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Proof of the Navier Stokes Equations

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Abstract

The Navier Stokes equation is used to describe the motion of viscous fluid. Here a proof shows the existence and smoothness of the Navier Stokes equations. Partial derivatives are used to show the existence for the momentum of the fluid in each coordinate x, y and z accounting for the viscosity and velocity of the fluid. Linear approximation is then used to show smoothness of the equations because of the forces acting on the fluid, such as gravity, pressure and friction.

Introduction

The Navier Stokes equations were developed by the French Physicist Claude-Louis Navier and George Gabriel Stokes in the 1800's. They are used to describe the motion of a viscous fluid and arise from Newtons second law of fluid motion. Mathematically they express the momentum where the conservation of mass remains constant over time (Wikipedia 2025). Described here is a proof of the Navier stokes equations showing the existence of the momentum for each coordinate x, y and z accounting for density, viscosity and velocity. Linear approximation is used for any given point (a) in the fluid where viscosity is set to zero allowing the determination of smoothness as result of the forces acting on the fluid. As a result, the velocity, u(x, t) does not grow large as x approaches infinity. The gradient of pressure is at any point (a) and at any point in time is due to forces applied to the fluid at that point (a) and at that point in time for fluid of any viscosity.

Method

From the equations to produce Reynolds number the following determines the turbulence of a fluid flowing through any given pipe.

$Re = Du\rho/v$

Where:

D = internal diameter of the pipe.

u = linear velocity of fluid.

ρ = density of the fluid.

v = viscosity of fluid.

When Reynolds number is less than 2100 laminar flow exists and when Reynolds number is between 2100 and not more than 4000 the fluid is in a transition from laminar flow to turbulent flow. When Reynolds number is greater than 4000 turbulent flow exists (CQU 2002, pp. 2-5).

Within a pipe the velocity of the fluid changes from zero, at the internal surface of the pipe, to whatever the bulk of the fluid velocity is beyond the boundary layer of the pipe. The velocity of the fluid at the internal surface of the pipe is zero and increases as measurements are taken towards the centre of the pipe. If measurements are taken for both laminar and turbulent flows the velocity of fluid in the centre of the pipe is moving faster than the velocity of the fluid at the internal surface of the wall of the pipe. For laminar flow the velocity of the fluid plotted against the distance from the wall of the pipe will produce a parabola, while turbulent flows will produce a flattened curve with smaller curves at the surface of the pipe which decrease as velocity increases (CQU 2002, pp. 2-6 - 2-7).

From the Navier Stokes equation adapted from (Fefferman n.d., p.1).

$u[\partial/\partial t + \sum_{j=1}^{n} \partial u/ \partial x_j] = u\Delta v - \partial p/\partial x_i + f_i(x, t) \quad (x \in \mathbb{R}^n, t \geq 0),$

Where

ρ = density of fluid.

t = time where $t \geq 0$.

u = velocity vector of a fluid $u(x, t) = u_i((x, t))_{1 \leq i \leq n} \in \mathbb{R}$

v = viscosity of fluid.

$f_i(x, t)$ = forces acting of the fluid including gravity.

$-\partial p/\partial x_i = -\nabla p$ = the pressure $p(x, t) \in \mathbb{R}$, gradient of the fluid defined for $x \in \mathbb{R}^n$.

Δ = the Laplacian in the space variables $\sum_{i=1}^{n} \partial^2/ \partial x_i^2$

For this proof the fluid is considered incompressible, and the following equation is given to show this.

div u = $\sum_{i=1}^{n} \partial u_i / \partial x_i = 0$

Initial conditions are as follows.

u(x, 0) = u°(x)

With v set equal to zero.

Ensuring that u(x, t) does not grow large as |x | → ∞.

Hence, we restrict our attention to forces acting on the fluid f_i (x, t) and initial conditions u°.

Showing the existence of the Navier Stokes equations.
As we are dealing with forces and initial conditions where viscosity is set to zero. Any given point in the fluid can be considered the same as any other given point in the fluid.
A continuous equation is defined for density and velocity at any given point (a) in the fluid (adapted from NASA 2024, p.1).

$\partial\rho/\partial t + \partial(\rho u)/\partial x + \partial(\rho v)/\partial y + \partial(\rho w)/\partial z = 0.$
For the x momentum of the fluid.

$x = \partial(\rho u)/\partial t + \partial(\rho u^2)/\partial x + \partial(\rho uv)/\partial y + \partial(\rho uw)/\partial z.$
For the y momentum of the fluid.

$y = \partial(\rho v)/\partial t + \partial(\rho uv)/\partial x + \partial(\rho v^2)/\partial y + \partial(\rho vw)/\partial z.$
For the z momentum of the fluid.

$z = \partial(\rho w)/\partial t + \partial(\rho uw)/\partial x + \partial(\rho vw)/\partial y + \partial(\rho w^2)/\partial z.$

When a force is increased or energy is applied to a fluid at any given point in the fluid an increase in pressure occurs until the applied force or energy reaches a steady state. During this process the volume of momentum, in the x, y and z coordinates from any given point in the fluid resembles that of a cone. The radius r, and height h, of the cone volume following the equation $1/3\pi r^2 h$, increases depending on the amount of force or energy applied. At the point of steady state, it is considered that this increase in radius and height has reached infinity and the change in pressure plateaus.

The volume of the momentum cone equals.

$1/3\pi r^2 h$

Let the radius equal x and the height of the momentum cone equal x – n. An increase in height of n results in the volume of the momentum cone equal to.

$1/3\pi(x^2)((x - n) + n)$
$= 1/3\pi x^3$

Once a steady state is reached the Navier Stokes equations become smooth as follows. Linearisation of the Navier Stokes equation to prove smoothness of these equations. Linear approximation of $f(x)$ near x = a (Washington 2005, p.727). When x approaches infinity at any given point (a) in the fluid and the initial condition of u° where v is set to zero we find that an increase in force smoothness still exists.

$L(x) = f(a) + f'(a)(x-a)$

The $f(a)$ at any given point (a) in a fluid.

$f(x) = - \nabla p$

Taking the derivative of $f(x)$ will equal the gradient.

$f'(x) = 0$

When x approaches infinity and proving that u(x, t) does not grow large.

$L(x) = f(a) + f'(a)(x-a)$
$f(\infty) = -\nabla p + (0)(\infty - \nabla p)$
$L(x) = -\nabla p$

Results

The existence of the Navier Stokes equations using partial differentiation shows that for all coordinates x, y and z the velocity, viscosity and density are accounted for in a flowing fluid by holding all other variables constant for any given point (a) in a flowing fluid within a pipe. The linear approximation shows that as x approaches infinity when viscosity is set to zero for any given point (a) in the fluid, the velocity does not grow large. The gradient is smooth because once the forces acting on the fluid reach a steady state at any given point in time for a fluid of any viscosity.

Conclusion

Proving the existence of the Navier Stokes equations using partial differentiation allows the variables, density, viscosity and velocity to be held constant and accounted for in each direction of the coordinates x, y and z of a flowing fluid. Using linear approximation for the function of x at any given point (a) where viscosity is zero, in a flowing fluid proves that when x approaches infinity the variables of velocity do not grow large, and the pressure gradient is smooth at any given point (a) at any given point in time for a fluid of any viscosity. When an increase in force or energy is applied to the fluid an increase in pressure occurs until the force or energy reaches a steady state. During this transition the x, y and z coordinates have a momentum volume resembling that of a cone from any given point in the fluid. Depending on the amount of force or energy applied to the fluid the radius and height of this momentum volume increases until a steady state is reached and is at infinity once again. Once the steady state is reached the equations smooth until an increase in force or energy is applied to the system. Therefore, the method herein proves the existence and smoothness of the Navier Stokes equations.

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Statements and Declarations

Data Availability
All data supporting the research is found within this research paper.

Conflicts of interest
The author declares no conflicts of interest.

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