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Lorentz Group by *SO*(3, *R*) in non-Minkowski Space

Euich Miztani
JEIN Institute for Fundamental Science, 5-14, Yoshida-Honmachi,
Sakyo-ku, Kyoto, 606-8317, Japan.
Nagoya International High School (NIHS), 1-16, Koji-honmachi,
Showa-ku, Nagoya, 466-0841, Japan.

Abstract

In the former paper [1], we verified that relativistic electro-magnetic field is phenomenally expressed by a simple rotation in real space or non-Minkowski space. Then, it'll be natural to get interested in the technical denotation by rotation matrix. In this paper, we discuss so to speak the Lorentz group which is denoted by $SO(2, R)$ or $SO(3, R)$.

LORENTZ TRANSFORMATION BY RELATIVISTIC ABERRATION

First of all, let us think of an observer in the static system κ and inertia one κ' as shown in Figure1 and 2. A ray of light to the observer declines to the x' axis by aberration and *principle of the constancy of the light speed* as velocity of the observer v approaches to light velocity.

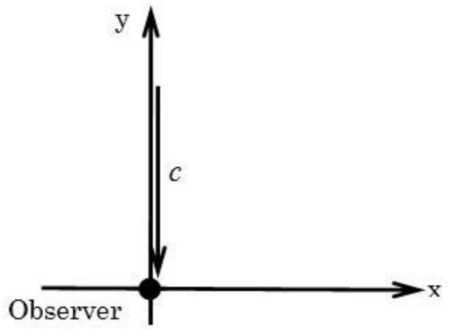

Figure 1

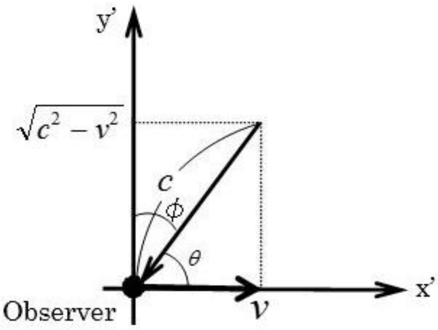

Figure 2

Based on Figure 2, we can denote as follows.

$$\sin\phi = v/c \tag{1}$$

$$\cos\phi = \sqrt{c^2 - v^2}/c = \sqrt{1-(v^2/c^2)} = 1/\beta \tag{2}$$

where ϕ is angle between the ray of light and y axis in κ' as shown in Figure 2. Since ϕ is clockwise, it is inverse rotation for θ. Now, let us think of relativistic magnetic flux density B'_x, B'_y, B'_z and electric field E'_x, E'_y, E'_z. They are as follows.

$$B'_x = B_x \tag{3}$$

$$B'_y = \beta\left(B_y + \frac{v}{c^2}E_z\right) \tag{4}$$

$$B'_z = \beta\left(B_z - \frac{v}{c^2}E_y\right) \tag{5}$$

$$E'_x = E_x \tag{6}$$

$$E'_y = \beta\left(E_y - vB_z\right) \tag{7}$$

$$E'_z = \beta\left(E_z + vB_y\right) \tag{8}$$

First of all, let us think of a specific case. Multiplying both sides of each equation (5) by $\sqrt{1-(c/v)^2}$, then $\sqrt{1-(v/c)^2}B'_z = \left(B_z - \frac{v}{c^2}E_y\right)$.

When $v = c$, LHS $= \sqrt{1-(c/c)^2}B'_z = B'_z\cos\frac{\pi}{2} = 0$ and

$$\text{RHS} = B_z - \frac{c}{c}\left(\frac{1}{c}E_y\right) = B_z - \frac{1}{c}E_y\sin\frac{\pi}{2} = B_z - \frac{1}{c}E_y. \quad \therefore B_z = \frac{1}{c}E_y.$$

It phenomenally suggests that the magnetic flux totally decline by relativistic aberration as shown in Figure 3 (: see also [1]).

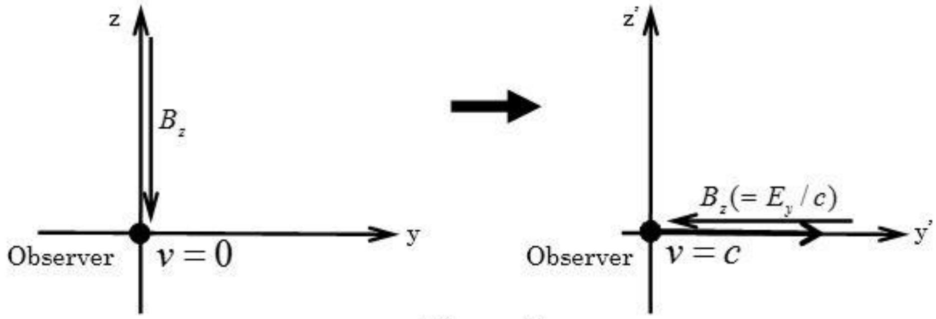

Figure 3

Identically, multiplying both sides of each equation (4) by $\sqrt{1-(c/v)^2}$, then

$\sqrt{1-(v/c)^2}\,B'_y=\left(B_y+\frac{v}{c^2}E_z\right)$.

When $v=c$, LHS $=\sqrt{1-(c/c)^2}\,B'_y=B'_y\cos\frac{\pi}{2}=0$ and

$$\text{RHS}=B_y+\frac{c}{c}\left(\frac{1}{c}E_z\right)=B_y+\frac{1}{c}E_z\sin\frac{\pi}{2}=B_y+\frac{1}{c}E_z.\ \ \therefore B_y=-\frac{1}{c}E_z.$$

Eq. (7) is $E_y=vB_z$ and Eq. (8) is $E_z=-vB_y$ when $v=c$ in the same manner. From the results,

$$\begin{pmatrix}B_y\\B_z\end{pmatrix}=\frac{1}{c}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}E_y\\-E_z\end{pmatrix}\ \text{and}\ \begin{pmatrix}E_y\\E_z\end{pmatrix}=v\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}-B_y\\B_z\end{pmatrix}.$$

They are equivalent to

$$\begin{pmatrix}B_y\\B_z\end{pmatrix}=\frac{1}{c}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}E_y\\E_z\end{pmatrix}\ \text{and}\ \begin{pmatrix}E_y\\E_z\end{pmatrix}=v\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}B_y\\B_z\end{pmatrix}.$$

Then, they are naturally related to $SO(2, R)$:

$$\begin{pmatrix}B_y\\B_z\end{pmatrix}=\frac{1}{c}\begin{pmatrix}\cos(\pi/2)&-\sin(\pi/2)\\\sin(\pi/2)&\cos(\pi/2)\end{pmatrix}\begin{pmatrix}E_y\\E_z\end{pmatrix}\ \text{and}\ \begin{pmatrix}E_y\\E_z\end{pmatrix}=v\begin{pmatrix}\cos(\pi/2)&\sin(\pi/2)\\-\sin(\pi/2)&\cos(\pi/2)\end{pmatrix}\begin{pmatrix}B_y\\B_z\end{pmatrix}.$$

Based on the fact above, let us discuss equations (3) to (8) in the same way. Firstly, let us think of the magnetic flux density. Solving the equation (5) for B_z, the expansion is denoted by equations (1) and (2) as follows.

$$B_z=\frac{1}{\beta}B'_z+\frac{v}{c^2}E_y=B'_z\cos\phi+\frac{1}{c}E_y\sin\phi.$$

Since $E/c=B'$ (: see also [1]),

$$B_z = B'_z \cos\phi + B'_y \sin\phi . \tag{9}$$

Likewise, solving the equation (5) for B_z, the expansion is

$$B_y = \frac{1}{\beta} B'_y - \frac{v}{c^2} E_z = B'_y \cos\phi - (E_z / c) \sin\phi .$$

Since $E / c = B'$,

$$B_y = B'_y \cos\phi - B'_z \sin\phi . \tag{10}$$

Denoting equations (9) and (10) by 2-by-2 matrix,

$$\begin{pmatrix} B_z \\ B_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B'_z \\ B'_y \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \begin{pmatrix} B'_y \\ B'_z \end{pmatrix} . \tag{11}$$

Multiplying both sides of each equation (11) by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\text{LHS} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_z \\ B_y \end{pmatrix} = \begin{pmatrix} B_y \\ B_z \end{pmatrix} \quad \text{and} \quad \text{RHS} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin\phi & \cos\phi \\ \cos\phi & -\sin\phi \end{pmatrix} \begin{pmatrix} B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B'_y \\ B'_z \end{pmatrix} .$$

$$\therefore \begin{pmatrix} B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B'_y \\ B'_z \end{pmatrix} . \tag{12}$$

Then, multiplying both sides of each equation (12) by $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^{-1}$,

$$\begin{pmatrix} B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B_y \\ B_z \end{pmatrix} . \tag{13}$$

Since $\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1}$ by our definition between ϕ (: clockwise) and θ (: counterclockwise) as shown in Fig. 2, Eq. (13) is

$$\begin{pmatrix} B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} B_y \\ B_z \end{pmatrix} . \tag{14}$$

Secondarily, let us think of the electric field E'_z. Solving Eq. (7) for E_y, the expansion is

$$E_y = \frac{1}{\beta} E'_y + vB_z = E'_y \cos\phi + vB_z .$$

Since $B = E' / c$,

$$E_y = E'_y \cos\phi + vB_z = E'_y \cos\phi + \frac{v}{c} E'_z = E' \cos\phi + E'_z \sin\phi \,. \tag{15}$$

Likewise, solving Eq. (8) for E_z, the expansion is

$$E_z = \frac{1}{\beta} E'_z - vB_y = E'_z \cos\phi - vB_y \,.$$

Since $B = E'/c$,

$$E_z = E'_z \cos\phi - \frac{v}{c} E'_y = E'_z \cos\phi - E'_y \sin\phi \tag{16}$$

Denoting equations (15) and (16) by 2-by-2 matrix,

$$\begin{pmatrix} E_y \\ E_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} E'_y \\ E'_z \end{pmatrix}. \tag{17}$$

Multiplying both sides of each equation (17) by $\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}^{-1}$,

$$\begin{pmatrix} E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}^{-1} \begin{pmatrix} E_y \\ E_z \end{pmatrix}. \tag{18}$$

Since $\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ by the definition between ϕ and θ, Eq. (18) is

$$\begin{pmatrix} E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_y \\ E_z \end{pmatrix}. \tag{19}$$

The rotation matrices of Eq. (14) and (19) can naturally make a special orthogonal group *SO*(2, *R*) in a series of Lorentz transformations.

Furthermore, including equations (3) and (6), the matrices of (14) and (19) are also denoted as

SO(3, *R*) as follows:

$$\begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{20}$$

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REFERENCES

[1] E. Miztani, Special Relativity from Geometric Viewpoint, Communications in Applied Geometry, Vol.1, No.1, 17-25, (2011)