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On Classes of Generalized Ruscheweyh Type Hormonic Functions

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Abstract

In this paper, we define and investigate the classes of complex valued harmonic functions using the generalized Rucsheweyh derivative D_{λ}^{n}. For functions belonging to these classes, we determine certain necessary and sufficient coefficient conditions. Furthermore, distributions bounds, extreme points, inclusion relations, convolution conditions and convex combinations are also obtained.

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INTRODUCTION

A complex-valued continuous function $\omega = f(z) = u(z) + iv(z)$ *defined on a domain* $\mathcal{G}$ *is harmonic if both* u and v *are real-valued harmonic functions on* $\mathcal{G}$, *that is* u and v *satisfy,respectively the Laplace equations*

$$\Delta u = u_{xx} + u_{yy} = 0 \text{ and } \Delta v = v_{xx} + v_{yy} = 0.$$

A one to one mapping $u = u(z), v = v(z)$ *from a region* $\mathcal{G}_1$ *in the* xy*-plane to a region* $\mathcal{G}_2$ *in the* uv*-plane is a harmonic mapping if* u and v *are harmonic*. It is well known that if $f = u + iv$ has continuous partial derivatives, then f is analytic if and only if the Cauchy-Riemann equations $u_x = v_y$ *and* $u_y = -v_x$ are satisfied. It follows that every analytic function is a complex-valued harmonic function. However , not every complex-valued harmonic functions is analytic, since no two solutions of the Laplace equation can be taken as the components u *and* v of an analytic function in $\mathcal{G}$, *they must be related by the Cauchy-Riemann equations*$u_x = v_y$ *and* $u_y = -v_x$.

A subject of considerable importance in harmonic mappings is the Jacobian.
If

$$J_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2$$

where$f = h + \bar{g}$ *is harmonic function in* $\mathcal{U} = \{z : |z| < 1\}$. *When* J_f *is positive in* $\mathcal{U}$, *the harmonic function* f *is called orientation –preserving or sense-preserving in* $\mathcal{U}$ *. A harmonic mapping is locally univalent in a neighborhood of a point* z_0 *if and only if* $J_f(z) \neq 0$ *at* z_0.

The first key insight into harmonic univalent mappings came from Clunie and Small [2] who observed that $f = h + \bar{g}$ *is locally univalent and orientation pre-serving if and only if*

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0.$$

This is equivalent to

$$|g'(z)| < |h'(z)|.$$

Let $\mathcal{H}$ denote the family of all harmonic, complex-valued orientation preserving, normalized and univalent mapping defined on $\mathcal{U}$. Thus, a function f in $\mathcal{H}$ admits the representation $f = h + \bar{g}$, where,

(1.1) $\quad h(z) = z + \sum_{j=2}^{\infty} a_j z^j$ *and* $g(z) = \sum_{j=1}^{\infty} b_j z^j$, $|b_1| < 1$.

Note that the family $\mathcal{H}$ of orientation preserving normalized, harmonic univalent functions reduces to the class S of normalized analytic univalent functions if the co-analytic part of $f = h + \bar{g}$ is analytically zero. Further, let $\overline{\mathcal{H}}$ denote the subfamily of $\mathcal{H}$ consists of harmonic functions $f = h + \bar{g}$ so that h *and* g are of the form

(1.2) $\quad h(z) = z + \sum_{j=2}^{\infty} |a_j| z^j$ *and* $g(z) = \sum_{j=1}^{\infty} |b_j| z^j$, $|b_1| < 1$.

The generalized Ruschweyh derivative $D_\lambda^n f$ [1] is defined as follows:

For $f \in \mathcal{A}(n), \lambda \geq 0$ *and* $n \in \mathbb{R}, n > -1$, we have

$$D_\lambda^n f(z) = \frac{z}{(1-z)^{n+1}} * D_\lambda f(z), \qquad z \in \mathcal{U},$$

where, $D_\lambda f(z) = (1-\lambda)f(z) + \lambda z f'(z)$.

For $= h + \bar{g}$, given by (1.1), we define the generalized Ruscheweyh derivative of the harmonic function by,

$$D_\lambda^n f\ (z) = D_\lambda^n h(z) + \overline{D_\lambda^n g(z)}.$$

The power series expansion of $D_\lambda^n f\ (z)$ is of the form

(1.3) $\quad D_\lambda^n f\ (z) = z + \sum_{j=2}^{\infty} C_\lambda(n,j) a_j\, z^j$,

where ,$C_\lambda(n,j) = [1 + (j-1)\lambda]\,\frac{(n+1)_{j-1}}{(1)_{j-1}}$.

Using the generalized Ruscheweyh derivative, we define the subclass $\mathcal{F}_H(n,\alpha,\beta,\lambda)$ as the family of harmonic functions $f = h + \bar{g}$ of the form (1.1), such that

(1.4) $$\Re\left\{(1-\alpha)\left[\frac{D_\lambda^n f\ (z)}{z}\right] + \alpha\left[\frac{\frac{\partial}{\partial\theta} D_\lambda^n f\ (z)}{\frac{\partial z}{\partial\theta}}\right]\right\} > \beta\,,\ \alpha \ge 0\,,$$

where, $0 \le \beta < 1$ and $z = re^{i\theta} \in\ \mathcal{U}$.

Further, $\mathcal{F}_{\overline{H}}(n,\alpha,\beta,\lambda) = \mathcal{F}_H(n,\alpha,\beta,\lambda) \cap \overline{\mathcal{H}}$.As$\alpha$ changes from 0 to 1, the family $\mathcal{F}_H(n,\alpha,\beta,\lambda)$ produces a passage from the class of harmonic functions $\mathcal{P}_H(n,\beta,\lambda) \equiv \mathcal{F}_H(n,0,\beta,\lambda)$ consisting of harmonic functions f where $\Re\left\{\left[\frac{D_\lambda^n f\ (z)}{z}\right]\right\} > \beta$ to the class of harmonic functions

$\mathcal{Q}_H(n,\beta,\lambda) \equiv \mathcal{F}_H(n,1,\beta,\lambda)$consisting of harmonic functions f where

$$\left\{\frac{\frac{\partial}{\partial\theta} D_\lambda^n f\ (z)}{\frac{\partial z}{\partial\theta}}\right\} > \beta.$$

2. PRIME RESULTS

In this section, certain sufficient condition for $f = h + \bar{g}$ given by (1.1) to be in the class $\mathcal{F}_H(n,\alpha,\beta,\lambda)$ and $\mathcal{F}_{\overline{H}}(n,\alpha,\beta,\lambda)$ are obtained. Further, a representation theorem, inclusion properties and distortion bounds are also obtained for functions belonging to class $\mathcal{F}_{\overline{H}}(n,\alpha,\beta,\lambda)$.

Theorem 2.1: Let $f = h + \bar{g}$be given by (1.1). If

(2.1) $\sum_{j=2}^{\infty} C_\lambda(n,j)|\alpha(j-1)+1||\,a_j| + \sum_{j=1}^{\infty} C_\lambda(n,j)|\alpha(j+1)-1||\,b_j| \le (1-\beta)$

then$f \in \mathcal{F}_H(n,\alpha,\beta,\lambda)$.

Proof. Let

$$\omega(z) = (1-\alpha)\left[\frac{D_\lambda^n f(z)}{z}\right] + \alpha\left\{\frac{\frac{\partial}{\partial\theta}D_\lambda^n f(z)}{\frac{\partial z}{\partial\theta}}\right\}.$$

It suffices to show that $|1-\beta+\omega| \geq |1+\beta-\omega|$. This is equivalent to show that if the condition (2.1), holds then,

$$(1-\beta)z + (1-\alpha)\left[D_\lambda^n h(z) + \overline{D_\lambda^n g(z)}\right] + \alpha[z(D_\lambda^n h(z))' - z(\overline{D_\lambda^n g(z)})']$$

$$-|(1+\beta)z - (1-\alpha)\left[D_\lambda^n h(z) + \overline{D_\lambda^n g(z)}\right] - \alpha[z(D_\lambda^n h(z))' - z(\overline{D_\lambda^n g(z)})']| : M(\alpha,\beta) \geq 0.$$

Substituting for $D_\lambda^n h(z)$ and $D_\lambda^n g(z)$ in $M(\alpha,\beta)$ yields.

$$M(\alpha,\beta) = |(1-\beta)z + \{\sum_{j=2}^{\infty}(\alpha(j-1)+1)a_j z^j - \sum_{j=1}^{\infty}(\alpha(j+1)-1)\bar{b}_j (\bar{z})^j\}C_\lambda(n,j)|$$

$$-\left|\beta z - \left\{\sum_{j=2}^{\infty}(\alpha(j-1)+1)a_j z^j - \sum_{j=1}^{\infty}(\alpha(j+1)-1) - \bar{b}_j (\bar{z})^j\}C_\lambda(n,j)\right\}\right|$$

$$\geq 2|z|\left[(1-\beta) - (\sum_{j=2}^{\infty}|\alpha(j-1)+1||a_j| + \sum_{j=1}^{\infty}|\alpha(j+1)-1||b_j|)C_\lambda(n,j)\right].$$

By (2.1), it follows that the last expression is non-negative. Hence $f \in \mathcal{F}_H(n,\alpha,\beta,\lambda)$.

We derive a necessary and sufficient condition for function $f = h + \bar{g}$ be given with (1.4).

Theorem 2.2. Let $f = h + \bar{g}$, be given by (1.1). then, $f \in \mathcal{V}_{\bar{H}}(n,\alpha,\lambda)$ if and only if

(2.2) $\sum_{j=2}^{\infty}\left(\frac{j-\alpha}{1-\alpha}|a_j| + \frac{j+\alpha}{1-\alpha}|b_j|\right)C_\lambda(n,j) \leq 1 - \frac{1+\alpha}{1-\alpha}b_1$

where ,$a_1 = 1$ and $0 \leq \alpha < 1$.

Proof: Since$\mathcal{V}_{\bar{H}}(n,\alpha,\lambda) \subset \mathcal{R}_H(n,\alpha,\lambda)$, we only need to prove the necessity part of the Theorem. For functions$f \in \mathcal{V}_{\bar{H}}(n,\alpha,\lambda)$, we notice that the condition

$$\frac{\partial}{\partial\theta}(arg D_\lambda^n f(z)) \geq \alpha$$

is equivalent to

$$\frac{\partial}{\partial\theta}(argD_\lambda^n f\ (z)) - \ \alpha = \mathcal{R}\left\{\frac{z(D_\lambda^n h\ (z))' - \overline{z(D_\lambda^n g(z))'}}{z(D_\lambda^n h\ (z)) - z(D_\lambda^n g\ (z))} - \alpha\right\} \geq 0.$$

That is,

$$(2.3)\quad \mathcal{R}\left[\frac{(1-\alpha)z+\left(\sum_{j=2}^{\infty} C_\lambda(n,j)(j-\alpha)|a_j|z^j - \sum_{j=1}^{\infty} C_\lambda(n,j)(j+\alpha)|b_j|\overline{z^j}\right)}{z+\left(\sum_{j=2}^{\infty} C_\lambda(n,j)|a_j|z^j + \sum_{j=1}^{\infty} C_\lambda(n,j)|b_j|\overline{z^j}\right)}\right] \geq 0.$$

The above condition must hold for all values of $z \in \mathcal{U}$. Choosing ϕ according to (1.4), we must have,

$$(2.4)\frac{(1-\alpha)-(1+\alpha)b_1-\left(\sum_{j=2}^{\infty}(j-\alpha)\ C_\lambda(n,j)|a_j|r^{j-1}+\sum_{j=2}^{\infty}(j+\alpha)\ C_\lambda(n,j)|b_j|r^{j-1}\right)}{1+|b_1|+\left(\sum_{j=2}^{\infty} C_\lambda(n,j)|a_j|+\sum_{j=2}^{\infty} C_\lambda(n,j)|b_j|\right)r^{j-1}} \geq 0.$$

If the condition (2.2) does not hold, then the numerator in (2.4) is negative for r sufficiently close to 1. Hence there exists $z_0 = r_0 \in (0,1)$ for which quotient of (2.4) is negative. This contradiction proves that $f \in \mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$.

Theorem 2.3: If $f \in \mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$, then

$$(2.5)|f(z)| \leq (1+|b_1|)r + C_\lambda(n,2)\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)r^2, \quad (|z| = r < 1)$$

and

$$(2.6)\qquad |f(z)| \geq (1+|b_1|)r - C_\lambda(n,2)\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2+\alpha}|b_1|\right)r^2, \quad (|z| = r < 1).$$

Proof: We will prove the inequality (2.6). The argument for (2.5) is similar. Let $f \in \mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$. Taking the absolute values of f , we have

$$|f(z)| \leq (1+|b_1|)r + \sum_{j=2}^{\infty}\left(|a_j| + |b_j|\right)|r|^j$$

$$\leq (1+|b_1|)r + r^2\sum_{j=2}^{\infty}\left(|a_j| + |b_j|\right).$$

That is

$$|f(z)| \leq (1+|b_1|)r + \frac{1-\alpha}{(2-\alpha)C_\lambda(n,2)}\left(\sum_{j=2}^{\infty}\frac{(2-\alpha)}{(1-\alpha)}C_\lambda(n,2)|a_j| + \sum_{j=2}^{\infty}\frac{(2-\alpha)}{(1-\alpha)}C_\lambda(n,2)|b_j|\right)r^2$$

$$\leq (1+|b_1|)r + \frac{1-\alpha}{(2-\alpha)C_\lambda(n,2)}\left[1-\frac{(1+\alpha)}{(1-\alpha)}|b_1|\right]r^2$$

$$\leq (1+|b_1|)r + \frac{1}{C_\lambda(n,2)}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right) r^2.$$

Corollary 2.4: Let f be the form (1.1), so that $f \in \mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$.Then,

(2.7) $\left\{\omega : |\omega| < \frac{2C_\lambda(n,2)-1-[C_\lambda(n,2)-1]\alpha}{(2-\alpha)C_\lambda(n,2)} - \frac{2C_\lambda(n,2)-1-[C_\lambda(n,2)-1]\alpha}{(2+\alpha)C_\lambda(n,2)}b_1\right\} \subset f(\mathcal{U}).$

We use the coefficient bounds to examine the extreme points for $f \in \mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$ and determine extreme point of $\mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$.

Theorem 2.5: The extreme points for $\mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$ are

(2.8) $\left\{z+\gamma_j x z^j + \overline{b_1 z}\right\} \cup \left\{z + \overline{b_1 z + \mu_j x z^j}\right\}$

where, $j \geq 2$ *and* $|x| = 1-|b_1|$

$$\gamma_j = \frac{(1-\alpha)}{(j-\alpha)\, C_\lambda(n,j)} \text{ and } \mu_j = \frac{(1+\alpha)}{(j+\alpha)\, C_\lambda(n,j)}.$$

Proof. Any functions f *in* $\mathcal{V}_{\overline{H}}(n,\alpha,\lambda)$ is of the form

$$f(z) = z + \sum_{j=2}^{\infty}|a_j|\, e^{i\beta_j} z^j + \overline{b_1 z} + \overline{\sum_{j=2}^{\infty}|b_j|\, e^{i\delta_j} z^j}$$

where the coefficients satisfy the inequality (2.1).

Let $h_1(z) = z$, $g_1(z)$=$b_1(z)$, $h_j(z) = z + \gamma_j e^{i\beta_j} z^j$, $g_j(z) = b_1 z + \mu_j e^{i\delta_j} z^j$, $for\ j = 2,3, \ldots.$

Writing$\chi_j = \frac{|a_j|}{\gamma_j} \eta_j = \frac{|b_j|}{\mu_j}$ $j = 2,3, \ldots ..$

and$\chi_1 = 1 - \sum_{j=2}^{\infty} \chi_j\, \eta_1 = 1 - \sum_{j=2}^{\infty} \eta_j$

we have ,

$$f(z) = \sum_{j=2}^{\infty} \left(\chi_j h_j(z) + \eta_j g_j(z)\right).$$

In particular , setting

$f_1(z) = z + \overline{b_1 z}$ and $f_j(z) = z + \gamma_j x z^j + \overline{b_1 z} + \overline{\mu_j y z^j}$, $(j \geq 2,\ |x| + |y| = 1 - |b_1|)$.

We see that the extreme points of $\mathcal{V}_{\overline{H}}(n, \alpha, \lambda)$ are contained in $\{f_j(z)\}$.

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