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2 - Outer Independent Monophonic Domination Number of a Graph

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Abstract

We initiate the study of 2 - outer independent monophonic domination in graphs. A set of vertices M of a graph G is called a 2- outer independent monophonic dominating set of G if M is a monophonic set and every vertex of $V(G)-M$ has at least 2 neighbors in M and the set $(G)-M$. The minimum cardinality of all 2 outer independent monophonic dominating sets of M is called the 2 - outer independent monophonic domination number and is denoted by $\gamma_{2m}^{oi}(G)$. For every pair k,p of integers with $3 \leq k \leq p$, there exists a connected graph G of order p such that $\gamma_{2m}^{oi}(G)=k$. Also for any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $m(G)=a$ and $\gamma_{2m}^{oi}(G)=b$.

Keywords: 2 - outer independent monophonic dominating set, 2 - outer independent monophonic domination number.

AMS Subject classification: 05C12

1. INTRODUCTION

Let $G = (V, E)$ be a graph and n be the number of vertices and m be the number of edges. Thus the cardinality of $V(G) = m$ and the cardinality of $E(G) = n$. We consider a finite undirected graph without loops or multiple edges. For the basic graph theoretic notations and terminology we refer to Buckley and Harary. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G . A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic.

The neighbourhood of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighbourhood. A vertex v is an extreme vertex if the subgraph induced by its neighbourhood is complete. A vertex v in a connected graph G is a cut vertex of G, if $G - v$ is disconnected. A vertex v in a connected graph G is said to be a semi-extreme vertex if $\Delta(< N(v) >) = |N(v)| - 1$. A graph G is said to be semi-extreme graph if every vertex of G is a semi-extreme vertex. An acyclic graph is called a tree.

A subset of $V(G)$ is independent if there is no edge between any two vertices of this set.The independence number of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. A *monophonic set* of G is a set $M \subseteq$ V(G) such that every vertex of G is contained in a monophonic path joining some pair of vertices in M. The *monophonic number* $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a *minimum monophonic set* of G. We say that a subset of V(G) is *independent* if there is no edge between any two vertices of this set. The *independence number* of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. A subset D ⊆ V(G) is a *dominating set* of G if every vertex of V(G) - D has a neighbor in D, while it is a 2-*dominating set* of G if every vertex of V (G) – D has at least two neighbors in D. The *domination* (2-*domination,* respectively) *number* of a graph G, denoted by $\gamma(G)$ ($\gamma_2(G)$ respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G.

2. 2 - OUTER INDEPENDENT MONOPHONIC DOMINATION NUMBER OF A GRAPH

Definition 2.1 A monophonic set $M \subseteq V(G)$ is said to be a 2 outer independent monophonic dominating set, abbreviated 2-OIMDS if it is a monophonic set and $\langle V(G) - S \rangle$ has at least 2 neighbours in S and $V(G) - S$ is independent. The minimum cardinality of a 2 - outer independent monophonic dominating set, denoted by $\gamma_{2m}^{oi}(G)$ is called the 2 - outer independent monophonic domination number of G.

Example 2.2 For the graph given in Fig.2.1, it is clear that $M_1 = \{v_1, v_5\}$ is the monophonic set of G so that $m(G) = 2$. It is verified that the set $M_2 = \{v_1, v_4, v_5\}$ is the minimum 2 - outer independent monophonic dominating set so that $\gamma_{2m}^{oi}(G) = 3$.

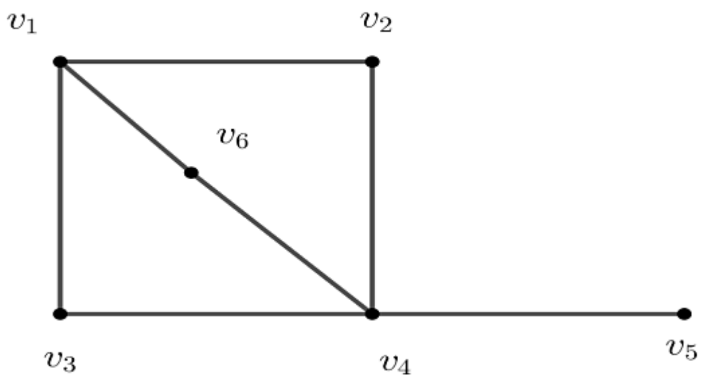

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Fig 1.1

3. PRELIMINARY RESULTS

Proposition 3.1. Let G be a graph. Then

(i) $\gamma_{2m}^{oi}(G) \geq \gamma_m(G)$

(ii) $\gamma_{2m}^{oi}(G) \geq \gamma_m^{oi}(G)$

Proof. (i) Every monophonic domination set of a graph lies in the 2- outer independent monophonic dominating set of a graph and thus $\gamma_m(G) \leq \gamma_{2m}^{oi}(G)$.

(ii) Every 2 - outer independent monophonic dominating set is a outer independent monophonic dominating set and so $\gamma_m^{oi}(G) \leq \gamma_{2m}^{oi}(G)$.

Theorem 3.2. For the complete graph $K_p, p \geq 2$, $\gamma_{2m}^{oi}(K_p) = p$.

Proof. Since every vertex of the complete graph $K_p, p \geq 2$ is an extreme vertex , the vertex set of K_p is the unique 2-OIMD set of K_p. Thus $\gamma_{2m}^{oi}(K_p) = p$.

Theorem 3.3. Let G be a connected graph of order $p \geq 2$, then $\gamma_{2m}^{oi} = p$ if and only if G is the complete graph on p vertices.

Proof. Suppose $G = K_p$.Then by theorem 2.2, $\gamma_{2m}^{oi} = p$. Conversely let $\gamma_{2m}^{oi} = p$. Suppose that G is not a complete graph, then $\gamma_{2m}^{oi} \leq p - 1$, which is a contradiction.

Theorem 3.4. For the complete bipartite graph $G = K_{p,q}$

(i) $\gamma_{2m}^{oi}(G) = 2$ if $p = q = 1$
(ii) $\gamma_{2m}^{oi}(G) = n$ if $p = 1, q \geq 2$
(iii) $\gamma_{2m}^{oi}(G) = \min\{p, q\}$ if $p, q \geq 2$

Observation 3.5. Every pendant vertex of a graph G belongs to every $\gamma_{2m}^{oi}(G)$-set.

Observation 3.6. Each simplicial vertex of G belongs to every 2 - outer independent monophonic dominating set of G.

Proposition 3.7. Let G be a connected graph. We have

(i) $\gamma_{2m}^{oi}(G)$=2 iff $G \in \{P_2, P_3, C_4\}$
(ii) $\gamma_{2m}^{oi}(G) = n$ iff $G = P_2$.

Proof. Obviously $\gamma_{2m}^{oi}(P_2) = 2 = n$ and $\gamma_{2m}^{oi}(P_3) = 2$, $\gamma_{2m}^{oi}(C_4) = 2$.
Assume that for some graph G, we have $\gamma_{2m}^{oi}(G) = 2$. Let M be a $\gamma_{2m}^{oi}(G)$-set. If all vertices of G belong to the set M, then the graph G has two vertices. Hence $G = P_2$.
Let x be a vertex of $V(G) - M$. The vertex x has to be dominated twice, thus $d(x) \geq 2$. Also, the vertex x has to be independent, the vertex x cannot have more than 2 neighbours in G.Thus $G = P_3$.
Assume that for a cycle C_4, we have $\gamma_{2m}^{oi}(G)$= 2. Let $M = \{u, y\}$ be a $\gamma_{2m}^{oi}(G)$-set. Also $\{x, z\}$ be a vertex of $V(G) - M$.The vertices of $V(G) - M$ has to be dominated twice, also the vertices has to be independent. Thus $G = C_4$.

4. BOUNDS

Proposition 4.1. Let G be a graph. For every vertex v of G, we have $\gamma_{2m}^{oi}(G) - 1 \leq \gamma_{2m}^{oi}(G - v) \leq \gamma_{2m}^{oi}(G) + d_G(v) - 1$.

Proof. Let M be a $\gamma_{2m}^{oi}(G)$-set. If $\notin M$, then observe that M is a 2 - OIMDS of the graph $G - v$. Now assume that $v \in M$. Also, $M \cup N_G(v)\backslash\{v\}$ is a 2 - OIMDS of the graph $G - v$. Therefore, $\gamma_{2m}^{oi}(G - v) \leq |M \cup N_G(v)\backslash\{v\}| \leq |M\{v\}| + |N_G(v)| = \gamma_{2m}^{oi}(G) + d_G(v) - 1$.

Now, let M' be any $\gamma_{2m}^{oi}(G - v)$ set. It is easy to see that $M' \cup \{v\}$ is a 2 - OIMDS of the graph G.Thus $\gamma_{2m}^{oi}(G) \leq \gamma_{2m}^{oi}(G - v) + 1$.

Proposition 4.2. Let G be a graph.For every edge e of G we have

$\gamma_{2m}^{oi}(G-e) \in \{\gamma_{2m}^{oi}(G)-1, \gamma_{2m}^{oi}(G), \gamma_{2m}^{oi}(G)+1\}$.

Proof. Let M be a $\gamma_{2m}^{oi}(G)$- set and let $e = xy$ be an edge of G. Since the set $V(G) - M$ is independent, some of the vertices x and y belongs to the set M. Without loss of generality we may assume that $x \in M$. If $y \in M$, then it is easy to see that M is a 2 - OIMDS of the graph $G - e$. If $y \notin M$, then $M \cup \{y\}$ is a 2 - OIMDS of $G - e$. Thus $\gamma_{2m}^{oi}(G-e) \le \gamma_{2m}^{oi}(G) + 1$. Now let M' be a $\gamma_{2m}^{oi}(G-e)$-set. If some of the vertices x and y belongs to the set M', then it is easy to observe that $M' \cup \{x\}$ is a 2 - OIMDS of the graph G.Therefore $\gamma_{2m}^{oi}(G) \le \gamma_{2m}^{oi}(G-e) + 1$.

Proposition. 4.3. Let G be a graph. If $e \notin E(G)$, then

$\gamma_{2m}^{oi}(G+e) \in \{\gamma_{2m}^{oi}(G)-1, \gamma_{2m}^{oi}(G), \gamma_{2m}^{oi}(G)+1\}$.

5. REALISATION RESULTS

Theorem 5.1. For every pair k, p of integers with $3 \le k \le p$, there exists a connected graph G of order p such that $\gamma_{2m}^{oi}(G) = k$.

Proof. Let $V(K_2) = \{x, y\}$ and $V(K_{p-k}) = \{u_1, u_2, \dots, u_{p-k}\}$. Let $H = K_{p-k} + K_2$. Let G be the graph obtained in Fig.1.2 from H by adding $k-2$ new vertices $\{v_1, v_2, \dots, v_{k-2}\}$ and joining each vertex $v_i (1 \le i \le k-2)$ with y. Let $M = \{v_1, v_2, \dots, v_{k-2}\}$ be the set of all extreme vertices of G. It is clear that M is not a monophonic set of G. Also every edge monophonic set contains M.

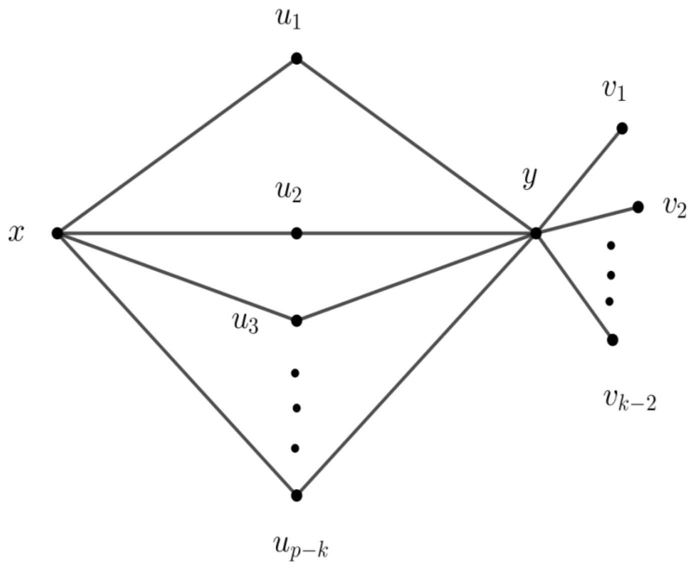

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Fig.1.2

Clearly $M_1 = M \cup \{x\}$ is the unique minimum monophonic set and monophonic domination set of G. But $V(G) - M_1$ is independent. Therefore, $M_2 = M_1 \cup \{y\}$ is the 2 - OIMDS of G, since $V(G) - M_2$ is dominated twice and so that $\gamma_{2m}^{oi}(G) = k - 2 + 1 + 1 = k$.

Theorem 5.2. For any positive integers $2 \leq a \leq b$, there exists a connected graph G such that $m(G) = a$ and $\gamma_{2m}^{oi}(G) = b$.

Proof. If $a = b$, take $G = K_a$.Then by theorem 2.2, $m(G) = a$, $\gamma_{2m}^{oi}(G) = b$.

If $a = 2, b = 3$, then for the graph G given in Fig 1.3. $m(G) = 2$ and $\gamma_{2m}^{oi}(G) = 3$. If $a = 2, b \geq 4$, let G be a graph given in Fig.1.4 obtained from the path on three vertices $P: v_1, v_2, v_3, v_4$ by adding $b - 3$ new vertices $u_1, u_2, \ldots, u_{b-3}$ and joining each $u_i (1 \leq i \leq b - 3)$ with v_1, v_2, v_3, each $u_i's$ are adjacent to each other. It is clear that $M = \{v_1, v_3\}$ is a monophonic set of G so that $m(G) = 2 = a$. Since $V(G) - M$ is independent, $M_1 = M \cup \{v_2, u_1, u_2, \ldots, u_{b-3}\}$ is the 2 - OIMDS of G, so that $\gamma_{2m}^{oi}(G) = 3 + b - 3 = b$.

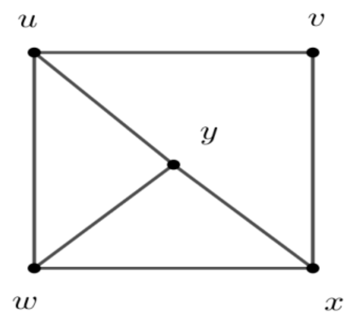

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Fig.1.3

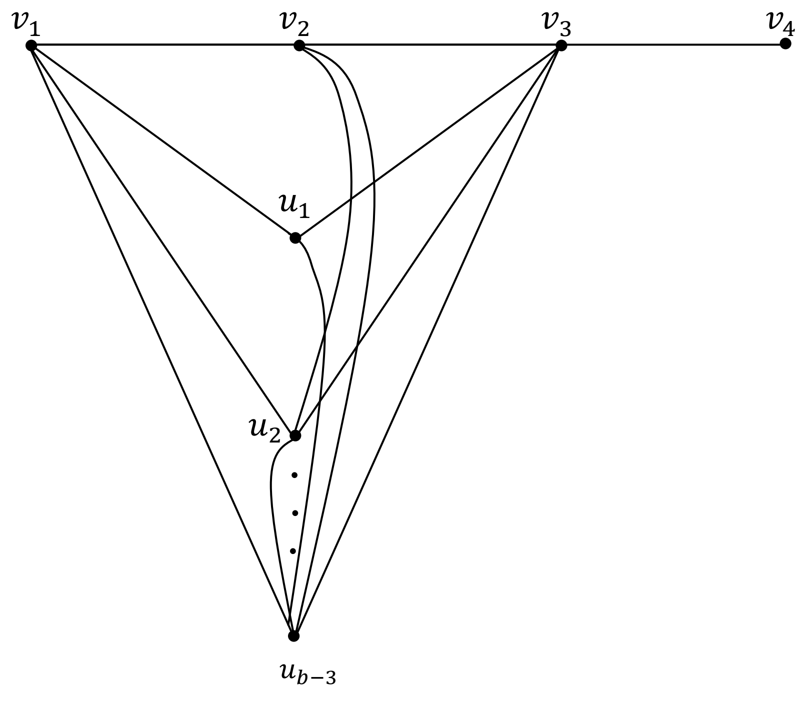

Fig.1.4

If $a \geq 3, b \geq 4, b \neq a+1$. Let G be the graph given in Fig 1.5**,** obtained from the path on three vertices $P: v_1, v_2, v_3$ by adding the new vertices $u_1, u_2, .., u_{b-a-1}$ and $x_1, x_2, \dots, x_{a-1}$ and joining each $u_i (1 \leq i \leq b-a-1)$ with v_1, v_2, v_3 and also joining each $x_i (1 \leq i \leq a-1)$ with v_1 and v_2.Also each $u_i's$ are adjacent to each other. First we show that $m(G) = a$. Since each $x_i (1 \leq i \leq a-1)$ is a simplicial vertex of G , by observation 3.6, each $x_i (1 \leq i \leq a-1)$ belongs to every monophonic set of G. Let $X = \{x_1, x_2, \dots, x_{a-1}\}$. Then X is not a monophonic set of G and so $m(G) \geq a$. However, $X_1 = X \cup \{v_3\}$ is a monophonic set of G and so $m(G) = a$. Next we show that $\gamma_{2m}^{oi}(G) = b$. Since $V(G) - X_1$ is not independent and let $X_2 = X_1 \cup \{v_2, u_1, u_2, \dots, u_{b-a-1}\}$. It is clear that X_2 is the 2 - OIMDS of G, so that $\gamma_{2m}^{oi}(G) = a + b - a - 1 + 1 = b$.

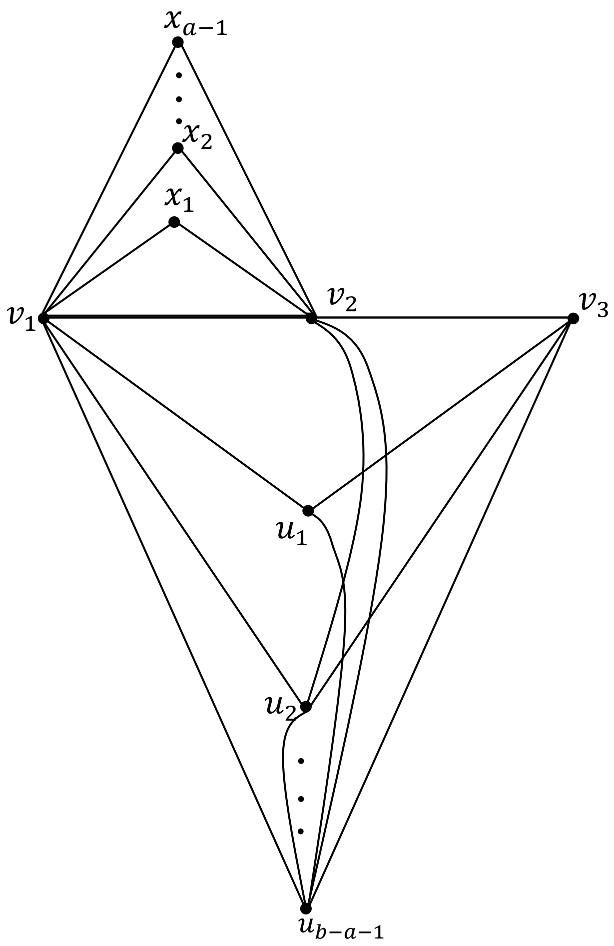

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Fig.1.5

If $a \geq 3, b \geq 4, b = a + 1$. Let $C_4: v_1, v_2, v_3, v_4, v_1$ be a cycle of order 4. Let G be a graph obtained from C_4 by adding the new vertices $x_1, x_2, \ldots, x_{a-1}$ and joining each $x_i (1 \leq i \leq a - 1)$ to v_1.The graph G is given in Fig 1.6**.** Let $X = \{x_1, x_2, \ldots, x_{a-1}\}$ be the set of simplicial vertices of G. It is clear that X is contained in every monophonic set of G by observation 3.6. It is easily seen that X is not a monophonic set of G. Let $X_1 = X \cup \{v_4\}$. It is clear that X_1 is the monophonic set of G and so that $m(G) = a - 1 + 1 = a$. Let $X_2 = X_1 \cup \{v_1\}$. It is easily seen that X_2 is the monophonic dominating set of G and $V(G) - X_2$ is independent and so that X_2 is the 2 - OIMDS of G, so that $\gamma_{2m}^{oi}(G) = a + 1 = b$.

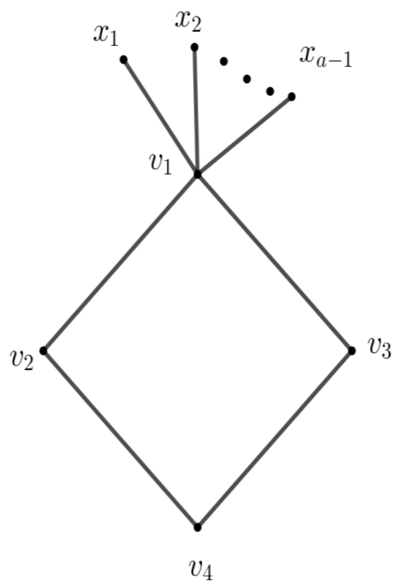

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Fig.1.6

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