Communications in Applied Geometry.
ISSN 2249-4286 Volume 3, Number 1 (2016) pp. 21-27

https://dx.doi.org/10.37622/CAG/3.1.2016.21-27

A Group Representation

Notations for "Projections and Dimensions"

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Abstract

In this paper, we verify a group representation we talk in [1].

Claim 1. *Equation of a group* $E_{lm}E_{mn} = E_{ln}$ *in [1] is automorphism.*

Proof **.** First of all, the automorphism (i) is proved as follows. Let A be E_{lm}, B be E_{mn}.

a1). If $0 \prec l \prec m \prec n$ (projecting into higher dimensions,)

For $f(A)f(B)$:

$$(f(A))(x) = x' = E_{lm}x = diag(\overbrace{1,1,\ldots,1}^{l}, \overbrace{D, D,\ldots,D}^{m-l})(x_1, x_2, \ldots, x_l, \overbrace{T, T, \ldots, T}^{m-l})^T$$

$$= (x_1, x_2, \ldots, x_l, \overbrace{DT, DT, \ldots, DT}^{m-l})^T .$$

Then, $$(f(B))(x') = E_{mn}x' = diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D, D,\ldots,D}^{n-m})(x_1, x_2, \ldots, x_l, \overbrace{DT, DT, \ldots, DT}^{m-l}, \overbrace{T, T, \ldots, T}^{n-m})^T$$

$$= (x_1, x_2, \ldots, x_l, \overbrace{DT, DT, \ldots, DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l}, \overbrace{D, D,\ldots,D}^{n-l})(x_1, x_2, \ldots, x_l, \overbrace{T, T, \ldots, T}^{n-l})^T$$

$$= E_{ln}x .$$

For $f(AB)$:

$$(f(AB))(x) = (E_{lm}E_{mn})x = (diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l},\overbrace{1,1,\ldots,1}^{n-m})diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D}^{n-m}))(x_1,x_{2,}\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= E_{\ln}x\,.$$

$$\therefore f(A)f(B) = f(AB)\,.$$

a2). If $0 \prec l \prec n \prec m$ (projecting into higher dimensions)

For $f(A)f(B)$:

$$(f(A))(x) = x' = E_{lm}x = diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l})(x_1,x_{2,}\ldots,x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l})^T\,.$$

Then,

$$(f(B))(x') = E_{mn}x' = E_{nm}^{-1}x' = diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n})(x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l},\overbrace{DT,DT,\ldots,DT}^{m-n})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l},\overbrace{T,T,\ldots,T}^{m-n})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T \quad (\because \text{ see also [1],})$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= E_{\ln}x\,.$$

For $f(AB)$:

$$(f(AB))(x) = (E_{lm}E_{mn})x = (diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l})diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n}))(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{m-l},\overbrace{T,T,\ldots,T}^{m-n})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l},\overbrace{T,T,\ldots,T}^{m-n})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= E_{\ln}x \, .$$

$$\therefore f(A)f(B) = f(AB) \, .$$

a3). If $0 \prec m \prec l \prec n$ (projecting into higher dimensions)

For $f(A)f(B)$:

$$(f(A))(x) = x' = E_{lm}x = E_{ml}^{-1}x = diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m})(x_1,x_2,\ldots,x_l)^T$$

$$= (x_1,x_2,\ldots,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T \, .$$

Then,

$$(f(B))(x')) = E_{mn}x' = diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D,}^{n-m})(x_1,x_2,\ldots,x_m,\overbrace{D^{-1}x_{m+1},D^{-1}x_{m+2}\ldots,D^{-1}x_l}^{l-m},\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= E_{\ln}x \, .$$

For $f(AB)$:

$$(f(AB))(x)) = (E_{lm}E_{mn})x$$

$$= (diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m},\overbrace{1,1,\ldots,1}^{n-l})diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D,D,\ldots,D,}^{n-m}))(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{n-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{n-l})^T$$

$$= E_{\ln}x \, .$$

$$\therefore f(A)f(B) = f(AB) \, .$$

b1). If $0 \prec n \prec m \prec l$ (projecting into lower dimensions)

For $f(A)f(B)$:

$$(f(A))(x) = x' = E_{lm}x = E_{ml}^{-1}x = diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m})(x_1,x_2,\ldots,x_l)^T$$

$$= (x_1,x_2,\ldots,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2},\ldots,D^{-1}x_l)^T .$$

Then,

$$(f(B))(x') = E_{mn}x' = E_{nm}^{-1}x' = diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n},\overbrace{1,1,\ldots,1}^{l-m})(x_1,x_2,\ldots,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2}\ldots,D^{-1}x_l)^T$$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T$$

$$= E_{nl}^{-1}x = E_{\ln}x .$$

For $f(AB)$:

$$(f(AB))(x)) = (E_{lm}E_{mn})x$$

$$= (diag(\overbrace{1,1,\ldots,1}^{m},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-m})diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n},\overbrace{1,1,\ldots,1}^{l-m}))(x_1,x_2,\ldots,x_l)^T$$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T$$

$$= E_{nl}^{-1}x = E_{\ln}x .$$

$$\therefore f(A)f(B) = f(AB) .$$

b2). If $0 \prec n \prec l \prec m$ (projecting into lower dimensions)

For $f(A)f(B)$:

$$(f(A))(x) = x' = E_{lm}x = diag(\overbrace{1,1,\ldots,1}^{l},\overbrace{D,D,\ldots,D}^{m-l})(x_1,x_2,\ldots,x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l})^T .$$

Then, $(f(B))(x') = E_{mn}x' = E_{nm}^{-1}x' = diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{m-n})(x_1,x_2,\ldots,x_l,\overbrace{DT,DT,\ldots,DT}^{m-l})^T$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l,\overbrace{T,T,\ldots,T}^{m-l})^T$$

$$= (x_1,x_2,\ldots,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},\ldots,D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n},\overbrace{D^{-1},D^{-1},\ldots,D^{-1}}^{l-n})(x_1,x_2,\ldots,x_l)^T$$

$$= E_{nl}^{-1} x = E_{\ln} x \ .$$

For $f(AB)$:

$$(f(AB))(x)) = (E_{lm} E_{mn}) x$$

$$= E_{mn} x' = E_{nm}^{-1} x' = (diag(\overbrace{1,1,\ldots,1}^{l}, \overbrace{D, D, \ldots, D}^{m-l}) diag(\overbrace{1,1,\ldots,1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{m-n}))(x_1, x_2, \ldots, x_l, \overbrace{T, \ldots, T}^{m-l})^T$$

$$= (x_1, x_2, \ldots, x_n, D^{-1} x_{n+1}, D^{-1} x_{n+2}, \ldots, D^{-1} x_l, \overbrace{T, T, \ldots, T}^{m-l})^T$$

$$= (x_1, x_2, \ldots, x_n, D^{-1} x_{n+1}, D^{-1} x_{n+2}, \ldots, D^{-1} x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-n})(x_1, x_2, \ldots, x_l)^T$$

$$= E_{nl}^{-1} x = E_{\ln} x \ .$$

$$\therefore f(A) f(B) = f(AB) \ .$$

b3). If $0 \prec m \prec n \prec l$ (projecting into lower dimensions)

For $f(A)f(B)$:

$$(f(A))(x) = x' = E_{lm} x = E_{ml}^{-1} x = diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-m})(x_1, x_2, \ldots, x_l)^T$$

$$= (x_1, x_2, \ldots, x_m, D^{-1} x_{m+1}, D^{-1} x_{m+2}, \ldots, D^{-1} x_l)^T \ .$$

Then, $(f(B))(x') = E_{mn} x' = diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D, D, \ldots, D}^{n-m}, \overbrace{1,1,\ldots,1}^{l-n})(x_1, x_2, \ldots, x_m, D^{-1} x_{m+1}, D^{-1} x_{m+2}, \ldots, D^{-1} x_l)^T$

$$= (x_1, x_2, \ldots, x_n, D^{-1} x_{n+1}, D^{-1} x_{n+2}, \ldots, D^{-1} x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-n})(x_1, x_2, \ldots, x_l)^T$$

$$= E_{nl}^{-1} x = E_{\ln} x \ .$$

For $f(AB)$:

$$(f(AB))(x)) = (E_{lm} E_{mn}) x$$

$$= (diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-m}) diag(\overbrace{1,1,\ldots,1}^{m}, \overbrace{D, D, \ldots, D}^{n-m}, \overbrace{1,1,\ldots,1}^{l-n}))(x_1, x_2, \ldots, x_l)^T$$

$$= (x_1, x_2, \ldots, x_n, D^{-1} x_{n+1}, D^{-1} x_{n+2}, \ldots, D^{-1} x_l)^T$$

$$= diag(\overbrace{1,1,\ldots,1}^{n}, \overbrace{D^{-1}, D^{-1}, \ldots, D^{-1}}^{l-n})(x_1, x_2, \ldots, x_l)^T$$

$$= E_{nl}^{-1} x = E_{\ln} x \ .$$

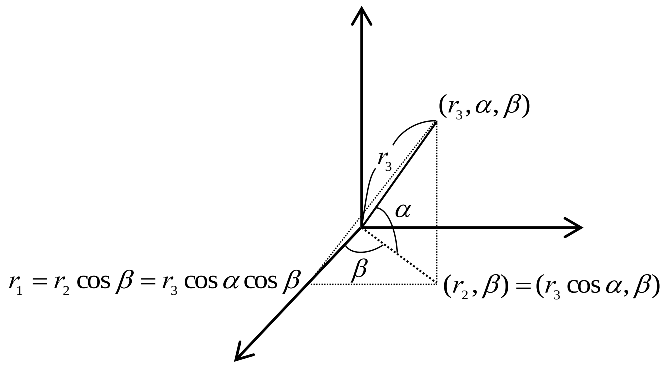
$\therefore f(A)f(B) = f(AB)$. □

Claim 2. $\{I_n\}$ *in the group is unique.*

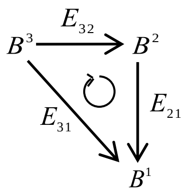
Proof. For $G := \{E_{mn}\}$, the scalar multiplication by 1 in field k holds as $s: 1\times G(= G\times 1) \to G$. It is compatible with the matrix multiplications in G. Then, $I_m E_{mn} = 1E_{mn} = E_{mn}1 = E_{mn} I_n = E_{mn}$. □

3. As shown in the graph below, a coordinate (r_3, α, β) in B^3 is projected to another one $(r_2, \beta) = (r_3 \cos\alpha, \beta)$ in B^2, $(r_2, \beta) = (r_3 \cos\alpha, \beta)$ in B^2 to $r_1 = r_2 \cos\beta = r_3 \cos\alpha \cos\beta$ in B^1, and (r_3, α, β) in B^3 to $r_1 = r_2 \cos\beta = r_3 \cos\alpha\cos\beta$ in B^1:

$$\begin{array}{ccc} B^3 \to B^2 & B^2 \to B^1 & B^3 \to B^1 \\ (r_3,\alpha,\beta) \mapsto (r_2,\beta) & (r_2,\beta) \mapsto r_1 & (r_3,\alpha,\beta) \mapsto r_1 \end{array}$$



The commutative diagram with morphisms(: $f \circ g = E_{32}E_{21} = E_{31}$) is



Likewise, projecting into lower dimensional space(, although we will further need to discuss it in the reverse case(: into higher dimensional one),) the commutative diagram with morphisms $\{E_{jk}\}$ above is possible to form it with any other combinations by:

$(r_n, \theta_1, \ldots, \theta_{n-1}) \in B^n$,
$(r_{n-1}(= r_n \cos\theta_1), \theta_2, \ldots, \theta_{n-1}) \in B^{n-1}$,
$(r_{n-2}(= r_n \cos\theta_1 \cos\theta_2), \theta_3, \ldots, \theta_{n-1}) \in B^{n-2}$,
…………………………………………,
$(r_2(= r_n \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n-2}), \theta_{n-1}) \in B^n$,
$r_1(= r_n \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n-1}) \in B^1$.

REFERENCES

[1] E. Miztani, Projections and Dimension, Communications in Applied Geometry, vol.1, number 1(2011), pp. 7-18