material plate having variable thickness under uniform compression by finite-element method," *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science,* vol. 221, no. 11, pp. 1241-1247, 2007.

[9] E. Winkler, *Die Lehre von der Elasticitaet und Festigkeit: mit besonderer Rücksicht auf ihre Anwendung in der Technik, für polytechnische Schulen, Bauakademien, Ingenieure, Maschinenbauer, Architecten, etc*. H. Dominicus, 1867.

[10] S. Abbasi, F. Farhatnia, and S. Jazi, "A semi-analytical solution on static analysis of circular plate exposed to non-uniform axisymmetric transverse loading resting on Winkler elastic foundation," *Archives of Civil and Mechanical engineering,* vol. 14, pp. 476-488, 2014.

[11] P. Pasternak, "On a new method of an elastic foundation by means of two foundation constants," *Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstuve i Arkhitekture,* 1954.

[12] M. Arefi and M. Allam, "Nonlinear responses of an arbitrary FGP circular plate resting on the Winkler-Pasternak foundation," *Smart Struct. Syst,* vol. 16, no. 1, pp. 81-100, 2015.

[13] M. Shariyat and M. Alipour, "A differential transform approach for modal analysis of variable thickness two-directional FGM circular plates on elastic foundations," 2010.

[14] M. Foyouzat, M. Mofid, and J. Akin, "Free vibration of thin circular plates resting on an elastic foundation with a variable modulus," *Journal of Engineering Mechanics,* vol. 142, no. 4, p. 04016007, 2016.

[15] M. Foyouzat and M. Mofid, "An analytical solution for bending of axisymmetric circular/annular plates resting on a variable elastic foundation," *European Journal of Mechanics-A/Solids,* vol. 74, pp. 462-470, 2019.

[16] A. B. Rad and M. Shariyat, "A three-dimensional elasticity solution for two-directional FGM annular plates with non-uniform elastic foundations subjected to normal and shear tractions," *Acta Mechanica Solida Sinica,* vol. 26, no. 6, pp. 671-690, 2013.

[17] F. Alinaghizadeh and M. Shariati, "Static analysis of variable thickness two-directional functionally graded annular sector plates fully or partially resting on elastic foundations by the GDQ method," *Journal of the Brazilian Society of Mechanical Sciences and Engineering,* vol. 37, no. 6, pp. 1819-1838, 2015.

[18] K. Liew, J.-B. Han, Z. Xiao, and H. Du, "Differential quadrature method for Mindlin plates on Winkler foundations," *International Journal of Mechanical Sciences,* vol. 38, no. 4, pp. 405-421, 1996.

[19] S. Hosseini-Hashemi, H. Akhavan, H. R. D. Taher, N. Daemi, and A. Alibeigloo, "Differential quadrature analysis of functionally graded circular and annular sector plates on elastic foundation," *Materials & Design,* vol. 31, no. 4, pp. 1871-1880, 2010.

[20] E. Arshid and A. R. Khorshidvand, "Free vibration analysis of saturated porous FG circular plates integrated with piezoelectric actuators via differential quadrature method," *Thin-Walled Structures,* vol. 125, pp. 220-233, 2018.

[21] F. Farhatnia, R. Saadat, and S. Oveissi, "Functionally Graded Sandwich Circular Plate of Non-Uniform Varying Thickness with Homogenous Core Resting on Elastic Foundation: Investigation on Bending via Differential Quadrature Method," *American Journal of Mechanical Engineering,* vol. 7, no. 2, pp. 68-78, 2019.

[22] D.-G. Zhang and Y.-H. Zhou, "A theoretical analysis of FGM thin plates based on physical neutral surface," *Computational Materials Science,* vol. 44, no. 2, pp. 716-720, 2008.

[23] C. Shu, "Application of differential quadrature method to structural and vibration analysis," in *Differential Quadrature and Its Application in Engineering*: Springer, 2000, pp. 186-223.

[24] C. W. Bert and M. Malik, "Differential quadrature method in computational mechanics: a review," 1996.

[25] M. Shariyat and M. Alipour, "Differential transform vibration and modal stress analyses of circular plates made of two-directional functionally graded materials resting on elastic foundations," *Archive of Applied Mechanics,* vol. 81, no. 9, pp. 1289-1306, 2011.