

The Center and the Barycenter

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Abstract

In the first part, we deal with the question of which points we have to connect to generate a non self-intersecting polygon. Afterwards, we introduce a *polyhole*, which is a generalization of a polygon. Roughly speaking a polyhole is a big polygon, where we cut out a finite number of small polygons.

In the second part, we present two ‘centers’, which we call *center* and *barycenter*. In the case that both centers coincide, we call these polygons *nice*. We show that if a polygon has two symmetry axes, it is nice. We yield examples of polygons with a single symmetry axis which are nice and which are not nice. We show that all symmetry axes intersect at one point.

In the third part, we introduce the *Spieker center* and the *Point center* for polygons. We define *beautiful* polygons and *perfect* polygons.

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1. INTRODUCTION

We look for a criterion to generate a simple polygon.

Let us assume a set of $k + 1$ points called *Points* $\subset \mathbb{R}^2$, where $Points = \{(x_1, y_1), (x_2, y_2), \dots, (x_{k-1}, y_{k-1}), (x_k, y_k), (x_{k+1}, y_{k+1})\}$. We joint the possible edges. We define the subset *Union* of \mathbb{R}^2 , $Union = \bigcup [(x_i, y_i), (x_{i+1}, y_{i+1})]$ for $i \in \{1, 2, \dots, k-1, k\}$. With the expression ‘ $[a, b]$ ’ we mean all points on the line segment between a and b and the boundaries a and b . We say that *Points* is *suitable* if and only if *Union* is homeomorphic to the circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

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Definition 1.1. We call *Union* a *polygon* if and only if it holds $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$ where $i, j \leq k$. We demand that there are no consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$; also $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ and $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ are not collinear. The element (x_i, y_i) is called a *vertex*. We name a polygon such that *Points* is suitable a *simple polygon*. If we have a simple polygon we include its interior, and it holds $(x_{k+1}, y_{k+1}) = (x_1, y_1)$ and $k > 2$.

An r -gon is a simple polygon with r vertices.

2. STARLIKE POLYGONS

We can start also with a circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. We presume a finite set called A of k different points on the circle, where $k > 2$ and $A := \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_{k-1}, \vec{a}_k\}$ is in counterclockwise order. We map A bijectively into a set $H := \{\vec{h}_1, \vec{h}_2, \dots, \vec{h}_{k-1}, \vec{h}_k\}$, $\vec{a}_i \mapsto \vec{h}_i$, such that the three points \vec{h}_i, \vec{a}_i and $(0, 0)$ are collinear and the norm of \vec{h}_i is positive. We demand that there are no collinear points $\vec{h}_i, \vec{h}_{i+1}, \vec{h}_{i+2}$, and also that $\vec{h}_{k-1}, \vec{h}_k, \vec{h}_1$ and $\vec{h}_k, \vec{h}_1, \vec{h}_2$ are not collinear. We keep the order as in A . We move all points with the same vector \vec{m} , i.e. $Z := \{\vec{z}_1, \vec{z}_2, \dots, \vec{z}_i, \dots, \vec{z}_{k-1}, \vec{z}_k\} \subset \mathbb{R}^2$ where $1 \leq i \leq k$ and $\vec{z}_i := \vec{h}_i + \vec{m}$. We keep the order. We call a polygon with a set of vertices $V := \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}, \vec{v}_k\}$ a *starlike polygon* if and only if V provided with an appropriate order can be constructed as it is just described for Z . We add $\vec{v}_{k+1} := \vec{v}_1$ into V . Please see the following Proposition 2.2. Note that V is a suitable set.

Question 2.1. *Is there an alternative description of starlike polygons? Is every convex simple polygon a starlike polygon?*

Proposition 2.2. *We get a simple polygon P if there is a finite set of points $V := \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{k-1}, \vec{v}_k, \vec{v}_{k+1}\} \subset \mathbb{R}^2$ constructed as above where $k > 2$ and*

$$P := \bigcap \{ \text{Circle} \subset \mathbb{R}^2 \mid V \subset \text{Circle}, \text{ where Circle is homeomorphic to the circle area } \{x^2 + y^2 \leq 1\}, \text{ and the points between } \vec{v}_i \text{ and } \vec{v}_{i+1}, 1 \leq i \leq k, \text{ are a subset of Circle} \}$$

Proof. The claim of the proposition is trivial since V is a suitable set. \square

It follows that a starlike polygon is a compact set, homeomorphic to any circle area, and for all $1 \leq i \leq k$ the vector \vec{v}_i is a boundary point. All starlike polygons are simple polygons.

A triangle is a starlike polygon, too.

Proposition 2.3. If we have a set of $k + 1$ points $Points := \{(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k), (x_{k+1}, y_{k+1})\} \subset \mathbb{R}^2$ which forms a polygon we get a simple polygon if and only if $Points$ is suitable.

Proof. Trivial. □

3. POLYHOLES

We define a subset of \mathbb{R}^2 , which we will call a *polyhole*. This geometric structure consists of a finite number of simple polygons $P, P_1, P_2, P_3, \dots, P_{m-1}, P_m$. From the polygon P we cut out polygons $P_1, P_2, P_3, \dots, P_{m-1}, P_m$.

Definition 3.1. Let $\{P, P_1, P_2, P_3, \dots, P_{m-1}, P_m\}$ be a set of simple polygons. A *polyhole* is defined as P without $P_1 \cup P_2 \cup P_3 \cup \dots \cup P_{m-1} \cup P_m$

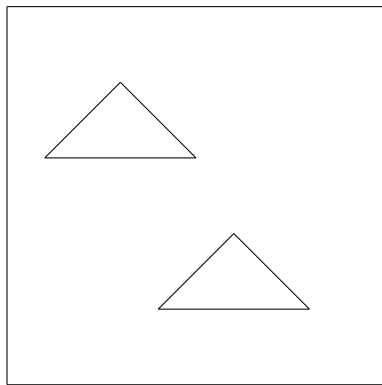


Figure 1:

The polyhole on the left-hand side consists of a square where we cut out two triangles.

A corresponding definition is possible for polytopes. To define them please see [1].

Definition 3.2. Let $\{P, P_1, P_2, P_3, \dots, P_{m-1}, P_m\}$ be a set of polytopes. A *polytopehole* is defined as P without $P_1 \cup P_2 \cup P_3 \cup \dots \cup P_{m-1} \cup P_m$.

Question 3.3. What is the barycenter of a polyhole, if it is realized with homogeneous material of constant thickness? What is the barycenter of a polytopehole of \mathbb{R}^3 , if it is realized with homogeneous material?

4. NICE POLYGONS

We assume a suitable set of points called $Points = \{(x_i, y_i) | i \in \{1, 2, \dots, k-1, k, k+1\}\}$. We define two ‘centers’, where the center $Cent$ is just the arithmetic mean of the first and second coordinates of the generating points, respectively.

We got the following formulas for the *barycenter* $B = (B_x, B_y)$ of a simple polygon from [2] or [3]. Please see also [4] and [5]. Area is the area of a simple polygon. Note that $Area \neq 0$ and that in [2] and [4] the barycenter is called a Centroid, and further that

B is the center of gravity of the polygon if it is realized with homogeneous material of constant thickness. Note that the order in the polygon is counterclockwise. We write

$$D_i = x_i \cdot y_{i+1} - x_{i+1} \cdot y_i, \text{ where } 1 \leq i \leq k \quad (4.1)$$

$$\text{Area} = \frac{1}{2} \cdot \sum_{i=1}^k D_i \quad (4.2)$$

$$B_x = \frac{1}{6 \cdot \text{Area}} \cdot \sum_{i=1}^k (x_i + x_{i+1}) \cdot D_i, \quad B_y = \frac{1}{6 \cdot \text{Area}} \cdot \sum_{i=1}^k (y_i + y_{i+1}) \cdot D_i \quad (4.3)$$

$$\text{Cent} = \frac{1}{k} \cdot \left(\sum_{i=1}^k x_i, \sum_{i=1}^k y_i \right) \quad (4.4)$$

Definition 4.1. Let us presume a simple polygon P . We call P *nice* if and only if it holds $B = \text{Cent}$.

Proposition 4.2. *Every triangle is nice*

Remark 4.3. When we use the term *symmetry axis* of a polygon P we mean a line segment s in the convex hull of P of maximal length, i.e. it holds for a symmetry axis t in the convex hull of P with more than one common point with s that $t \subset s$.

Proposition 4.4. *If a simple polygon has two different symmetry axes, it is nice*

Proof. The proposition is an easy consequence of the fact that both the center of gravity and Cent must be on a line determined by a symmetry axis. Please see Lemma 4.11.

Lemma 4.5. The property of being nice or being not nice remains with these operations.

- *Revolving P by an arbitrary angle around any point*
- *Shifting P by an arbitrary vector*

Proof. We assume $B \neq \text{Cent}$. Let us revolve P by an arbitrary angle around any point. There is a positive distance d between B and Cent . It will be kept since a rotation is a distance-preserving map. Hence the distance between the image points of B and Cent is also d . After the rotation still P is not nice.

In the case $B = \text{Cent}$ we have nothing to show. □

Lemma 4.6. Both operations which we have mentioned above in Lemma 4.5 are distance-preserving operations. Therefore the shape of a polygon is kept after these operations.

Proof. Trivial. □

In a polygon, we fix four real numbers.

Definition 4.7. We define

min_x := minimum of the set of the first coordinates of the set of the vertices *Points* of P .

min_y := minimum of the second coordinates of *Points*,

max_x := maximum of the first coordinates of *Points*,

max_y := maximum of the second coordinates of *Points*.

Let $min_x < max_x$ and $min_y < max_y$. Note that in a simple polygon P , these conditions are fulfilled. We define a simple polygon called $Rectangle(P)$ by four vertices (max_x, max_y) , (min_x, max_y) , (min_x, min_y) , (max_x, min_y) .

Remark 4.8. In a simple polygon P it holds that both P and the convex hull of *Points* are in $Rectangle(P)$.

Definition 4.9. Let $s = \{\vec{a} + r \cdot \vec{d} \mid r \in [v, w]\}$ for fixed real numbers v, w be a symmetry axis of a polygon. We define $l(s)$ as the line $\{\vec{a} + r \cdot \vec{d} \mid r \in \mathbb{R}\}$.

Remark 4.10. It holds that s is a subset of $l(s)$.

Lemma 4.11. Let s be a symmetry axis of a simple polygon P . Both B and $Cent$ are on the line segment $l(s) \cap Rectangle(P)$.

Proof. We assume a simple polygon P with a symmetry axis s and centers B and $Cent$. Note that B is the center of gravity of P . Hence B must be on $l(s)$, since s is a symmetry axis of P . For the same reason, B is also in $Rectangle(P)$.

We use Lemma 4.6. We map P by a rotation and a shift parallel the vertical y axis into a second polygon P' with a symmetry axis s' and centers B' and $Cent'$ such that s' is on the x axis. Assume a vertex (x', y') of P' . Since s' is a symmetry axis and it is on the x axis either $y' = 0$ or there is a second vertex $(x', -y')$ of P' . If we add all vertices together we get $Cent' = (c', 0)$ with any real number c' , i.e. $Cent'$ is on the x axis. This means that $Cent'$ is on $l(s')$. Since P' has the same shape as P we get that $Cent$ is on $l(s)$.

It is easy to show that $Cent$ is a point in $Rectangle(P)$: It holds

$$min_x = \frac{1}{k} \cdot \sum_{i=1}^k min_x < \frac{1}{k} \cdot \sum_{i=1}^k x_i < \frac{1}{k} \cdot \sum_{i=1}^k max_x = max_x \quad (4.5)$$

If we consider correspondingly the second coordinate of $Cent$ we get $min_y < \frac{1}{k} \cdot \sum_{i=1}^k y_i < max_y$, and it follows that $Cent$ is in $Rectangle(P)$.

We get that $Cent$ is in $l(s) \cap Rectangle(P)$. Lemma 4.11 has been proved. \square

Two symmetry axes intersect in one point. It is both B and $Cent$. The proof of Proposition 4.4 is finished. \square

Corollary 4.12. In a simple polygon with multiple symmetry axes all symmetry axes intersect in one point. It is both B and $Cent$. It follows that a simple polygon with more than a single symmetry axis is nice.

Note that one symmetry axis is not sufficient, as the kite defined by $(0, 0), (1, -1), (3, 0), (1, 1)$ shows, since $\frac{5}{4} \neq \frac{4}{3}$. It is not nice.

We give an example of a polygon with a single symmetry axis which is nice, but not a triangle.

We present a 5-gon. It has centers $B = Cent$, it is about $(0.50, 0.85)$. It has vertices

$$(0, 0), (1, 0), (1, 1), \left(\frac{1}{2}, 1 + \frac{1}{2} \cdot \sqrt{6}\right), (0, 1). \text{ We get } B = Cent = \left(\frac{1}{2}, \frac{1}{10} \cdot (6 + \sqrt{6})\right). \quad (4.6)$$

This example proves the conjecture that besides triangles only polygons with two or more symmetry axes are nice is wrong.

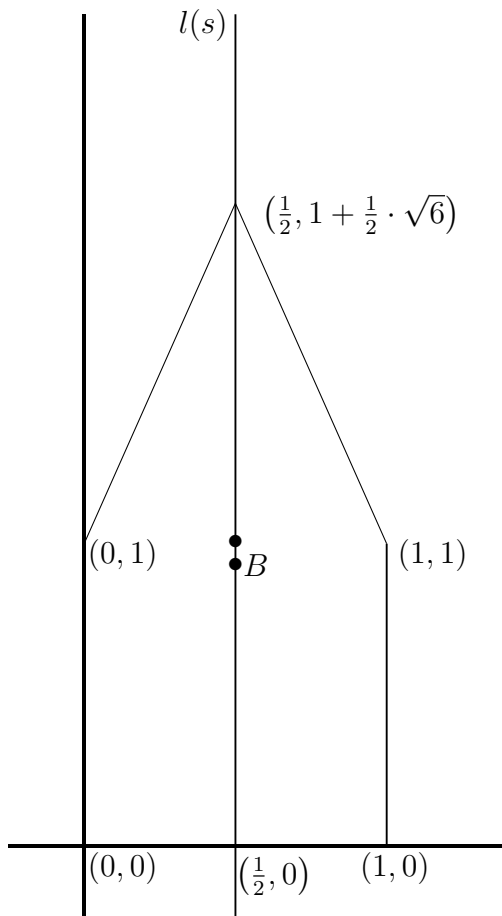


Figure 2:

On the left-hand side is a nice 5-gon. It has sidelengths 1 and $\frac{1}{2} \cdot \sqrt{7}$. We show a piece of the symmetry line $l(s)$, the axes, and two centers, i.e. above the Spieker center and below $B = Cent$.

5. SPIEKER CENTER AND POINT CENTER

In a triangle the Spieker center is well-known. We have gotten the formulas of the Spieker center from [5]. Please see also [6]. The Spieker center of a triangle $A = (x_A, y_A), B = (x_B, y_B), C = (x_C, y_C)$ is its barycenter, at which the triangle is formed by a wire of constant diameter. The interior of the triangle is not regarded. The Spieker center is outside the wire. The sidelengths of the triangle are l_1, l_2 and l_3 , where sides with lengths l_1 and l_3 intersect in A , while sides with lengths l_1 and l_2 intersect in B . The coordinates of the Spieker center $(spieker_x, spieker_y)$ are

$$spieker_x = \frac{(l_1 + l_3) \cdot x_A + (l_1 + l_2) \cdot x_B + (l_2 + l_3) \cdot x_C}{2 \cdot (l_1 + l_2 + l_3)} \quad \text{and} \quad (5.1)$$

$$spieker_y = \frac{(l_1 + l_3) \cdot y_A + (l_1 + l_2) \cdot y_B + (l_2 + l_3) \cdot y_C}{2 \cdot (l_1 + l_2 + l_3)} \quad (5.2)$$

The concept of the *Spieker center* can easily be generalized to polygons. We assume the polygon is made from a wire of constant diameter. The shape of the polygon is formed by the wire. We look for its center of gravity; it is generally outside the wire. We consider a new polygon, constructed by k mass centers. Therefore it also has k vertices. We compute the *Point center* of the new polygon. The Point center of an r -gon is defined by the imagination that the masses are in the vertices of the polygon. Let $m_1, m_2, \dots, m_{r-1}, m_r$ be r masses. The polygon has the Point center $Point = (point_x, point_y)$, where

$$point_x = \frac{1}{M} \cdot \sum_{i=1}^r m_i \cdot a_i \quad (5.3)$$

$$point_y = \frac{1}{M} \cdot \sum_{i=1}^r m_i \cdot b_i, \text{ and} \quad (5.4)$$

$$M = \sum_{i=1}^r m_i \text{ is the sum of the masses and} \quad (5.5)$$

$$(a_1, b_1), (a_2, b_2), \dots, (a_{r-1}, b_{r-1}), (a_r, b_r) \text{ are the vertices of the polygon.} \quad (5.6)$$

To calculate the Spieker center of a given polygon we have to consider a new polygon, constructed by k mass centers of the k edges. Therefore it also has k vertices. We assume that in the new polygon, the masses are on these k vertices. The Spieker center

of the given polygon is the Point center of the new polygon. The formulas are

$$spieker_x = \frac{1}{U} \cdot \sum_{i=1}^k l_i \cdot \left(\frac{1}{2} \cdot (x_i + x_{i+1}) \right) = \frac{1}{2 \cdot U} \cdot \sum_{i=2}^{k+1} (l_i + l_{i-1}) \cdot x_i \quad (5.7)$$

$$spieker_y = \frac{1}{U} \cdot \sum_{i=1}^k l_i \cdot \left(\frac{1}{2} \cdot (y_i + y_{i+1}) \right) = \frac{1}{2 \cdot U} \cdot \sum_{i=2}^{k+1} (l_i + l_{i-1}) \cdot y_i \quad (5.8)$$

$$\text{where } l_i = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2} \quad \text{and} \quad U = \sum_{i=1}^k l_i \quad (5.9)$$

We define $l_{k+1} := l_1$. Note the indices in the formulas! Note that it holds $(x_{k+1}, y_{k+1}) = (x_1, y_1)$. The variable ‘ l_i ’ means the length of one edge of the polygon. Every edge $l_i := [(x_i, y_i), (x_{i+1}, y_{i+1})]$ has a center of gravity $\frac{1}{2} \cdot ((x_i, y_i) + (x_{i+1}, y_{i+1}))$. U is the perimeter of the polygon.

As an example, we take the 5-gon above. Its Spieker center is about $(0.50, 0.93)$. In exact coordinates it is

$$\left(\frac{1}{2}, \frac{1}{6 + 2 \cdot \sqrt{7}} \cdot \left(2 + 2 \cdot \sqrt{7} + \frac{1}{2} \cdot \sqrt{42} \right) \right) = \left(\frac{1}{2}, \sqrt{7} - 2 + \frac{1}{8} \cdot \left(3 \cdot \sqrt{42} - \sqrt{294} \right) \right). \quad (5.10)$$

Definition 5.1. Let us presume a simple polygon P . We call P *beautiful* if and only if it holds that B equals the Spieker center. We call P *perfect* if and only if all three centers are the same, i.e. it holds that B equals both $Cent$ and the Spieker center.

Lemma 5.2. Let s be a symmetry axis of a simple polygon P . The Spieker center is on $s \cap Rectangle(P)$.

Proof. The line segment s is a symmetry axis both for the entire polygon and its contour. Because the Spieker center is the barycenter of the contour it must be on s . The statement that the Spieker center is in $Rectangle(P)$ is trivial. \square

Proposition 5.3. Let a simple polygon has two or more symmetry axes. Then it is perfect.

Proof. The three points B , $Cent$ and the Spieker center all are on the line determined by a symmetry axis. There is only a single possibility that all points are on every line. \square

Conjecture 5.4. A triangle is perfect if and only if it is an equilateral triangle.

The next conjecture contains the previous one.

Conjecture 5.5. An r -gon is perfect if and only if it is an regular r -gon.

Conjecture 5.6. *A simple polygon is beautiful if and only if it is perfect. In other words, it holds $B = \text{Spieker center}$ if and only if $B = \text{Spieker center}$ and $B = \text{Cent}$.*

Conjecture 5.7. *A simple polygon is beautiful if and only if it has more than one symmetry axis.*

We end the paper with an infinite number of questions. We define that a *connection* between two points A and B is a continuous map $f : [0, 1] \rightarrow \mathbb{R}^2$ such that $f(0) = A$ and $f(1) = B$.

Question 5.8. *We assume a set of n different fixed points in \mathbb{R}^2 . The first set of questions is: Are there connections between all points such that the union of the connections is homeomorphic to a circle? The second is the same, except that only line segments are allowed as connections. In the next questions, we may use a subset of the set of points. We ask whether there are connections such that these connections are homeomorphic to m circles. In the last set of questions, we ask the same, except that only line segments are allowed.*

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