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# Projections and Dimensions

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**Abstract**

In this paper, we consider a relationship between projection and dimension in geometry. We discuss the relation from projective geometrical viewpoint by paying attention to higher and lower dimensional spaces. Then, we categorize those projections among different dimensional ones by their characteristics. We also express the classification by group theory and verify that there exists the automorphism.

**Keywords:** freedom degree, higher dimensional space, lower dimensional space, projective geometry, homogeneous coordinates

**2000 Mathematics Subject Classification:** 51N15, 51N25, 51N35.

## Introduction

This paper is mainly focused on a viewpoint of geometry projected into higher and lower dimensional spaces. First of all, about projection into higher dimensional space, for example, mapping any point in the space of the first dimension into second dimension, we will commonly consider that it is kept on *point*. Similarly, the point is *point* all in dimensional spaces. However, we could unnecessarily say any point mapped into higher dimensional space is point. As such a case, for example, we could pick up that $x = a$ indicates a point in one-dimensional space but $x = a$ in two-dimensional space is considered equation of a straight line. We discuss it from the section 2.1 to 2.3 in detail.

Contrarily, thinking of projection into lower dimensional space, we discuss it with homogeneous coordinates. For example, let us think of an equation $f(x, y) = x^2 + y^2 - 1 = 0$. If substituting $x = x_1 / x_0$ and $y = x_2 / x_0$ for the equation, then it is expanded as $(x_1 / x_0)^2 + (x_2 / x_0)^2 - 1 = 0$. Hence $x_1^2 + x_2^2 - x_0^2 = 0$. The expanded equation $x_1^2 + x_2^2 - x_0^2 = 0$ is a quadric surface in the three-dimensional space though the original equation $f(x, y) = x^2 + y^2 - 1 = 0$ indicates a circle in the two-dimensional space. Such a mapping between homogeneous and non-homogeneous coordinates could be considered the projection into lower dimensional space from our viewpoint. We further discuss it in the section 2.4.

Finally, we discuss that a series of projections among higher and lower dimensional spaces makes a group. Especially, conclusion in the section 2.5 will request them necessity of description by transformation group. In Chapter 3, we discuss how to describe what is discussed in Chapter 2.

## Concept and Definition

### Degree of Freedom

First of all, we designate any point in the orthogonal coordinates. Assuming the point $x = a$ is mapped into the two-dimension, it will be commonly considered as *point* also in the two-dimension as shown in Figure 1. However, let us contemplate the mapped point. It originally has only one-*degree of freedom* along the x-axis. Nonetheless, the second degree of freedom has already been assumed in the above two-dimension; Because the location of the point in the two-dimension is $(x, y) = (a,0)$ as shown in Figure 1, the second degree of freedom is already given as 0 or the y-coordinate is set as 0.

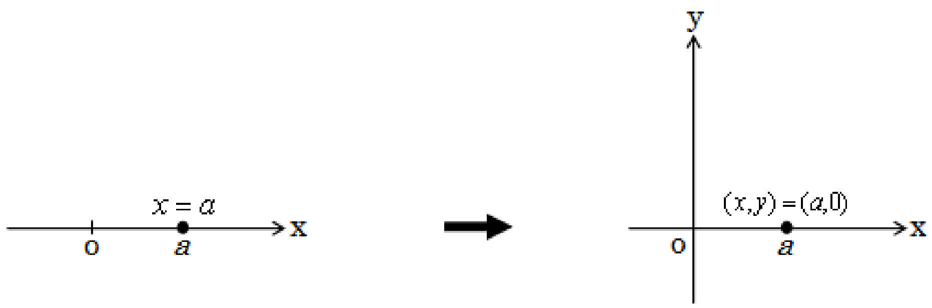


**Figure 1**

If the mapped point is related to the second dimension by keeping its original one-degree of freedom, we cannot decide the y-coordinate. To avoid the decision, the mapped point should occupy overall y-coordinates to keep its original 1-dimension's character as shown in Figure 2. It is straight line which is parallel to the y-axis because it is only the way to exclude the second degree of freedom. Therefore, $x = a$ is point in the 1-dimension, but equation of a straight line in the two-dimension. It is uncontradicted in orthodox mathematics.

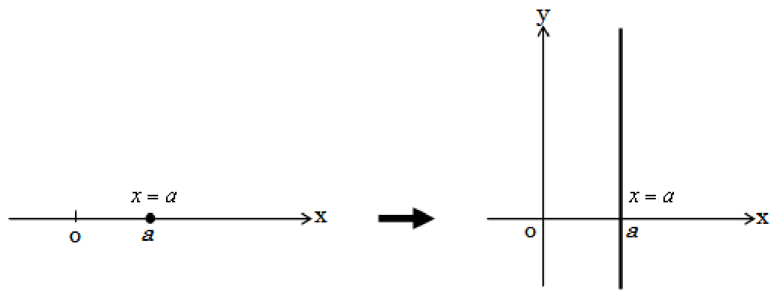


**Figure 2**

Furthermore, about *mapping* we have discussed, in other words, we could say like that any point of the one-dimension is *projected* into the two-dimension in the broadest sense of the word(; the term *projection* is generally used for mapping from higher to lower dimensional space or within the same dimension though.) Henceforth, we say *project* instead of *map* in a series of the process.

**The Projection of Any Point in the Coordinate**

Let us think about and define the above understanding with easy examples of any point in the one and two-dimension projected into the two and three-dimension in accordance with the style of Euclid's element.

About any point in the one-dimensional coordinate;

**Definition 1**. If projecting any point of the one-dimension into the two-dimension, then it is a line that is parallel to y-axis of the two-dimension.

For example, it is the equation of a straight line $x = a$ as discussed in the section 2.1.

**Definition 2**. If projecting any point of the one-dimension into the three-dimension, then it is a plane that is parallel to y-z plane in the three-dimension.

For example, it is $x = a$ where $a$ is the distance between the plane and y-z plane as shown in Figure 3.

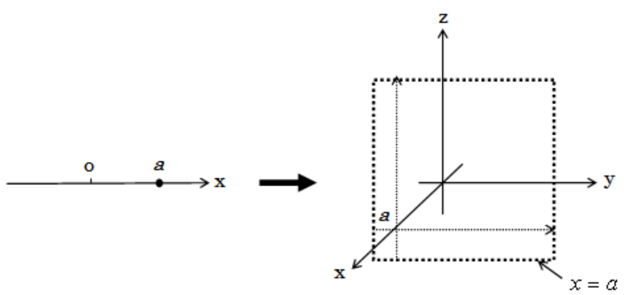


**Figure 3**

About any point in the two-dimensional space;

**Definition 3**. If projecting any point of the two-dimension into the three-dimension, then it is a line that is parallel to z-axis of the 3-dimension.

For example, it is $x = a$, $y = b$ where the coordinates of the line is $(x, y) = (a, b)$ as shown in Figure 4.

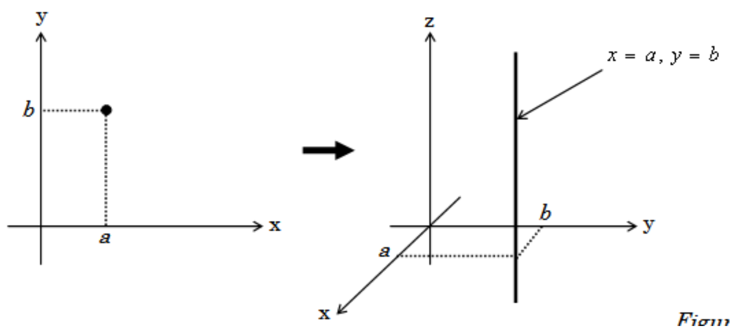


**Figure 4**

**Definition 4**. If projecting any line(; any set of points) of the two-dimension into the three-dimension, then it is a plane in the three-dimension.

For example, it is $y = x$ as shown in Figure 5.

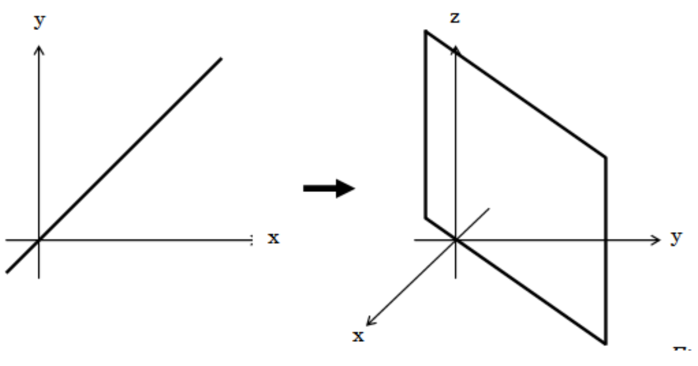


**Figure 5**

**Application for All Other Coordinates**

It is possible that all other kinds of coordinates are defined by the same way. For example, projection of any point of two-dimensional polar coordinates into three-dimensional spherical coordinates is expressed by a circle with attention to no degree of freedom of the variable $\varphi$ as shown in Figure 6.

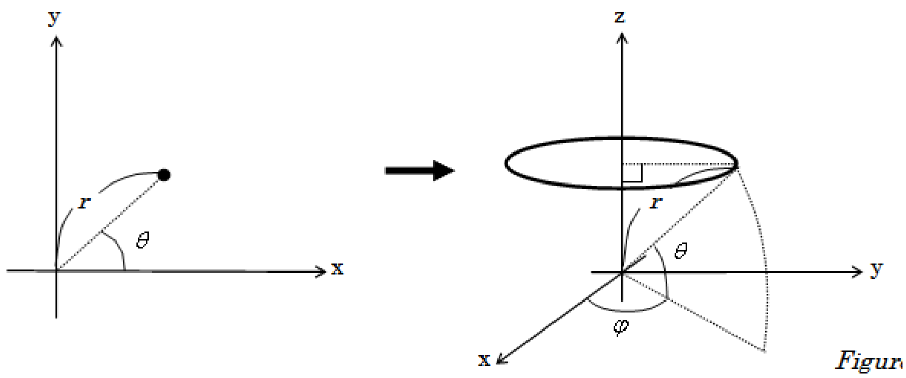


**Figure 6**

Hence, in the polar and sphere coordinates, we define any point of one and two-dimensional coordinate according to our discussion in the section 2.2;

**Definition 5**. If projecting any point of one-dimensional coordinate into two-dimensional polar ones, then it is a circle.

**Definition 6**. If projecting any point of one-dimensional coordinate into three-dimensional spherical ones, then it is a sphere.

About any point of the two-dimensional polar coordinates;

**Definition 7**. If projecting any point of two-dimensional polar coordinates into three-dimensional spherical ones, then it is a circle as shown in Figure 6.

**Definition 8**. If projecting any circle of two-dimensional polar coordinates into three-dimensional spherical ones, then it is a sphere.

## Projection into Lower Dimensions

Contrarily, let us think of projection into lower dimensional space such from our viewpoint. As for already known case, projection of a curve of the three-dimensional orthogonal coordinates into the x-y plane as in shown Figure 7 seems to be such a case. It will be certainly a common image of projective geometry. However, from our viewpoint, it is not projected into lower dimensional space. Because, the mapping is $f(x,y,z) \mapsto g(x,y,0)$ ; it is projected within the same dimensional space.

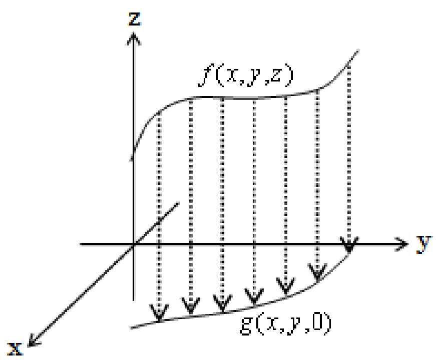


**Figure 7**

Instead, we propose mapping between homogeneous coordinates and non-homogeneous ones as an example. To explain what it suggests, let us think of a quadric equation $f(x,y)=x^2+y^2-1=0$ . Substituting $x=x_0/x_2$ and $y=x_1/x_2$ for the equation, then $(x_0/x_2)^2+(x_1/x_2)^2-1=0$ ,$\therefore\ x_0{}^2+x_1{}^2-x_2{}^2=0\ (=g(x_0,x_1,x_2))$ . The expanded equation $g(x_0,x_1,x_2)$ indicates a quadric surface in three-dimensional space, but the original equation $f(x,y)$ does a circle in two-dimensional space. The original equation merely shows a part of its character though, it actually has a quite different character in higher dimensional space. Now, let us take the projective transformation from another viewpoint. Assume that $g(x_0,x_1,x_2)$ is projected into two-dimensional space, then it is $(x_1/x_0)^2+(x_2/x_0)^2-1=x^2+y^2-1=f(x,y)$ . It means that $f(x,y)$ virtually has three variables or three-degrees of freedom in the two-dimension; $g(x_0,x_1,x_2)$ keeps its *original three-degrees of freedom* also after projecting into the two-dimension. Therefore, the relation between $g(x_0,x_1,x_2)$ and $f(x,y)$ is projection into lower dimensional space from our viewpoint. Thus, we take mapping from homogeneous coordinates to non-homogeneous ones as projection into lower dimensions.

## Classification

Thinking of these projections among higher and lower dimensional spaces, we have an opportunity to classify such projections by the corresponding manner of Erlangen program. It is as follows;

1. Projection into higher dimensional space;

Our discussion on the section 2.1 to 2.3 comes under this category.

2. Projection into lower dimensional space;

Mapping from homogeneous coordinates to non-homogeneous ones as discusses on the section 2.4.

3. Projection within the same dimensional space;
Erlangen program

## Expression by Group

We have discussed projective geometry from our viewpoint in the chapter 2 . Additionally, we will also need to pay attention to that it should make a group. Because, the conclusion in the section 2.5. will naturally request our discussion necessity of description by transformation group as well as Erlangen program. Therefore, let us discuss it in this chapter.

## Projective Operator

First of all, let us describe our discussion in the chapter 2 by matrix. Now, we introduce an operator to project any point into higher and lower dimensions. For example, operating between any point *A* of one-dimensional space and the projected point *A'* in two-dimensional one according to **Definition 1** as discussed in the section 2.2, expression with the operator is

$$A' = E_{12}A, \ \therefore \begin{pmatrix} a \\ DT \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a \\ T \end{pmatrix} \qquad (3.1)$$

$E_{12}$ denotes the operator projecting any point of one-dimensional space into two-dimensional one, *D* the matrix element operating the projection, and *T* temporary variable to adjust the point to the two-dimension after the operation. *DT* therefore denotes all real numbers of y-coordinate. Thus, the equation (3.1) stands for **Definition 1**. Operating to project any point of one-dimensional space into three-dimensional one identically, it is

$$A' = E_{13}A = \begin{pmatrix} 1 & & O \\ & D & \\ O & & D \end{pmatrix} \begin{pmatrix} a \\ T \\ T \end{pmatrix} = \begin{pmatrix} a \\ DT \\ DT \end{pmatrix} \qquad (3.2).$$

The operation expressed by the equation (3.2) stands for **Definition 2**. Expressing the inverse operator, then it is as follows;

$$E_{13}^{-1} = \begin{pmatrix} 1 & & O \\ & D^{-1} & \\ O & & D^{-1} \end{pmatrix} \ (3.3), \ \therefore E_{13}E_{13}^{-1} = I \qquad (3.4),$$

where *I* is unit matrix. For example, if inverting the projected point by the operation (3.2) into one-dimensional space, then the expression is

$$A = E_{13}^{-1}A' = \begin{pmatrix} 1 & & O \\ & D^{-1} & \\ O & & D^{-1} \end{pmatrix} \begin{pmatrix} a \\ DT \\ DT \end{pmatrix} = \begin{pmatrix} a \\ T \\ T \end{pmatrix} = a \qquad (3.5$$

Contrarily, let us think of projection into lower dimensions. It is operated by the inverse operator. For example, operation projecting any point of $g(x_0, x_1, x_2)$ into another of $f(x, y)$ as discussed in the section 2.4 is expressed by inverse as follows;

$$x' = E_{32}x = E_{23}^{-1}x = \begin{pmatrix} 1 & & O \\ & 1 & \\ O & & D^{-1} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ D^{-1}x_2 \end{pmatrix} = \begin{pmatrix} x_0/x_2 \\ x_1/x_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.6).$$

It is expression of mapping from homogeneous coordinates to non-homogeneous ones from our viewpoint. The inverse mapping is

$$x = E_{23}x' = \begin{pmatrix} 1 & & O \\ & 1 & \\ O & & D \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ D^{-1}x_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \quad (3.7).$$

Furthermore, general expression of the operator is as follows;

a) If $0 \prec l \prec m$,

$$E_{lm} = diag(\overbrace{1,1,...,1}^{l}, \overbrace{D, D,..., D}^{m-l}) \quad (3.8)$$

$$E_{lm}^{-1} = diag(\overbrace{1,1,...,1}^{l}, \overbrace{D^{-1}, D^{-1},..., D^{-1}}^{m-l}) \quad (3.9)$$

$$\therefore E_{lm}E_{lm}^{-1} = E_{lm}^{-1}E_{lm} = I \quad (3.10)$$

b). If $0 \prec m \prec l$,

$$E_{lm} = E_{ml}^{-1} \quad (3.11)$$

$$\therefore E_{ml}E_{lm} = E_{lm}E_{ml} = I \quad (3.12)$$

Subindices of the operator $l$ and $m$ denote dimensions before and after projection.

**Group of Dimensions**

**Theorem**. The operator $E_{ij}$ as discussed in the former section 3.1 makes a group, it is indicated by formulae as follows ;

i. $E_{ln} = E_{lm}E_{mn}$, ii. $(E_{kl}E_{lm})E_{mn} = E_{kl}(E_{lm}E_{mn})$, iii. $E_{lm}E_{lm}^{-1} = E_{lm}^{-1}E_{lm} = I$,

iv. $l = m \rightarrow E_{lm} = I, \therefore E_{ij}I = IE_{ij} = E_{ij}$.

a₁). If $0 \prec l \prec m \prec n$ (; projecting into higher dimensions,)

$$x' = E_{lm}x = diag(\overbrace{1,1,...,1}^{l}, \overbrace{D, D,..., D}^{m-l})(x_1, x_2, ..., x_l, \overbrace{T, T,..., T}^{m-l})^T$$

$$= (x_1, x_2,..., x_l, \overbrace{DT, DT,..., DT}^{m-l})^T,$$

$$\therefore E_{mn}x' = diag(\overbrace{1,1,...,1}^{m}, \overbrace{D, D,..., D}^{n-m})(x_1, x_2,..., x_l, \overbrace{DT, DT,..., DT}^{m-l}, \overbrace{T, T,..., T}^{n-m})^T$$

$$= (x_1, x_2, ..., x_l, \overbrace{DT, DT, ..., DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,...,1}^{l}, \overbrace{D, D, ..., D}^{n-l})(x_1, x_2, ..., x_l, \overbrace{T, T, ..., T}^{n-l})^T$$

$$= E_{\ln} x .$$

a2). If $0 \prec l \prec n \prec m$ (; projecting into higher dimensions,)

$$x' = E_{lm} x = diag(\overbrace{1,1,...,1}^{l}, \overbrace{D, D, ..., D}^{m-l})(x_1, x_2, ..., x_l, \overbrace{T, T, ..., T}^{m-l})^T$$

$$= (x_1, x_2, ..., x_l, \overbrace{DT, DT, ..., DT}^{m-l})^T ,$$

$$\therefore E_{mn} x' = E_{nm}^{-1} x' = diag(\overbrace{1,1,...,1}^{m}, \overbrace{D^{-1}, D^{-1}, ..., D^{-1}}^{m-n})(x_1, x_2, ..., x_l, \overbrace{DT, DT, ..., DT}^{m-l}, \overbrace{DT, DT, ..., DT}^{m-n})^T$$

$$= (x_1, x_2, ..., x_l, \overbrace{DT, DT, ..., DT}^{n-l}, \overbrace{T, T, ..., T}^{m-n})^T$$

$$= (x_1, x_2, ..., x_l, \overbrace{DT, DT, ..., DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,...,1}^{l}, \overbrace{D, D, ..., D}^{n-l})(x_1, x_2, ..., x_l, \overbrace{T, T, ..., T}^{n-l})^T$$

$$= E_{\ln} x .$$

a3). If $0 \prec m \prec l \prec n$ (; projecting into higher dimensions,)

$$x' = E_{lm} x = E_{ml}^{-1} x = diag(\overbrace{1,1,...,1}^{m}, \overbrace{D^{-1}, D^{-1}, ..., D^{-1}}^{l-m})(x_1, x_2, ..., x_l)^T$$

$$= (x_1, x_2, ..., x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, ..., D^{-1}x_l)^T ,$$

$$\therefore E_{mn} x' = diag(\overbrace{1,1,...,1}^{m}, \overbrace{D, D, ..., D,}^{n-m})(x_1, x_2, ..., x_m, \overbrace{D^{-1}x_{m+1}, D^{-1}x_{m+2} ..., D^{-1}x_l}^{i-m}, \overbrace{T, T, ..., T}^{n-l})^T$$

$$= (x_1, x_2, ..., x_l, \overbrace{DT, DT, ..., DT}^{n-l})^T$$

$$= diag(\overbrace{1,1,...,1}^{l}, \overbrace{D, D, ..., D}^{n-l})(x_1, x_2, ..., x_l, \overbrace{T, T, ..., T}^{n-l})^T$$

$$= E_{\ln} x .$$

b1). If $0 \prec n \prec m \prec l$ (; projecting into lower dimensions,)

$$x' = E_{lm} x = E_{ml}^{-1} x = diag(\overbrace{1,1,...,1}^{m}, \overbrace{D^{-1}, D^{-1}, ..., D^{-1}}^{l-m})(x_1, x_2, ..., x_l)^T$$

$$= (x_1, x_2, ..., x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2}, ..., D^{-1}x_l)^T ,$$

$$\therefore E_{mn} x' = E_{nm}^{-1} x' = diag(\overbrace{1,1,...,1}^{n}, \overbrace{D^{-1}, D^{-1}, ..., D^{-1}}^{m-n}, \overbrace{1,1,...,1}^{l-m})(x_1, x_2, ..., x_m, D^{-1}x_{m+1}, D^{-1}x_{m+2} ..., D^{-1}x_l)^T$$

$$= (x_1, x_2, ..., x_n, D^{-1}x_{n+1}, D^{-1}x_{n+2}, ..., D^{-1}x_l)^T$$

$$= diag(\overbrace{1,1,...,1}^{n},\overbrace{D^{-1},D^{-1},...,D^{-1}}^{l-n})(x_1,x_2,...,x_l)^T$$
$$= E_{nl}^{-1}x = E_{\ln}x\ .$$

b2). If $0 \prec n \prec l \prec m$ (; projecting into lower dimensions,)

$$x' = E_{lm}x = diag(\overbrace{1,1,...,1}^{l},\overbrace{D,D,...,D}^{m-l})(x_1,x_2,...,x_l,\overbrace{T,T,...,T}^{m-l})^T$$
$$= (x_1,x_2,...,x_l,\overbrace{DT,DT,...,DT}^{m-l})^T\ ,$$
$$\therefore E_{mn}x' = E_{nm}^{-1}x' = diag(\overbrace{1,1,...,1}^{n},\overbrace{D^{-1},D^{-1},...,D^{-1}}^{m-n})(x_1,x_2,...,x_l,\overbrace{DT,DT,...,DT}^{m-l})^T$$
$$= (x_1,x_2,...,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},...,D^{-1}x_l,\overbrace{T,T,...,T}^{m-l})^T$$
$$= (x_1,x_2,...,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},...,D^{-1}x_l)^T$$
$$= diag(\overbrace{1,1,...,1}^{n},\overbrace{D^{-1},D^{-1},...,D^{-1}}^{l-n})(x_1,x_2,...,x_l)^T$$
$$= E_{nl}^{-1}x = E_{\ln}x\ .$$

b3). If $0 \prec m \prec n \prec l$ (; projecting into lower dimensions,)

$$x' = E_{lm}x = E_{ml}^{-1}x = diag(\overbrace{1,1,...,1}^{m},\overbrace{D^{-1},D^{-1},...,D^{-1}}^{l-m})(x_1,x_2,...,x_l)^T$$
$$= (x_1,x_2,...,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2},...,D^{-1}x_l)^T\ ,$$
$$\therefore E_{mn}x' = diag(\overbrace{1,1,...,1}^{m},\overbrace{D,D,...,D}^{n-m},\overbrace{1,1,...,1}^{n-l})(x_1,x_2,...,x_m,D^{-1}x_{m+1},D^{-1}x_{m+2}...,D^{-1}x_l)^T$$
$$= (x_1,x_2,...,x_n,D^{-1}x_{n+1},D^{-1}x_{n+2},...,D^{-1}x_l)^T$$
$$= diag(\overbrace{1,1,...,1}^{n},\overbrace{D^{-1},D^{-1},...,D^{-1}}^{l-n})(x_1,x_2,...,x_l)^T$$
$$= E_{nl}^{-1}x = E_{\ln}x\ .$$

The associative law (ii) goes as follow;
since $(E_{kl}E_{lm})E_{mn} = E_{km}E_{mn} = E_{kn}$ and $E_{kl}(E_{lm}E_{mn}) = E_{kl}E_{\ln} = E_{kn}$
from the formula (i), $\therefore (E_{kl}E_{lm})E_{mn} = E_{kl}(E_{lm}E_{mn})$.

Poof of the formula (iii) follows the rule as discussed in the section 3.1(; see also the general expressions from (3.8) to (3.12).)

Finally, about proof of the formula (iv), since it means projection within the same dimensional space, it results in equivalent with unit matrix. □

Furthermore, since the group of dimensions is automorphism from the formula (i), we could consider it as a representation of group.

## Open Problems

These problems below are unsolved problems and suggestions for further works brought by our discussion in this paper:

1. About the practical case of projection from lower dimensional space into higher dimensional one, we need to consider simultaneous equations with graphical viewpoint. For example, let us consider a system of polynomial equations $f(x,y,z)=x^2+y^2+z^2-1=0$ and $g(x,y)=x-y=0$. Although we take two equations $h_1(x,z)=2x^2+z^2-1=0$ and $h_2(y,z)=2y^2+z^2-1=0$ from $f(x,y,z)$ and $g(x,y)$, these equations seem to make no sense. Because, the equations of ellipse $h_1(x,z)$ and $h_2(y,z)$ are not intersections of the equation of sphere $f(x,y,z)$ and the equation of line $g(x,y)$.

However, assuming the equation $g(x,y)$ as a plane according to **Definition 4** as discussed in the section 2.2, it would make sense; When the curve of intersection by $f(x,y,z)$ and $g(x,y)$ is projected into x-z plane and y-z plane, $h_1(x,z)$ and $h_2(y,z)$ are the projected ellipses as shown in Figure 8(; They are drawn only in the first quadrants of graphs.) Hence the coordinates of the curve of intersection by the plane and sphere is ( $h_1(x,z)$ , $h_2(y,z)$ , $z$ ), $\therefore (x,y,z)=(\pm\sqrt{(1-z^2)/2},\pm\sqrt{(1-z^2)/2},z)$ and the range is $-1\le z\le 1$. If $z=0$, then $(x,y,z)=(\pm\sqrt{1/2},\pm\sqrt{1/2},0)$. It actually indicates intersections of the line and sphere from the conventional viewpoint(; by considering $g(x,y,z)=x-y=0,\ z=0$.)

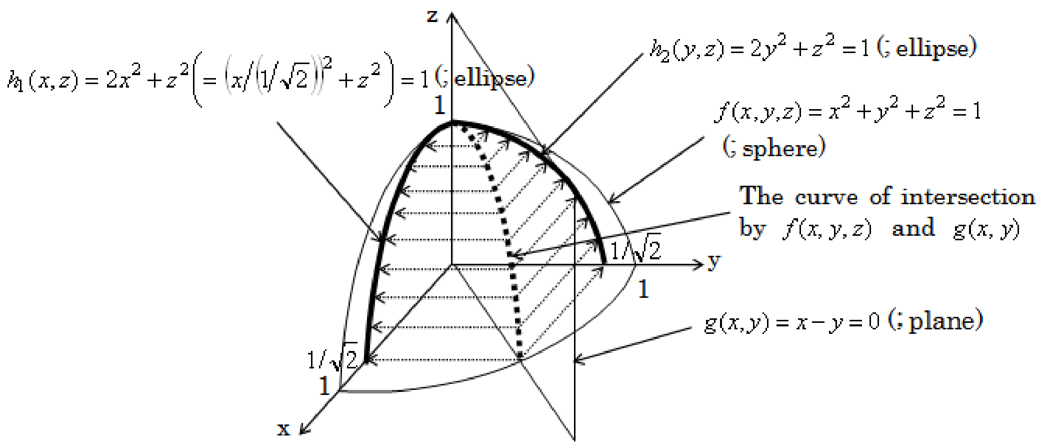


**Figure 8**

Based on the results above, we could say that there are other simultaneous equations supposed to have no solution so far.

2. Our argument of dimensionality could be beneficial also for higher dimensional physics like super string theory. Because, group of dimensions as discussed in the section 3.2 could make issues of higher dimensional spaces such in particle physics more concise and systematic.

## Acknowledgment
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