

One Tile Suffices

VOLKER W. THÜREY*

Bremen, Germany

Abstract

We have found for all k larger than four a new possibility to tile the plane completely with k -gons. We use infinite many copies of a single tile. The proofs are not by written words, but by pictures. Amongst others, we use the well-known tiling with hexagons.

Keywords: Tiling; plane

MSC-2020: 51P99

We think that it is useful to repeat the definition of a *simple polygon*.

A simple polygon with k vertices consists of k different points of the plane $(x_1, y_1), (x_2, y_2), \dots, (x_{k-1}, y_{k-1}), (x_k, y_k)$, called *vertices*, and the straight lines between (x_i, y_i) and (x_{i+1}, y_{i+1}) for $1 \leq i \leq k-1$, called *edges*. Also the straight line between (x_k, y_k) and (x_1, y_1) belongs to the polygon. We demand that it is homeomorphic to a circle, and that there are no three consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$ for $1 \leq i \leq k-2$. Also the points $(x_k, y_k), (x_1, y_1), (x_2, y_2)$ and $(x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)$ are not collinear.

We call this just described simple polygon a *k-gon*.

Note that we deal exclusively with simple polygons.

Theorem 1. Let k be a natural number larger than 2. Then there exists a tiling of the plane \mathbb{R}^2 by k -gons. We need infinite copies of only a single tile.

*E-mail ID: volker@thuerey.de

Proof. This theorem is well-known. Please see [1], p. 11.

There is another proof. For $k = 3$ and $k = 4$ and $k = 6$ the theorem is trivial. For $k = 5$ please see Figure 1. We take a regular 6-gon and cut it into identical halves. See also the tiling in the case $k = 6$.

Now let k be a natural number larger than 6.

- Possibility 1: $k \equiv 0 \pmod{4}$.

The numbers k are 8, 12, 16, ...

See Figure 2. As an example, we show one tile for the case $k = 12$. The measures for the big square are 4×4 , while the two small squares have measures of 2×2 .

- Possibility 2: $k \equiv 1 \pmod{4}$.

The sequence of the numbers of k is 9, 13, 17, ...

See Figure 3. There we show one tile for the case $k = 13$.

The big square has sidelengths of 4, while the small square has sidelengths of 2. The two horizontal edges on the left have lengths 4 and 2, respectively. They have a distance of 1. The two sloped edges both have a length of $\sqrt{2}$.

- Possibility 3: $k \equiv 2 \pmod{4}$.

The sequence of the numbers of k is 10, 14, 18, ...

See Figure 4. We show a 14-gon. The square has a sidelength of 4, the rectangle has measures of 2×4 . The triangle on the left has sidelengths 4, 2, and $\sqrt{20}$. The triangle on the right has sidelengths 4 and $\sqrt{32}$.

- Possibility 4: $k \equiv 3 \pmod{4}$.

The sequence of the numbers of k is 7, 11, 15, ...

See Figure 5. Here we show a 15-gon.

The square also has a sidelength of 4, both rectangles have measures of 2×4 . □

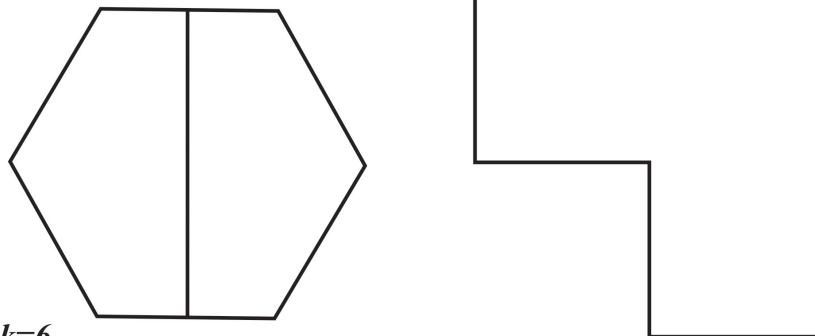


Figure 1:
 $k = 5$ and $k=6$

Figure 2:
 $k = 12$

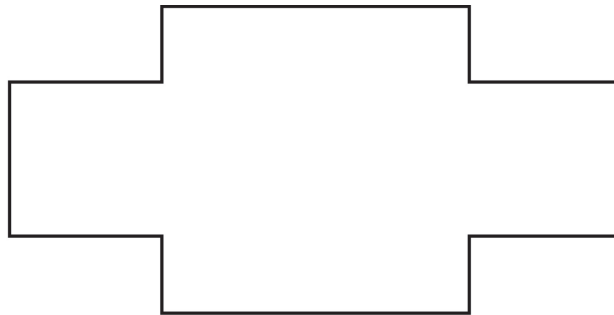


Figure 3:
 $k = 13$

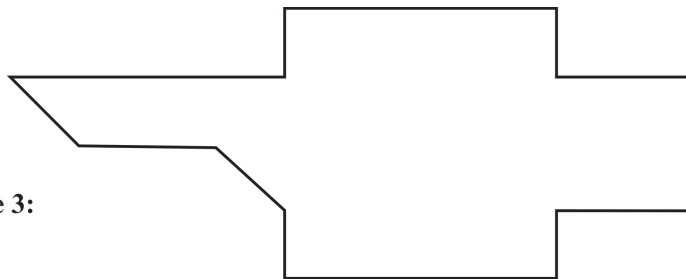


Figure 4:
 $k = 14$

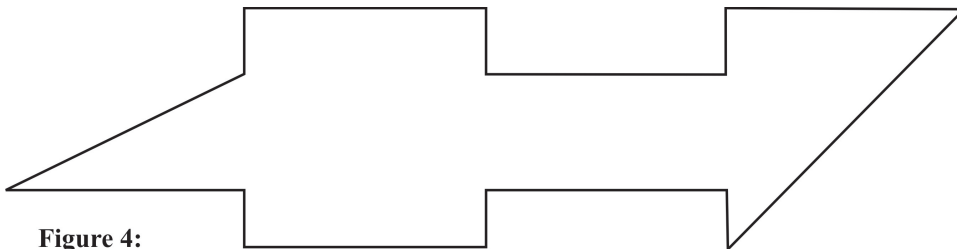
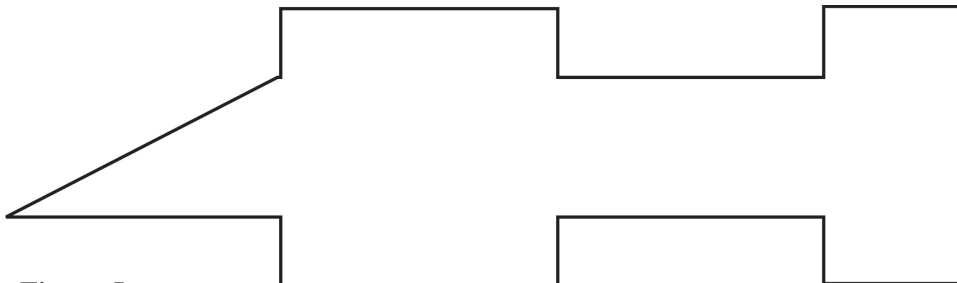


Figure 5:
 $k = 15$



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REFERENCES

- [1] <http://www.willimann.org/A07020-Parkettierungen-Theorie.pdf>
- [2] <http://www.mathematische-Basteleien.de/parkett2.htm>
- [3] Ehrhard Behrends: *Parkettierungen der Ebene*, Springer (2019)