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Unicity results on L-Functions with linear Differential Polynomials Sharing a Small Functions

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Abstract

In this research article we investigate the uniqueness of linear differential polynomials generated by an $\mathsf{L}(z)$-function and a meromorphic function with finitely many poles. By applying the concept of weighted sharing of small and rational functions, the study establishes new uniqueness results that extend, generalize, and improve upon earlier findings, including those of F. Liu, X.M. Li and H.X. Yi [[12]].

Keywords : L-function, linear differential polynomials, small function, rational function, finite weight

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1 INTRODUCTION AND PRELIMINARIES

Consider two non-constant meromorphic functions defined in $\mathbb{C}$: $\mathsf{f}(z)$ and $\mathsf{g}(z)$. Nevanlinna value distribution theory's fundamental findings and standard notations are assumed to be familiar to the reader (see [5],[7],[3],[26],[23]). If the zeros of $\mathsf{f} - a$ and $\mathsf{g} - a$ have the same multiplicities for some $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a CM (counting multiplicities); if not, we say that f and g share the value a IM (ignoring multiplicities). The Nevanlinna characteristic function of f is denoted by $T(r, \mathsf{f})$, and $S(r, \mathsf{f})$ is a small quantity defined by $S(r, \mathsf{f}) = o\{T(r, \mathsf{f})\}$ as $(r \to \infty, r \notin E)$, where E represents any set of positive real numbers with a finite linear measure.

Assume that $\mathcal{L}$ is a L-function, which is a meromorphic function on the complex plane $\mathbb{C}$ associated with one or more equations. " An $\mathcal{L}$-series is a Dirichlet series that may be analytically continued as a $\mathcal{L}$-function and usually converges on a half-plane. The $\mathcal{L}$-function is the function on the complex plane that is its analytic continuation. We first differentiate it from the $\mathcal{L}$-series, which is an infinite series representation (e.g., the Dirichlet series for the Riemann zeta function) (see.[19],[20]). Typically, the constructions start with a $\mathcal{L}$-series that is first explicitly defined as a Dirichlet series and then extended into an Euler product that is prime number indexed. There is at most one analytical definition of the $\mathcal{L}$-function to the remainder of the complex plane $\mathbb{C}$ atmost one pole. The Selberg class is an axiomatic definition of a class of $\mathcal{L}$-functions. $\mathcal{L}$, consists essentially of those Dirichlet series that satisfy the Riemann hypothesis as well as the Riemann zeta function, $\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

We state the following standard definitions of Nevanlinna theory and it is important to note that all the definitions discussed also applies to the $\mathcal{L}$-function.

In addition we need the following definitions.

Definition 1 ([19]). *The Selberg class S consists of the functions ξ satisfying the following axioms:*

1. *(Dirichlet series) $\mathcal{F}(s) = \sum_{n=1}^{\infty} a(n)/n^s$ where $s \in \mathbb{C}$, absolutely convergent for $\sigma > 1$.*
2. *(Analytic continuation) There exists a non-negative integer k such that $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order.*
3. *(Functional equation) There exist an integer $k \geq 0$, positive real numbers Q, λ_j, complex numbers v_j with $Re\, v_j \geq 0$ and ω with $|\omega| = 1$, such that $\wedge_{\mathcal{L}}(s)$ defined by*

$$\wedge_{\mathcal{L}}(s) = \mathcal{L}(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + v_j) = \gamma(s)\mathcal{F}(s),$$

   satisfies the functional equation $\wedge_{\mathcal{L}}(s) = \overline{\omega \wedge_{\mathcal{L}} (1-\overline{s})}$. We would call the function $\gamma(s)$ the γ factor.

4. *(Ramanujan conjecture) For every $\epsilon > 0, a(n) = O(n^\epsilon)$.*
5. *(Euler product hypothesis) L can be written over prime as*

$$\mathcal{L}(s) = \prod_p exp\left(\sum_{k=1}^{\infty} b(p^k)/p^{ks}\right)$$

with suitable coefficients $b(^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$ where the product is taken over all prime numbers p.
The degree $d_{\mathcal{L}}$ of an $\mathcal{L}$ is defined to be

$$d_{\mathcal{L}} = 2\sum_{j=1}^{K} \lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer as in axiom (iii) above.

Definition 2 ([9]). *Let $\mathsf{f}(z)$ be a meromorphic function defined in the complex plane. Let n be a positive integer and $\alpha \in \mathbb{C} \cup \{\infty\}$. By $N(r, \alpha; \mathsf{f}(z) \mid\leq n)$ we denote the counting function of the α - points of $\mathsf{f}(z)$ with multiplicity less than or equal to n and by $\bar{N}(r, \alpha; \mathsf{f}(z) \mid\leq n)$ the reduced counting function. Also by $N(r, \alpha; \mathsf{f}(z) \mid\geq n)$ we denote the counting function of the α - points of $\mathsf{f}(z)$ with multiplicity greater than or equal to n and by $\bar{N}(r, \alpha; \mathsf{f}(z) \mid\geq n)$ the reduced counting function. We define*

$$N_n(r, \alpha; \mathsf{f}(z)) = \bar{N}(r, \alpha; \mathsf{f}(z)) + \bar{N}(r, \alpha; \mathsf{f}(z) \mid\geq 2) + \ldots + \bar{N}(r, \alpha; \mathsf{f}(z) \mid\geq n)$$

Definition 3 ([9]). *Let $\mathsf{f}(z)$ be a meromorphic function defined in $\mathbb{C}$ and $p(z)$ be a small function of $\mathsf{f}(z)$ or a rational function. Then we denote the notations by $N(r, p; \mathsf{f}(z) \mid\leq m), \bar{N}(r, p; \mathsf{f}(z) \mid\leq m), N(r, p; \mathsf{f}(z) \mid\geq m), \bar{N}(r, p; \mathsf{f}(z) \mid\geq m), N_m(r, p; \mathsf{f}(z))$ etc, the counting functions $N(r, 0; \mathsf{f}(z) - p \mid\leq m), \bar{N}(r, 0; \mathsf{f}(z) - p \mid\leq m), N(r, 0; \mathsf{f}(z) - p \mid\geq m), \bar{N}(r, 0; \mathsf{f}(z) - p \mid\geq m), N_m(r, 0; \mathsf{f}(z) - p)$ respectively.*

Definition 4 ([8]). *Let $\mathsf{f}(z)$ and $\mathsf{g}(z)$ be two meromorphic functions defined in the complex plane and n be an integer (≥ 0) or infinity. We denote by $E_n(\alpha; \mathsf{f}(z))$ the set of all zeros of $\mathsf{f}(z) - \alpha$ and $\alpha \in \mathbb{C} \cup \{\infty\}$ and a zero of multiplicity k is counted k times if $k \leq n$ and $n + 1$ times if $k > n$, we say that $\mathsf{f}(z)$ and $\mathsf{g}(z)$ share α with weight n if $E_n(\alpha; f) = E_n(\alpha; \mathsf{g}(z))$. We say that $\mathsf{f}(z)$ and $\mathsf{g}(z)$ share (α, n) to mean that f, g share α with weight n. Clearly $\mathsf{f}(z), \mathsf{g}(z)$ share α IM or CM if and only if $\mathsf{f}(z)$ and $\mathsf{g}(z)$ share $(\alpha, 0)$ or (α, ∞) respectively.*

Definition 5 ([13]). *Let $\mathsf{f}(z)$ be a meromorphic function defined in the complex plane and $p(z)$ be a rational function or a small function of $\mathsf{f}(z)$. Then we denote by $E_{m)}(p; \mathsf{f}(z)), \overline{E}_{m)}(p; \mathsf{f}(z))$ and $E_m(p; \mathsf{f}(z))$ the sets $E_{m)}(r, 0; \mathsf{f}(z) - p), \bar{E}_{m)}(r, 0; \mathsf{f}(z) - p)$ and $E_m(r, 0; \mathsf{f}(z) - p)$ respectively. We write f, g share (p, n) to mean that $f - p$ and $g - p$ share the value 0 with weight n. Clearly, if $\mathsf{f}(z), \mathsf{g}(z)$ share (p, n) then f, g share (p, m) for all integers $m, 0 \leq m < n$. Also we note that f, g share p IM or CM if and only if $\mathsf{f}(z), \mathsf{g}(z)$ share $(p, 0)$ or (p, ∞) respectively.*

Definition 6 ([7]). *Let* $\mathsf{f}(z)$ *and* $\mathsf{g}(z)$ *be two non-constant meromorphic functions share a value* α *IM. Denote by* $\overline{N}_*(r,\alpha;\mathsf{f}(z),\mathsf{g}(z))$ *the counting function of the* α*-points of* $\mathsf{f}(z)$ *and* $\mathsf{g}(z)$ *with different multiplicities, where each* α *point is counted only once.*

Definition 7 ([14]). *Let* $\mathsf{f}(z)$ *and* $\mathsf{g}(z)$ *be two non-constant meromorphic functions share a value* α *IM. Denote by* $\overline{N}(r,\alpha;\mathsf{f}(z) \mid> \mathsf{g}(z))$ *the counting function of the* α *- points of* $\mathsf{f}(z)$ *and* $\mathsf{g}(z)$ *with multiplicities with respect to* $\mathsf{f}(z)$ *is greater than the multiplicities with respect to* $\mathsf{g}(z)$*, where each* α*-points is counted only once.*

Definition 8 ([14]). *Let* $\mathsf{f}(z)$ *and* $\mathsf{g}(z)$ *be two non-constant meromorphic functions share a value* α *IM. We denote by* $\overline{N}_E(r,\alpha;\mathsf{f}(z),\mathsf{g}(z) \mid> m)$ *the counting function of the* α*-points of* $\mathsf{f}(z)$ *and* $\mathsf{f}(z)$ *with multiplicities greater than* m *and the multiplicities with respect to* $\mathsf{f}(z)$ *is equal to the multiplicities with respect to* $\mathsf{f}(z)$*, where each* α*-points is counted only once.*

Definition 9 ([6]). *Let* $\mathsf{f}(z) \in \xi$*. Then we define its linear differential polynomial* $d_k[f]$ *as*

$$d_k[\mathsf{f}(z)] = a_0\mathsf{f}(z) + a_1\mathsf{f}'(z) + a_2\mathsf{f}''(z) + \ldots + a_k\mathsf{f}(z)^{(k)},$$

where $a_0, a_1, a_2, \ldots, a_{k-1}$ *and* $a_k \neq 0$ *are complex constants.*

Results due to Steuding [[20]], which actually holds without the Euler product hypothesis.

Theorem 1 ([20]). *If two* $L-$*functions* L_1 *and* L_2 *with* $a(1) = 1$ *share a complex value* $C \neq \infty$ *CM, the* $L_1 = L_2$.

Li[[11]] proved the following result to deal with q question posed by Chung Chun Yang[cf. [11]].

Theorem 2 ([**?**]). *Let* a *and* b *be two distinct finite values, and let* f *be a meromorphic function in the complex plane such the* f *has finitely many poles in the complex plane. If* f *and a nonconsatnt* $L-$ *function* L *share* a *CM and* b *IM, then* $L = f$.

In 1997, Lahiri[[10]] posed the following question, what can be said about the relationship between two meromorphic functions f and g, when two differential polynomials, generated by f and g respectively, share some nonzero finite value? IN this direction, Fang[[2]] and Yang Hua[[24]] respectively proved the following results.

Theorem 3 ([2]). *Let* f *and* g *be two nonconstnat entire functions, and let* n, k *be two positive integers such that* $n \geq 2k + 4$*. If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* 1 *CM, then neither* $f = c_1e^{cz}, g = c_2e^{-cz}$*, where* C_1, C_2 *and* c *are three constants satisfying* $(c_1c_2)^{(n)}(nc)^{2k} = 1$*, or* $f = tg$ *for a constant* t *such that* $t^n = 1$.

Theorem 4 ([24]). *Let f and g be two nonconstnat meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n f^{'}$ and $g^n g^{'}$ share 1 CM, then neither $f = c_1 e^{cz}, g = c_2 e^{-cz}$, where C_1, C_2 and c are three constants satisfying $(c_1 c_2)^{(n+1)} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In 2017, F. Liu, X.M. Li and H.X. Yi [[12]] proved the following uniqueness theorems.

Theorem 5 ([12]). *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and f be a non-constant meromorphic function. If $\{f^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share $(1, \infty)$, then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

Theorem 6 ([12]). *Let $j \geq 1$ and $k \geq 1$ be integers such that $j > 3k + 6$. Also let L be an L-function and f be a non-constant meromorphic function. If $\{f^j\}^{(k)}$ and $\{L^j\}^{(k)}$ share (z, ∞), then $f \equiv \alpha L$ for some non-constant α satisfying $\alpha^j = 1$.*

Now it will be interesting to study the above Theorems 5, 6 by considering more general form of difference-differential polynomial. The main motivation of this paper is the fact that the L - function where L - function has only one possible pole at $s = 1$ in $\mathbb{C}$.

Question 1. Can we take linear difference-differential polynomial of the form $[f^n(f-1)^m d_k[f]]^{(k)}$ and $[^n(L-1)^m d_k[L]]^{(k)}$ in Theorem 5 and Theorem 6?

2 MAIN RESULTS

Theorem 7. *Let f be a transcendental meromorphic function having finitely many zeros and L be an L - function, n, k, m be positive integers. If $[f^n(f-1)^m d_k[f]]^{(k)}$ and $[L^n(L-1)^m d_k[L]]^{(k)}$ share $(\alpha(z), l)$ and f, L share $(\infty, 0)$, where $\alpha(z)$ is a small function of f and L then*
(1) $l = 0$ and $n + m > 6k + 3m + 11$,
(2) $l = 1$ and $n > \frac{1}{2}(9k + 5m + 16)$,
(3) $l \geq 2$ and $n > 4k + 2m + 6$.

Then one of the following holds:
(i) $[f^n(f-1)^m d_k[f]]^{(k)} \equiv [L^n(L-1)^m d_k[L]]^{(k)}$,
(ii) $[f^n(f-1)^m d_k[f]]^{(k)} [L^n(L-1)^m d_k[L]]^{(k)} \equiv [\alpha(z)]^2$.

Theorem 8. *Let f be a transcendental meromorphic function having finitely many zeros and L be an L - function, n, k, m be positive integers. If $[f^n(f-1)^m d_k[f]]^{(k)}$ and $[L^n(L-1)^m d_k[L]]^{(k)}$ share $(R(z), l)$ and f, L share $(\infty, 0)$, where $R(z)$ is a rational function of f and L then*
(1) $l = 0$ and $n > 6k + 4m + 11$,
(2) $l = 1$ and $n > \frac{1}{2}(9k + 5m + 16)$,
(3) $l \geq 2$ and $n > 4k + 2m + 6$.

Then one of the following holds:
(i) $[f^n(f-1)^m d_k[f]]^{(k)} \equiv [L^n(L-1)^m d_k[L]]^{(k)}$,
(ii) $[f^n(f-1)^m d_k[f]]^{(k)} [L^n(L-1)^m d_k[L]]^{(k)} \equiv (R(z))^2$.

Example 1. *Let us consider* $L = \zeta$ *and* $f = -\zeta$, *where* ζ *is Riemann zeta function which has a simple pole. By hypothesis of the theorem* $F = [f^n(f-1)^m d_k[f]]^{(k)}$ *and* $L = [L^n(L-1)^m d_k[L]]^{(k)}$ *share* $(\alpha(z), l)$ *and the conditions are satisfied for different weights* $l = 0, l = 1$ *and* $l \geq 2$.
Remark 1. *Theorem 7 and Theorem 8 are the extension of Theorems* $5-6$ *respectively.*

3 AUXILIARY LEMMAS

In this section, we present some necessary Lemmas.
Denote H by the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$$

Lemma 1 ([20]). *Let* L *be an* L*-function with degree* q*. Then*

$$T(r, L) = \frac{q}{\pi} r \log r + O(1)$$

Lemma 2 ([13]). *Let* L *be an* L*-function,*

$$N(r, \infty \cdot L) = S(r, L) = O(\log r)$$

Lemma 3 ([14]). *Let* f *be a non-constant meromorphic function and* L *be an* L*-function. If* f *and* L *share* $(\infty, 0)$ *then*

$$\overline{N}(r, \infty; f) = \bar{N}(r, \infty; L) = S(r, L) = O(\log r)$$

Lemma 4 ([26]). *Let* $f(z) = \frac{\alpha_0+\alpha_1 z+....+\alpha_n z^n}{\beta_0+\beta_1 z+....+\beta_m z^m}$ *be a non-constant rational function defined in the complex plane* $\mathbb{C}$, *where* $\alpha_0, \alpha_1, \ldots, \alpha_n (\neq 0)$ *and* $\beta_0, \beta_1, \ldots., \beta_m (\neq 0)$ *are complex constants. Then,*

$$T(r, f) = \max(m, n) \log r + O(1)$$

Lemma 5 ([26]). *Let* f *be a transcendental meromorphic function of hyper order* $\rho_2(f) < 1$*. Then for any* $\alpha \in \mathbb{C} - \{0\}$.

$$T(r,f(z+\alpha))=T(r,f)+S(r,f)$$
$$N(r,\infty;f(z+\alpha))=N(r,\infty;f)+S(r,f)$$
$$N(r,0;f(z+\alpha))=N(r,0;f)+S(r,f)$$

Lemma 6 ([18]). *Let F and G be two non-constant meromorphic functions sharing $(1,1)$ and $(\infty,0)$. If $H\not\equiv 0$, then*

$$\begin{aligned}T(r,F)\leq &N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\bar{N}(r,F)+\bar{N}(r,G)\\&+\bar{N}_*(r,\infty;F,G)+\frac{1}{2}\bar{N}(r,0;F)+S(r,F)+S(r,G)\\T(r,G)\leq &N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\bar{N}(r,G)+\bar{N}(r,F)\\&+\bar{N}_*(r,\infty;F,G)+\frac{1}{2}\bar{N}(r,0;G)+S(r,F)+S(r,G)\end{aligned}$$

Lemma 7 ([18]). *Let F and G be two non-constant meromorphic functions sharing $(1,0)$ and $(\infty,0)$. If $H\not\equiv 0$, then*

$$\begin{aligned}T(r,F)\leq &N_2(r,0;F)+N_2(r,0;G)+3\bar{N}(r,F)+2\bar{N}(r,G)\\&+\bar{N}_*(r,\infty;F,G)+2\bar{N}(r,0;F)+\bar{N}(r,0;G)+S(r,F)+S(r,G)\\T(r,G)\leq &N_2(r,0;F)+N_2(r,0;G)+3\bar{N}(r,G)+2\bar{N}(r,F)\\&+\bar{N}_*(r,\infty;F,G)+2\bar{N}(r,0;G)+\bar{N}(r,0;F)+S(r,F)+S(r,G)\end{aligned}$$

Lemma 8 ([1]). *Let F and G be two non-constant meromorphic functions sharing $(1,l)$ and $(\infty,0)$ where $2\leq l<\infty$ and $H\not\equiv 0$ then*

$$\begin{aligned}T(r,F)\leq &N_2(r,0;F)+N_2(r,0;G)+\bar{N}(r,F)+\bar{N}(r,G)+\bar{N}_*(r,\infty;F,G)-m(r,1,G)\\&-N_E(r,1;F\mid>3)-\bar{N}(r,1;G>F)+S(r,F)+S(r,G)\\T(r,G)\leq &N_2(r,0;F)+N_2(r,0;G)+\bar{N}(r,F)+\bar{N}(r,G)+\bar{N}_*(r,\infty;F,G)-m(r,1,F)\\&-N_E(r,1;G\mid>3)-\bar{N}(r,1;F>G)+S(r,F)+S(r,G)\end{aligned}$$

Lemma 9 ([25]). *Let F be a non-constant meromorphic function and k,p be two positive integers, then*

$$T\left(r,F^{(k)}\right)\leq T(r,F)+k\bar{N}(r,\infty;F)+S(r,F),$$
$$N_p\left(r,0;F^{(k)}\right)\leq T\left(r,F^{(k)}\right)-T(r,F)+N_{p+k}(r,0;F)+S(r,F),$$
$$N_p\left(r,0;F^{(k)}\right)\leq N_{p+k}(r,0;F)+k\bar{N}(r,\infty;F)+S(r,F),$$
$$N\left(r,0;F^{(k)}\right)\leq N(r,0;F)+k\bar{N}(r,\infty;F)+S(r,F)$$

Lemma 10 ([25]). *Let f be a non-constant meromorphic function, define then polynomial $P(f) = a_0 + a_1 f + \ldots, + a_n f^n$, where $a_0, \ldots, a_n$ are complex constants and $a_n \neq 0$, then*

$$T(r, P(f)) = nT(r, f) + S(r, f)$$

Lemma 11 ([5]). *Let $f(z)$ be a meromorphic function and $a \in \mathbb{C}$. Then*

$$T\left(r, \frac{1}{f}\right) = T(r, f) + O(1)$$
$$T\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1)$$

Lemma 12 ([6]). *Let f be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and L be a $L-$ function with $\rho_2(L) < 1$. Let $F_1 = [f^n(f-1)^m d_k[f]]$. where n, m are positive integers and c is a complex constant. Then*

$$(n + m + 1 - k)T(r, f) \leq T\left(r, F_1\right) + S(r, f)$$

Proof. The proof of the Lemma follows from [Lemma 3.14, [6]].

4 PROOF OF THE MAIN RESULTS

Proof of Theorem 7. Let $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$ where $F_1 = f^n(f-1)^m d_k[f]$ and $L_1 = L^n(L-1)^m d_k[L]$ respectively. Then F and G share $(1, l)$ and share $(\infty, 0)$ except for zeros and poles of $\alpha(z)$. Clearly by Lemma 1, L is a transcendental meromorphic function. We have by Lemmas 9 and 12

$$\begin{aligned} N_2(r, 0; F) &\leq N_2\left(r, 0; F_1^{(k)}\right) + S(r, f) \\ &\leq T\left(r, F_1^{(k)}\right) - T\left(r, F_1\right) + N_{k+2}\left(r, 0; F_1\right) + S(r, f) \\ &\leq T\left(r, \frac{F_1^{(k)}}{\alpha(z)}\right) - (n + m + 1 - k)T(r, f) + N_{k+2}\left(r, 0; F_1\right) + S(r, f) \end{aligned} \tag{4.1}$$

Hence from inequality 4.1, we get

$$(n + m + 1 - k)T(r, f) \leq T(r, F) - N_2(r, 0; F) + N_{k+2}\left(r, 0; F_1\right) + S(r, f) \tag{4.2}$$

Similarly,

$$(n+m+1-k)T(r,L) \leq T(r,G) - N_2(r,0;G) + N_{k+2}(r,0;L_1) + S(r,f) \quad (4.3)$$

Now we have to consider the following two cases.

Case 1. *Let $H \not\equiv 0$. In this case we have to consider the following three subcases.*

Subcase 1.1 Let $l=0$. Hence by Lemmas 2, 3 and 7 and inequality 4.2 we have,

$$\begin{aligned}(n+m+1-k)T(r,f) \leq & N_2(r,0;F) + N_2(r,0;G) + 2\bar{N}(r,\infty;G) \\ & + 3\bar{N}(r,\infty;F) + \bar{N}_*(r,\infty;F,G) \\ & \pm 2\bar{N}(r,0;F) + \bar{N}(r,0;G) - N_2(r,0;F) \\ & + N_{k+2}(r,0;F_1) + S(r,f) + S(r,L) \\ \leq & N_2(r,0;F) + N_2(r,0;G) + 2\bar{N}(r,0;F) \\ & + \bar{N}(r,0;G) - N_2(r,0;F) \\ & + N_{k+2}(r,0;F_1) + S(r,f) + S(r,L) \\ \leq & N_2\left(r,0;F_1^{(k)}\right) + N_2\left(r,0;L_1^{(k)}\right) + 2\bar{N}\left(r,0;F_1^{(k)}\right) \\ & + \bar{N}\left(r,0;L_1^{(k)}\right) - N_2\left(r,0;F_1^{(k)}\right) \\ & + N_{k+2}(r,0;F_1) + S(r,f) + S(r,L) \\ \leq & N_2\left(r,0;L_1^{(k)}\right) + 2\bar{N}\left(r,0;F_1^{(k)}\right) \\ & + \bar{N}\left(r,0;L_1^{(k)}\right) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,L) \\ \leq & N_{k+2}(r,0;L_1) + 2N_{k+1}(r,0;F_1) \\ & + N_{k+1}(r,0;L_1) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,L) \\ & \leq (2k+2m+5)T(r,L) + (3k+3m+7)T(r,f) \\ & + S(r,f) + S(r,L)\end{aligned}$$

Hence

$$\begin{aligned}(n+m+1-k)T(r,f) \leq & (2k+2m+5)T(r,L) \\ & +(3k+3m+7)T(r,f) + S(r,f) + S(r,L) \quad (4.4)\end{aligned}$$

Similarly,

$$\begin{aligned}(n+m+1-k)T(r,L) \leq & (2k+2m+5)T(r,f) \\ & +(3k+3m+7)T(r,L) + S(r,f) + S(r,L) \quad (4.5)\end{aligned}$$

From inequalities 4.4 and 4.5 we get

$$\begin{aligned}(n+m+1-k)[T(r,f)+T(r,L)] \leq (5k+5m+12)[T(r,f)\\ +T(r,L)]+S(r,f)+S(r,L)\end{aligned} \tag{4.6}$$

which is a contradiction from 4.6 as $n > 6k+4m+11$.

Subcase 1.2 Let $l = 1$. Hence by Lemmas 2, 3 and 6 and inequality 4.6 we have,

$$\begin{aligned}(n+m+1-k)T(r,f) \leq & N_2(r,0;G)+\frac{3}{2}\bar{N}(r,\infty;F)+\bar{N}(r,\infty;G)\\ & +\bar{N}_*(r,\infty;F,G)+\frac{1}{2}\bar{N}(r,0;F)\\ & +N_{k+2}(r,0;F_1)+S(r,f)+S(r,L)\\ \leq & N_2\left(r,0;L_1^{(k)}\right)+\frac{1}{2}N_{k+1}(r,0;F_1)\\ & +\bar{N}\left(r,0;L_1^{(k)}\right)+N_{k+2}(r,0;F_1)\\ & +S(r,f)+S(r,L)\\ \leq & (2k+2m+5)T(r,L)+\frac{1}{2}(3k+3m+8)T(r,f)\\ & +S(r,f)+S(r,L)\end{aligned}$$

Hence

$$\begin{aligned}(n+m+1-k)T(r,f) \leq & (2k+2m+5)T(r,f)\\ & +\frac{1}{2}(3k+3m+8)T(r,L)+S(r,f)+S(r,L)\end{aligned} \tag{4.7}$$

Similarly

$$\begin{aligned}(n+m+1-k)T(r,L) \leq & (2k+2m+5)T(r,L)\\ & +\frac{1}{2}(3k+3m+8)T(r,f)+S(r,f)+S(r,L)\end{aligned} \tag{4.8}$$

From inequalities 4.7 and 4.8 we arrive at a contradiction as $n > \frac{1}{2}(9k+5m+16)$.

Subcase 1.3

Let $2 \leq l < 1$. Hence by Lemmas 2, 3 and 8 and inequality 4.2.

$$\begin{aligned}(n+m+1-k)T(r,L) &\leq N_2(r,0;F) + N_2(r,0;G) + \bar{N}(r,F) + \bar{N}(r,G)\\ &+ \bar{N}_*(r,\infty;F,G)\\ &- m(r,1,F) - N_E(r,1;G\,|>3) - \bar{N}(r,1;F>G)\\ &+ S(r,f) + S(r,L)\\ &\leq N_2\left(r,0;F_1^{(k)}\right) + N_2\left(r,0;L_1^{(k)}\right)\\ &+ S(r,f) + S(r,L)\\ &\leq (k+m+3)T(r,f) + (k+m+3)T(r,L) + S(r,f)\\ &+ S(r,L)(n+m+1-k)T(r,L)\\ &\leq (k+m+3)T(r,f) + (k+m+3)T(r,L)\\ &+ S(r,f) + S(r,L) \qquad (4.9)\end{aligned}$$

Similarly

$$(n+m+1-k)T(r,f) \leq (k+m+3)T(r,L) + (k+m+3)T(r,f) + S(r,f) + S(r,L) \qquad (4.10)$$

From inequalities 4.9 and 4.10 we arrive at a contradiction as $l \geq 2$ and $n > 3k+m+5$.
Case 2 Let $H \equiv 0$. Then

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \equiv 0$$

Integrating both sides we get

$$F - 1 = \frac{G-1}{b - c(G-1)}, \qquad (4.11)$$

where $b \neq 0$ and c are constants. Now we have to consider the following subcases.
Subcase 2.1
Let $c = 0$. Then from 4.11 we have,

$$F - 1 = \frac{G-1}{b}. \qquad (4.12)$$

If $b \neq 1$ then from 4.12

$$\bar{N}(r,0;F) = \bar{N}(r,1-b;G) \qquad (4.13)$$

By Lemmas 2 and 9, using Second Fundamental Theorem of Nevanlinna and from inequality 4.3 we have

$$\begin{aligned}(n+m+1-k)T(r,L) \leq & T(r,G) - N_2(r,0;G) + N_{k+2}\left(r,0;L_1\right) + S(r,L)\\ \leq & \bar{N}(r,0;G) + \bar{N}(r,1-b;G) + \bar{N}(r,\infty;G) - N_2(r,0;G)\\ & + N_{k+2}\left(r,0;L_1\right) + S(r,L)\\ \leq & \bar{N}(r,0;G) + \bar{N}(r,0;F) - N_2(r,0;G)\\ & + N_{k+2}\left(r,0;L_1\right) + S(r,L)\\ \leq\leq & \bar{N}\left(r,0;F_1^{(k)}\right) + \bar{N}\left(r,0;L_1^{(k)}\right) + N_{k+2}\left(r,0;L_1\right) + S(r,L)\\ \leq & N_{k+1}\left(r,0;F_1\right) + N_{k+1}\left(r,0;L_1\right) + N_{k+2}\left(r,0;L_1\right) + S(r,L)\\ \leq & (2k+2m+5)T(r,L) + (k+m+2)T(r,f)\\ & + S(r,f) + S(r,L)\end{aligned}$$

Hence

$$\begin{aligned}(n+m+1-k)T(r,L) \leq & \ (2k+2m+5T(r,L)\\ & + (k+m+2)T(r,f) + S(r,f) + S(r,L) \qquad (4.14)\end{aligned}$$

Similarly

$$\begin{aligned}(n+m+1-k)T(r,f) \leq & \ (2k+2m+5)T(r,f)\\ & + (k+m+2)T(r,L) + S(r,f) + S(r,L) \qquad (4.15)\end{aligned}$$

From the inequalities 4.14 and 4.15 we arrive at a contradiction as $n > 4k+2m+6$. Hence $b = 1$ and therefore we get from 4.12

$$[f^n(f-1)^m d_k[f]]^{(k)} \equiv [L^n(L-1)^m d_k[L]]^{(k)}$$

Subcase 2.2 Let $c \neq 0$ and $b = -c$.
If $c = 1$, then from 4.11 we have $FG \equiv 1$. Hence

$$[f^n(f-1)^m d_k[f]]^{(k)} \left[L^n(L-1)^m d_k[L]\right]^{(k)} = [\alpha(z)]^2$$

If $c \neq 1$, then from 4.11 we have,

$$\frac{1}{F} = \frac{-cG}{(1-c)G-1}$$

Hence $\overline{N}(r,0;F) = N\left(r, \frac{1}{1-c}; G\right)$.

Now proceeding as in subcase 2.1, we arrive at a contradiction. If $c = 1$, then from 4.11 we have

$$F \equiv \frac{-b}{G - b - 1} \tag{4.16}$$

Hence by Lemma 3 we have from 4.16,

$$\overline{N}(r, b+1; G) = \overline{N}(r, F) = \overline{N}(r, f) + S(r, L) = S(r, L)$$

Now proceeding as in subcase 2.1, we arrive at a contradiction. If $c \neq 1$, then from 4.11, we have

$$F - \left(1 - \frac{1}{c}\right) \equiv \frac{-b}{c^2 \left(G - \frac{b+c}{c}\right)}$$

Therefore by Lemma 3 we have

$$\overline{N}\left(r, \frac{b+c}{c}; G\right) = \overline{N}(r, F) = \bar{N}(r, f) + S(r, L) = S(r, L)$$

Hence proceeding as in subcase we arrive at a contradiction.
This completes the proof of the Theorem 7

Proof of Theorem 8

Since f and L are transcendental meromorphic function and $R(z)$ is a rational function therefore $R(z)$ is a small function of f and L. Thus, Theorem 8 can be proved in a similar way as Theorem 7.

5 CONCLUSION

We have investigate the value distribution of a L - function and an arbitrary meromorphic function using the concept of weighted sharing when certain type of linear difference-differential polynomials $f^n(f-1)^m d_k[f]$ and $L^n(L-1)^m d_k(L)$ share a small and rational function. L - functions can be analytically continued as meromorphic functions in $\mathbb{C}$ and it has only one possible pole at $s = 1$ in $\mathbb{C}$ is the main concept of this paper. Our results extends earlier results due to F. Liu, X.M. Li and H.X. Yi [[12]] .

6 OPEN QUESTIONS

1. Can the condition for n in Theorem 2.1 and Theorem 2.2 be still reduced?
2. Can the difference polynomials in Theorems 2.1-2.2 be replaced by linear differential polynomials of the form $f^n P(f) H[f]$ by using weakly weighted sharing and truncated weighted sharing?

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