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General Manpower Two Phase Machine System With Exponential Production and General Recruitment and General Sales

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Abstract

The machine attended by manpower system has two phases. In phase 1 it has exponential life time distribution with parameter λ_1.If it has not failed in an exponential random time with parameter μ_1, it moves to phase 2. In phase 2 it has failure time distribution which is an exponential with parameter λ_2.If it does not fail in an exponential time with parameter μ_2, it moves to phase 1 and so on it oscillates between phase 1 and phase 2 till it fails. The manpower system is also exposed to a failure process. The entire system fails when one of them fails. During the operation time, the system produces products for sale. When the system fails, the recruitments, the repairs and sales are attended. We study two models. In Model-I, the vacancies caused by departure of employees are filled up one by one and in Model-II, when the operation time is more than a threshold time, the recruitment are done all together and when the operation time is less than the threshold time, the recruitments are done one by one. Joint Laplace transform of the pdf of the operation time, the repair time of the machine, the recruitment time and sales time, their expectations and the covariance of operation time and recruitment time are presented with numerical illustrations .

Keywords: Manpower and Machine system, attrition, shocks, Joint Laplace transform.

Introduction

Nowadays we find that labour has become a buyer market as well as seller market. Any company normally runs on commercial basis wishes to keep only the optimum level of any resources needed to meet company’s requirement at any time during the course of the business and manpower is not an exception. In an organization, the total flow out of the Manpower System (MPS) is termed as shortage, The flow out of the MPS of an organization happens due to resignation, dismissal and death. The shortages that have occurred due to the outflow of manpower should be compensated by recruitment. But recruitment cannot be made frequently since it involves cost. Therefore, the MPS is allowed to undergo Cumulative Shortage Process (CSP). The breakdown point can be interpreted as that point at which immediate recruitment is necessitated

The shortage of MPS depends on individual propensity to leave the organization, which in turn depends on various factors as discussed before. Manpower Planning models have been studied by Grinold and Marshall [2], For statistical approach one may refer to Bartholomew [1]. Lesson [6] has given methods to compute shortages (Resignations, Dismissals, Deaths). Markovian models are designed for shortage and promotion in MPS by Vassiliou [11]. Subramanian. [10] has made an attempt to provide optimal policy for recruitment, training, promotion and shortages in manpower planning models with special provisions such as time bound promotions, cost of training and voluntary retirement schemes. For other manpower models one may refer Setlhare [9]. For three characteristics system in manpower models one may refer to Mohan and Ramanarayanan [8].

Esary et al. [3] have discussed that any component or decice, when exposed to shocks which cause damage to the device or system, is likely to fail when the total accumulated damage exceeds a level called threshold. Stochastic analysis of manpower levels affecting business with varying recruitment rate, are presented by .Hari Kumar, Sekar and Ramanarayanan [4]. Manpower System with Erlang departure and one by one recruitment, is discussed by Hari Kumar [5]. For the study of Semi Markov Models in Manpower planning one may refer Meclean [7].

In this paper, two models of Manpower and Machine System with recruitment, production and sales are considered. In Model-I, the vacancies caused by departure of employees are filled up one by one. In Model-II, when the operation time is more than a threshold time the recruitments are done all together and when the operation time is less than the threshold time, the recruitments are done one by one. Joint Laplace transform of the pdf of the operation time, the repair time of the machine, the recruitment time and sales time, their expectations and the covariance between operation time and the recruitment time are presented with numerical illustrations.

Model -I

Assumptions

[1] Inter-departure time of employees are independent and identically distributed random variables with Cdf F(x) and pdf f(x). The Manpower system collapses with probability p and survives with probability q when an employee departs, with p + q=1.

[2] The machine attended by manpower system has two phases. In phase 1 it has exponential life time distribution with parameter λ_1. If it has not failed

in an exponential random time with parameter μ_1, it moves to phase 2. In phase 2 it has failure time distribution which is exponential with parameter λ_2. If it does not fail in an exponential time with parameter μ_2, it moves to phase 1 and so on it oscillates between phase 1 and phase 2 till it fails. We take $\lambda_1>0$, $\lambda_2>0$, $\mu_1>0$ and $\mu_2>0$ permitting failures in the two phases and transitions from one phase to another phase.

[3] The manpower-machine system fails when one of them fails.

[4] When the system fails the vacancies caused by the departure of employees are filled up one by one with recruitment time V which has Cdf V(y) and pdf v(y).

[5] The repair time of the machine when the system fails due to machine failure is R_i with Cdf R_i (z) and pdf ri(z) for i=1,2 when the failure occurs in phase 1 or phase 2 respectively. When the manpower system fails and the machine is working in phase 2 its maintenance time distribution is R_3 (z) with pdf r_3(z).When the machine is working in phase 1 and the manpower system fails then there is no repair or maintenance.

[6] When the manpower and machine system is in operation, products are produced one at a time with exponential inter-production time distribution with parameter μ.

[7] The sales time G of a product is general with Cdf G(w) and pdf g (w).

[8] When the manpower machine system fails the repairs, recruitments and sales are under taken.

Analysis

To study the above model, pdf of failure times and probability of survival of the two phase machine system are required. It is easier to consider the life time distribution of the machine as a phase-type distribution of time from the starting until absorption in the absorbing state. The two phase machine system has three states namely phase 1, phase 2 and failed state. The transition rate matrix is given by

$$\begin{matrix}1\\2\\3\end{matrix}\begin{pmatrix} -(\lambda_1+\mu_1) & \mu_1 & \lambda_1 \\ \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ 0 & 0 & 0 \end{pmatrix} \quad .....(1)$$

Here state 3 is the absorbing state and we assume at time 0 the state is 1. It is easy to see that the model given above includes several 2-unit systems as special cases. State 1 may be considered as the state in which the two units are good. State 2 may be considered as the state in which there is one good unit and the failed unit is under repair.

The machine system becomes a 2 unit parallel system with repair when $\lambda_1=0$, $\mu_1=2\lambda_2$ and the repair rate is μ_2.

When $\lambda_1=0$, it becomes a standby 2-unit system with repair where μ_1 is the failure rate of a unit when two units are good and λ_2 is the failure of the machine when one unit is good and the repair rate is μ_2.

It becomes a machine with a hyper exponential life time when λ_1=0 and μ_2 =0 with any starting probability vector.

The life time has Coxian distribution when μ_2 =0 and $\mu_1=\alpha(\lambda_1+\mu_1)$ and $\lambda_1=\beta(\lambda_1+\mu_1)$ where $\alpha+\beta=1$.

For the sake of generality we consider λ_1, λ_2 as failure rates and μ_1 and μ_2 as phase transition rates of the machine. We now find various probabilities and probability densities of the system. Let for i = 1,2 and j = 1,2

$P_{ij}(t)$ = P(At time t the machine is in phase j/ at time 0 it is in phase i) ....(2)

It can be seen that $P_{1,1}(t)=e^{-(\lambda_1+\mu_1)t}+\int_0^t \mu_1 e^{-\mu_1 u}e^{-\lambda_1 u}P_{2,1}(t-u)du$ .....(3)

The equation (3) is written considering the fact that the probability that at time t, the machine is in phase 1 given at time t = 0 it was in state 1, is the sum of the probabilities that

1. the machine remains in state 1 during (0,t) and
2. the machine moves to state 2 and at time t, it is in state 1.

It can also be seen that $P_{2,1}(t)=\int_0^t \mu_2 e^{-\mu_2 u}e^{-\lambda_2 u}P_{1,1}(t-u)du$ .....(4)

The right side of equation (4) is the probability that the machine moves to state 1 at u without a failure during (0,u) and it is in state1 at time t.

Using similar arguments we can write the following

$$P_{1,2}(t)=\int_0^t \mu_1 e^{-\mu_1 u}e^{-\lambda_1 u}P_{2,2}(t-u)du \quad .....(5)$$

$$P_{2,2}(t)=e^{-(\lambda_2+\mu_2)t}+\int_0^t \mu_2 e^{-\mu_2 u}e^{-\lambda_2 u}P_{1,2}(t-u)du \quad .....(6)$$

Using Laplace transform the above probabilities may be found explicitly

$$P_{1,1}(t)=\frac{1}{2}e^{-at}\left(e^{bt}+e^{-bt}\right)+\frac{1}{4b}(\lambda_2-\lambda_1+\mu_2-\mu_1)e^{-at}\left(e^{bt}-e^{-bt}\right) \quad ....(7)$$

$$P_{1,2}(t)=\frac{\mu_1}{2b}e^{-at}\left(e^{bt}-e^{-bt}\right) \quad ....(8)$$

$$P_{2,2}(t)=\frac{1}{2}e^{-at}\left(e^{bt}+e^{-bt}\right)+\frac{1}{4b}(\lambda_1-\lambda_2+\mu_1-\mu_2)e^{-at}\left(e^{bt}-e^{-bt}\right) \quad ....(9)$$

$$P_{2,1}(t)=\frac{\mu_2}{2b}e^{-at}\left(e^{bt}-e^{-bt}\right) \quad ....(10)$$

Here $a=\frac{1}{2}\left(\lambda_1+\lambda_2+\mu_1+\mu_2\right)$ and $b=\frac{1}{2}\sqrt{\left(\lambda_1-\lambda_2+\mu_1-\mu_2\right)^2+4\mu_1\mu_2}$

$$a^2-b^2=\lambda_1\lambda_2+\lambda_1\mu_2+\lambda_2\mu_1 \quad ......(11)$$

The machine can fail from state 1 and state 2. The probability density function of time to failure $p_{1,3}(t)$ is written noting the failure rates λ_1,λ_2 in the states as follows

$$p_{1,3}(t) = \lambda_1 P_{1,1}(t) + \lambda_2 P_{1,2}(t) \quad ....(12)$$

The probability distribution function of time to failure T, (time to absorption in state 3 starting at time 0 in state 1) is given by

$$P(T \le t) = \int_0^t p_{13}(u)du$$

$$=1-\left(\frac{a+b-\lambda_1}{2b}\right)e^{-(a-b)t} + \left(\frac{a-b-\lambda_1}{2b}\right)e^{-(a+b)t} \quad ....(13)$$

To study the model 1 we need the joint pdf of four variables $\left(X,\hat{V},\hat{R},\hat{S}\right)$ namely

1) X is the operation time of manpower and machine system, which is the minimum of life times of the two system.
2) $\hat{V}$ is the total recruitment time of employees caused by the departures.
3) $\hat{R}$ is repair time of the machine and
4) $\hat{S}$ is the total sales times.

We note when n employees have left and k products have been produced then

$$\hat{V} = V_1 + V_2 + \ . \ . \ . + V_n \text{ and}$$

$$\hat{S} = G_1 + G_2 + \ . \ . \ . + G_k$$

The repair time is $\hat{R}=R_1,\ R_2,\ R_3$ according as the machine fails in phase 1, the machine fails in phase 2 or the machine is in phase 2, when the manpower system fails.

The joint pdf of $\left(X,\hat{V},\hat{R},\hat{S}\right)$ is

$$f(x,y,z,w) = \left\{ \begin{array}{l} \left(\lambda_1 P_{11}(x) r_1(z) + \lambda_2 P_{12}(x) r_2(z) \sum_{n=0}^{\infty} \left(F_n(x) - F_{n+1}(x)\right) q^n v_n(y)\right) + \\ \left(P_{11}(x) + P_{12}(x) r_3(z)\right) \sum_{n=1}^{\infty} f_n(x) p q^{n-1} v_n(y) \end{array} \right\} \left(\sum_{k=0}^{\infty} e^{-\mu x} \frac{(\mu x)^k}{\underline{|k}} g_k(w)\right) \quad ....(14)$$

There are two term inside the flower bracket written considering the two cases namely

1) Machine fails when manpower system is working with n departures
2) The manpower system fails on the n-th departure and the machine is working.
3) When the machine fails in phase 1 or phase 2 its corresponding repair densities are provided in the terms and when the manpower system fails and the machine is in phase 2 the machine is provided the maintenance and its pdf is presented in the term there. The last bracket presents the case of k productions and k sales. The suffix letter of v(.), g(.) and f(.) indicates the corresponding number fold convolutions of the function with itself. $F_j(x)$ is the

j fold Cdf convolution (Stieltjes convolution of F(x) with itself).$P_{1,1}(x)$ and $P_{1,2}(x)$ are given equation (7) and (8).

We find the quadruple Laplace transform of f(x,y,z,w) as follows

$$f^*(\xi,\eta,\varepsilon,\delta)=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\xi x-\eta y-\varepsilon z-\delta w} f(x,y,z,w)dxdydzdw \quad \ldots (15)$$

Here * indicates Laplace transform.

Now, equation (15) using the structure of equation (14) becomes a single integral as follows.

$$f^*(\xi,\eta,\varepsilon,\delta)= \int_0^\infty e^{-\xi x} e^{-\mu x(1-g^*(\delta))} \left\{ \begin{array}{c} \left(\lambda_1 P_{11}(x) r_1^*(\varepsilon) + \lambda_2 P_{12}(x) r_2^*(\varepsilon) \sum_{n=0}^{\infty} \left(F_n(x) - F_{n+1}(x)\right) q^n v^{*n}(\eta)\right) + \\ \left(P_{11}(x) + P_{12}(x) r_3^*(\varepsilon)\right) \sum_{n=1}^{\infty} f_n(x) p q^{n-1} v^{*n}(\eta) \end{array} \right\} dx \ldots (16)$$

Upon simplification we obtain

$$f^*(\xi,\eta,\varepsilon,\delta)=$$

$$\frac{1}{\left(1-qv^*(\eta) f^*(\chi_1)\right)} \left\{ \begin{array}{l} \frac{\left(1-f^*(\chi_1)\right)}{\chi_1} \left[\left(\frac{\lambda_1}{2} + \frac{\lambda_1}{4b}(\lambda_2 - \lambda_1 + \mu_2 - \mu_1) \right) r_1^*(\varepsilon) + \lambda_2 \frac{\mu_1}{2b} r_2^*(\varepsilon) \right] + \\ pv^*(\eta) f^*(\chi_1) \left[\left(\frac{1}{2} + \frac{1}{4b}(\lambda_2 - \lambda_1 + \mu_2 - \mu_1) \right) + \frac{\mu_1}{2b} r_3^*(\varepsilon) \right] \end{array} \right\} +$$

$$\frac{1}{\left(1-qv^*(\eta) f^*(\chi_2)\right)} \left\{ \begin{array}{l} \frac{\left(1-f^*(\chi_2)\right)}{\chi_2} \left[\left(\frac{\lambda_1}{2} - \frac{\lambda_1}{4b}(\lambda_2 - \lambda_1 + \mu_2 - \mu_1) \right) r_1^*(\varepsilon) - \lambda_2 \frac{\mu_1}{2b} r_2^*(\varepsilon) \right] + \\ pv^*(\eta) f^*(\chi_2) \left[\left(\frac{1}{2} - \frac{1}{4b}(\lambda_2 - \lambda_1 + \mu_2 - \mu_1) \right) - \frac{\mu_1}{2b} r_3^*(\varepsilon) \right] \end{array} \right\} \ldots (17)$$

Here $\chi_1=\xi+\mu\left(1-g^*(\delta)\right)+a-b$ and

$$\chi_2=\xi+\mu\left(1-g^*(\delta)\right)+a+b \quad \ldots.(18)$$

We can find various expectation from (17). We first find E ($\hat{R}$) the expected repair time / maintenance time of the machine.

$$E\left(\hat{R}\right)=-\frac{\partial}{\partial\varepsilon} f^*(\xi,\eta,\varepsilon,\delta)\Big|_{\xi=\eta=\varepsilon=\delta=0}$$

$$E\left(\hat{R}\right)=$$

$$\frac{1}{\left(1-qf*\left(a-b\right)\right)}\left\{\begin{array}{r}\frac{\left(1-f*\left(a-b\right)\right)}{\left(a-b\right)}\left[\frac{\left(a-b\right)}{2b}\left(a+b-\lambda_1\right)E\left(R_1\right)-\frac{\lambda_2\mu_1}{2b}\left(E\left(R_1\right)-E\left(R_2\right)\right)\right]+\\ pf*\left(a-b\right)\frac{\mu_1}{2b}E\left(R_3\right)\end{array}\right\}$$

$$+\frac{1}{\left(1-qf*\left(a+b\right)\right)}\left\{\begin{array}{r}\frac{\left(1-f*\left(a+b\right)\right)}{\left(a+b\right)}\left[\frac{\left(a+b\right)}{2b}\left(\lambda_1-a+b\right)E\left(R_1\right)+\frac{\lambda_2\mu_1}{2b}\left(E\left(R_1\right)-E\left(R_2\right)\right)\right]-\\ pf*\left(a+b\right)\frac{\mu_1}{2b}E\left(R_3\right)\end{array}\right\} \quad ...(20)$$

For further analysis we consider the Laplace transform $f*\left(\xi,\eta,0,\delta\right)$

$$f^*\left(\xi,\eta,\varepsilon,\delta\right)=$$

$$\frac{1}{\left(1-qv^*\left(\eta\right)f*\left(\chi_1\right)\right)}\left\{\frac{\left(1-f*\left(\chi_1\right)\right)}{\chi_1}\left(\frac{a-b}{2b}\left(a+b-\lambda_1\right)\right)+pv^*\left(\eta\right)f*\left(\chi_1\right)\left(\frac{\left(a+b-\lambda_1\right)}{2b}\right)\right\}+$$

$$\frac{1}{\left(1-qv^*\left(\eta\right)f*\left(\chi_2\right)\right)}\left\{\frac{\left(1-f*\left(\chi_2\right)\right)}{\chi_2}\left(\frac{a+b}{2b}\left(\lambda_1-a+b\right)\right)+pv^*\left(\eta\right)f*\left(\chi_2\right)\left(\frac{\lambda_1-a+b}{2b}\right)\right\}$$

....(21)

This is the Laplace transform of pdf $\left(X,\hat{V},\hat{S}\right)$

$$E\left(\hat{S}\right)=-\frac{\partial}{\partial\delta}f^*\left(\xi,\eta,0,\delta\right)\Big|_{\xi=\eta==\delta=0}$$ gives

$$E\left(\hat{S}\right)=\mu E\left(G\right)\frac{\left(1-f*\left(a-b\right)\right)\left(a+b-\lambda_1\right)}{\left(1-qf*\left(a-b\right)\right)2b\left(a-b\right)}+\mu E\left(G\right)\frac{\left(1-f*\left(a+b\right)\right)\left(\lambda_1-a+b\right)}{\left(1-qf*\left(a+b\right)\right)2b\left(a+b\right)}$$

.....(22)

$$E\left(X\right)=-\frac{\partial}{\partial\xi}f^*\left(\xi,\eta,0,\delta\right)\Big|_{\xi=\eta==\delta=0}$$ gives

$$E\left(X\right)=\frac{\left(1-f*\left(a-b\right)\right)\left(a+b-\lambda_1\right)}{\left(1-qf*\left(a-b\right)\right)2b\left(a-b\right)}+\frac{\left(1-f*\left(a+b\right)\right)\left(\lambda_1-a+b\right)}{\left(1-qf*\left(a+b\right)\right)2b\left(a+b\right)} \quad ...(23)$$

Similarly,

$$E\left(\hat{V}\right)=-\frac{\partial}{\partial\eta}f^*\left(\xi,\eta,0,\delta\right)\Big|_{\xi=\eta==\delta=0}$$

We get

$$E\left(\hat{V}\right)=E(V)\left[\frac{(a+b-\lambda_1)\,f^*(a-b)}{\left(1-qf^*(a-b)\right)2b}+\frac{(\lambda_1-a+b)\,f^*(a+b)}{\left(1-qf^*(a+b)\right)2b}\right] \quad ...(24)$$

The product moment of X and $\hat{V}$ is given by

$$E\left(X\hat{V}\right)=\frac{\partial^2}{\partial\xi\partial\eta}f(\xi,\eta,0,\delta)\Big|_{\xi=\eta=\varepsilon=0}$$

We get

$$E\left(X\hat{V}\right)=$$

$$E(V)\left[\begin{array}{l}\frac{(\lambda_1-a-b)}{2b}\frac{f^{*'}(a-b)}{\left(1-qf^*(a-b)\right)^2}+q\frac{(a+b-\lambda_1)}{2b}\frac{f^*(a-b)\left(1-f^*(a-b)\right)}{(a-b)\left(1-qf^*(a-b)\right)^2}+\\ \frac{(a-b-\lambda_1)}{2b}\frac{f^{*'}(a+b)}{\left(1-qf^*(a+b)\right)^2}+q\frac{(\lambda_1-a+b)}{2b}\frac{f^*(a+b)\left(1-f^*(a+b)\right)}{(a+b)\left(1-qf^*(a+b)\right)^2}\end{array}\right] \quad ...(25)$$

The covariance of X and $\hat{V}$ is given by

$$Cov\left(X,\hat{V}\right)=E\left(X\hat{V}\right)-E(X)E\left(\hat{V}\right)$$

We can write down $Cov\left(X,\hat{V}\right)$ using equations (25), (24) and (23)

Model-II

In this section we treat the previous model-1 with all assumption (1), (2), (3), (5), (6),(7) (8) except the assumption (4) for manpower recruitment pattern.

Assumption For Manpower Recruitment

(4.1) When the operation time X is more than a threshold time U, the recruitments are done all together. It is assigned to an agent whose service time V_1 to fill up all vacancies has Cdf $V_1(y)$ and pdf $v_1(y)$.

(4.2) When the operation time X is less than the threshold time U, the recruitments are done one by one and the recruitment time V_2 has Cdf $V_2(y)$ and pdf $v_2(y)$.

(4.3) The threshold U has exponential distribution with parameter θ.

Analysis

Using the argument given for Model-1, the joint pdf of $\left(X,\hat{V},\hat{R},\hat{S}\right)$ (The Operation Time, Recruitment time of employees, Repair time of the machine, Sale Time) may be obtained as follows.

$$f(x,y,z,w)=$$
$$\left\{\begin{array}{l}\left(\lambda_1 P_{1,1}(x)r_1(z)+\lambda_2 P_{1,2}(x)r_2(z)\right)\sum_{n=0}^{\infty}\left(F_n(x)-F_{n+1}(x)\right)q^n\left[\left(1-e^{-\theta x}\right)v_1(y)+e^{-\theta x}v_{2,n}(y)\right]\\ +\left(P_{1,1}(x)+P_{1,2}(x)r_3(z)\right)\sum_{n=1}^{\infty}f_n(x)pq^{n-1}\left[\left(1-e^{-\theta x}\right)v_1(y)+e^{-\theta x}v_{2,n}(y)\right]\end{array}\right\}\times$$
$$\left(\sum_{k=0}^{\infty}e^{-\mu x}\frac{(\mu x)^k}{\lfloor k}g_k(w)\right) \quad ....(26)$$

We use the same arguments given for Model-1 for all terms in equation (26) except the square brackets appearing in first and second terms inside the flower bracket.

The pdf $v_n(y)$ appearing in two places in equation (14) is replaced by $\left(1-e^{-\theta x}\right)v_1(y)+e^{-\theta x}v_{2,n}(y)$ indicating the cases namely, the operation time is greater than the threshold X > U and the operation time is less than the threshold X < U. The function $v_{2,n}(y)$ is the n-fold convolution of $v_2(y)$ with itself.

The Laplace transform of the pdf of four variables $(X,\hat{V},\hat{R},\hat{S})$ is given by

$$f^*(\xi,\eta,\varepsilon,\delta)=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-\xi x-\eta y-\varepsilon z-\delta w}f(x,y,z,w)dxdydzdw \quad ... \quad (27)$$

Here * indicates Laplace transform.

Now, equation (27) using equation (26) reduces to a single integral as follows.

$$f^*(\xi,\eta,\varepsilon,\delta)=$$
$$\int_0^{\infty}e^{-\xi x}e^{-\mu x\left(1-g^*(\delta)\right)}\left\{\begin{array}{c}\left(\lambda_1 P_{1,1}(x)r_1^*(\varepsilon)+\lambda_2 P_{1,2}(x)r_2^*(\varepsilon)\sum_{n=0}^{\infty}\left(F_n(x)-F_{n+1}(x)\right)q^n\right)\\ +\ \left(P_{1,1}(x)+P_{1,2}(x)r_3^*(\varepsilon)\right)\ \sum_{n=1}^{\infty}f_n(x)pq^{n-1}\end{array}\right\}\times$$
$$\left[\left(1-e^{-\theta x}\right)v^*_1(\eta)+e^{-\theta x}v_2^{*n}(\eta)\right]dx \quad ...(28)$$

This reduces to the following after simplification

$$f^*\left(\xi,\eta,\varepsilon,\delta\right)=$$
$$\left[\left(\frac{\lambda_1}{2}+\frac{\lambda_1}{4b}\left(\lambda_2-\lambda_1+\mu_2-\mu_1\right)\right)r_1^*\left(\varepsilon\right)+\lambda_2\frac{\mu_1}{2b}r_2^*\left(\varepsilon\right)\right]\times$$
$$\left\{\frac{\left(1-f^*\left(\chi_1\right)\right)v_1^*\left(\eta\right)}{\chi_1\left(1-qf^*\left(\chi_1\right)\right)}-\frac{\left(1-f^*\left(\chi_1+\theta\right)\right)v_1^*\left(\eta\right)}{\left(\chi_1+\theta\right)\left(1-qf^*\left(\chi_1+\theta\right)\right)}+\frac{\left(1-f^*\left(\chi_1+\theta\right)\right)}{\left(\chi_1+\theta\right)\left(1-qv_2^*\left(\eta\right)f^*\left(\chi_1+\theta\right)\right)}\right\}+$$
$$\left[\left(\frac{1}{2}+\frac{1}{4b}\left(\lambda_2-\lambda_1+\mu_2-\mu_1\right)\right)+\frac{\mu_1}{2b}r_3^*\left(\varepsilon\right)\right]\times$$
$$\left\{\frac{pf^*\left(\chi_1\right)v_1^*\left(\eta\right)}{\left(1-qf^*\left(\chi_1\right)\right)}-\frac{pf^*\left(\chi_1+\theta\right)v_1^*\left(\eta\right)}{\left(1-qf^*\left(\chi_1+\theta\right)\right)}+\frac{pf^*\left(\chi_1+\theta\right)v_2^*\left(\eta\right)}{\left(1-qv_2^*\left(\eta\right)f^*\left(\chi_1+\theta\right)\right)}\right\}+$$
$$\left[\left(\frac{\lambda_1}{2}-\frac{\lambda_1}{4b}\left(\lambda_2-\lambda_1+\mu_2-\mu_1\right)\right)r_1^*\left(\varepsilon\right)-\lambda_2\frac{\mu_1}{2b}r_2^*\left(\varepsilon\right)\right]\times$$
$$\left\{\frac{\left(1-f^*\left(\chi_2\right)\right)v_1^*\left(\eta\right)}{\chi_2\left(1-qf^*\left(\chi_2\right)\right)}-\frac{\left(1-f^*\left(\chi_2+\theta\right)\right)v_1^*\left(\eta\right)}{\left(\chi_2+\theta\right)\left(1-qf^*\left(\chi_2+\theta\right)\right)}+\frac{\left(1-f^*\left(\chi_2+\theta\right)\right)}{\left(\chi_2+\theta\right)\left(1-qv_2^*\left(\eta\right)f^*\left(\chi_2+\theta\right)\right)}\right\}+$$
$$\left[\left(\frac{1}{2}-\frac{1}{4b}\left(\lambda_2-\lambda_1+\mu_2-\mu_1\right)\right)-\frac{\mu_1}{2b}r_3^*\left(\varepsilon\right)\right]\times$$
$$\left\{\frac{pf^*\left(\chi_2\right)v_1^*\left(\eta\right)}{\left(1-qf^*\left(\chi_2\right)\right)}-\frac{pf^*\left(\chi_2+\theta\right)v_1^*\left(\eta\right)}{\left(1-qf^*\left(\chi_2+\theta\right)\right)}+\frac{pf^*\left(\chi_2+\theta\right)v_2^*\left(\eta\right)}{\left(1-qv_2^*\left(\eta\right)f^*\left(\chi_2+\theta\right)\right)}\right\}\quad ...(29)$$

Here $\chi_1=\xi+\mu\left(1-g^*\left(\delta\right)\right)+a-b$ and $\chi_2=\xi+\mu\left(1-g^*\left(\delta\right)\right)+a+b$

Since there is only change in the manpower recruitment pattern to fill up the vacancies caused by manpower loss, $E\left(X\right)$, $\mathrm{E}\left(\hat{R}\right)$ and $\mathrm{E}\left(\hat{S}\right)$ remain the same as of Model -1.

The Laplace transform of pdf of $\left(X,\hat{V}\right)$ is given by

$$f^*(\xi,\eta,0,0)=$$
$$\frac{(a-b)}{2b}(a+b-\lambda_1)\times$$
$$\left\{\frac{(1-f^*(\chi_3))v_1^*(\eta)}{\chi_3(1-qf^*(\chi_3))}-\frac{(1-f^*(\chi_3+\theta))v_1^*(\eta)}{(\chi_3+\theta)(1-qf^*(\chi_3+\theta))}+\frac{(1-f^*(\chi_3+\theta))}{(\chi_3+\theta)(1-qv_2^*(\eta)f^*(\chi_3+\theta))}\right\}+$$
$$\frac{1}{2}(a+b-\lambda_1)\times$$
$$\left\{\frac{pf^*(\chi_3)v_1^*(\eta)}{(1-qf^*(\chi_3))}-\frac{pf^*(\chi_3+\theta)v_1^*(\eta)}{(1-qf^*(\chi_3+\theta))}+\frac{pf^*(\chi_3+\theta)v_2^*(\eta)}{(1-qv_2^*(\eta)f^*(\chi_3+\theta))}\right\}+$$
$$\left(\frac{a+b}{2b}\right)(\lambda_1-a+b)\times$$
$$\left\{\frac{(1-f^*(\chi_4))v_1^*(\eta)}{\chi_4(1-qf^*(\chi_4))}-\frac{(1-f^*(\chi_4+\theta))v_1^*(\eta)}{(\chi_4+\theta)(1-qf^*(\chi_4+\theta))}+\frac{(1-f^*(\chi_2+\theta))}{(\chi_4+\theta)(1-qv_2^*(\eta)f^*(\chi_4+\theta))}\right\}+$$
$$\frac{1}{2b}(\lambda_1-a+b)\times$$
$$\left\{\frac{pf^*(\chi_4)v_1^*(\eta)}{(1-qf^*(\chi_4))}-\frac{pf^*(\chi_4+\theta)v_1^*(\eta)}{(1-qf^*(\chi_4+\theta))}+\frac{pf^*(\chi_4+\theta)v_2^*(\eta)}{(1-qv_2^*(\eta)f^*(\chi_4+\theta))}\right\} \quad ...(30)$$

Here $\chi_3=\xi+a-b$ and $\chi_4=\xi+a+b$ ...(31)

Expected recruitment time is

$$E(\hat{V})=-\frac{\partial}{\partial\eta}f^*(\xi,\eta,0,\delta)\Big|_{\xi=\eta==\delta=0}$$

We get

$$E(\hat{V})=E(V_1)\left[\begin{array}{l}\frac{(a+b-\lambda_1)}{2b}\left(1-\frac{a-b}{a-b+\theta}-\frac{\theta pf^*(a-b+\theta)}{(a-b+\theta)(1-qf^*(a-b+\theta))}\right)+\\ \frac{(\lambda_1-a+b)}{2b}\left(1-\frac{a+b}{a+b+\theta}-\frac{\theta pf^*(a+b+\theta)}{(a+b+\theta)(1-qf^*(a+b+\theta))}\right)\end{array}\right]+$$
$$\mathrm{E}(V_2)\left[\begin{array}{l}\frac{(a+b-\lambda_1)f^*(a-b+\theta)\left(p\theta+(a-b)(1-qf^*(a-b+\theta))\right)}{(a-b+\theta)(1-qf^*(a-b+\theta))^2\,2b}+\\ \frac{(\lambda_1-a+b)f^*(a+b+\theta)\left(p\theta+(a+b)(1-qf^*(a+b+\theta))\right)}{(a+b+\theta)(1-qf^*(a+b+\theta))^2\,2b}\end{array}\right] \quad ...(32)$$

The product moment of X and $\hat{V}$ is given by

$$E\left(X\hat{V}\right)=\frac{\partial^2}{\partial\xi\partial\eta}f\left(\xi,\eta,0,0\right)\Big|_{\xi=\eta=0}$$

We get,

$$E\left(X\hat{V}\right)=$$

$$E(V_1)\frac{(a-b)(a+b-\lambda_1)}{2b}\left[\sum_{i=0}^{1}(-1)^i\left(\frac{(1-f^*(\alpha_i)(1-qf^*(\alpha_i))+p\alpha_i f^{*'}(\alpha_i))}{\left[\alpha_i(1-qf^*(\alpha_i))\right]^2}\right)\right]-$$

$$\frac{(a+b-\lambda_1)}{2b}pE(V_1)\sum_{i=0}^{1}(-1)^i\frac{f^{*'}(\alpha_i)}{(1-qf^*(\alpha_i))^2}+E(V_1)\frac{(a+b)(\lambda_1-a+b)}{2b}\times$$

$$\left[\sum_{i=0}^{1}(-1)^i\left(\frac{(1-f^*(\beta_i)(1-qf^*(\beta_i))+p\beta_i f^{*'}(\beta_i))}{\left[\beta_i(1-qf^*(\beta_i))\right]^2}\right)\right]-$$

$$E(V_2)\frac{(\lambda_1-a+b)}{2b\alpha_1(1-qf^*(\alpha_1))^2}\times$$

$$\left[\frac{f^{*'}(\alpha_1)}{(1-qf^*(\alpha_1))}\left((a-b)(1-qf^*(\alpha_1))+\theta p(1+qf^*(\alpha_1))\right)-(a-b)q\frac{f^*(\alpha_1)}{\alpha_1}(1-f^*(\alpha_1))\right]+$$

$$E(V_2)\frac{(a-b-\lambda_1)}{2b\beta_1(1-qf^*(\beta_1))^2}\times$$

$$\left[\frac{f^{*'}(\beta_1)}{(1-qf^*(\beta_1))}\left((a+b)(1-qf^*(\beta_1))+\theta p(1+qf^*(\beta_1))\right)-(a+b)q\frac{f^*(\beta_1)}{\beta_1}(1-f^*(\beta_1))\right] \quad ...(33)$$

Here $\alpha_0=a-b;\ \alpha_1=a-b+\theta;\ \beta_0=a+b;\ \beta_1=a+b+\theta.$ When $\theta_1=0,$ the coefficeints $\mathrm{E}(V_1)$ becomes zero.

The equation (33) becomes the same as (25).

The covariance of X and $\hat{V}$ is given by

$$Cov\left(X,\hat{V}\right)=E\left(X\hat{V}\right)-E(X)E\left(\hat{V}\right)$$

This can be written using equation (33), (32) and (23)

Numerical Illustration

Model-I

Assumptions

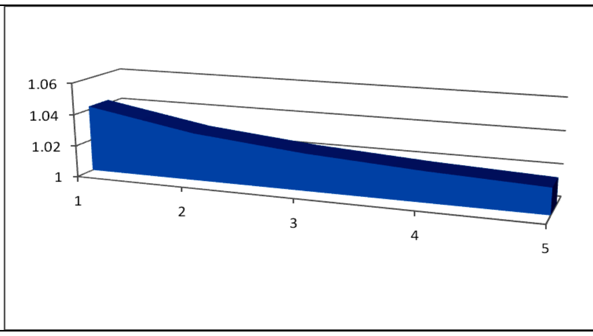
We assume fixed values for

$a=2,$ b=3, $\mathrm{E}(R_1)=10,\ \mathrm{E}(R_2)=07,\ \mathrm{E}(R_3)=13,\ \mathrm{E}(G)=8,\ \mathrm{E}(V)=15,$ p=0.2, q=0.8, $\lambda_1=10,\ \lambda_2=15,\ \mu_1=10,\ \mu_2=15$

We provide the different values for the parameter $\delta=10,\ 20, 30,\ 40,\ 50$

Table and Graph of $E\left[X\right]$

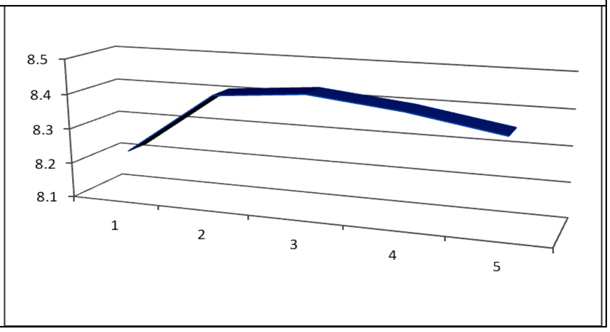
δ	$E\left[X\right]$	Graph of $E\left[X\right]$
10	1.042105	
20	1.030075	
30	1.023392	
40	1.019139	
50	1.016194	



As the value of δ increases, the value of $E\left[X\right]$ decreases .

Table and Graph of $E\left[\hat{R}\right]$

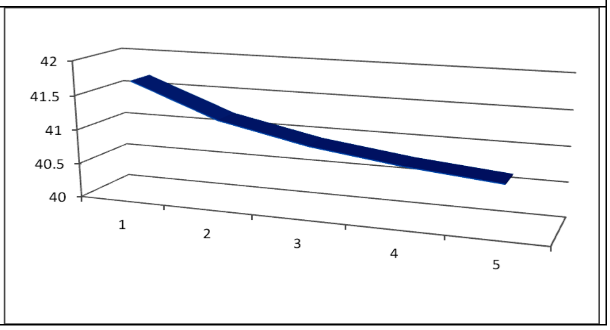
δ	$E\left[\hat{R}\right]$	Graph of $E\left[\hat{R}\right]$
10	8.227868	
20	8.408163	
30	8.429168	
40	8.400627	
50	8.356606	



As the value of δ increases, the value of $E\left[\hat{R}\right]$ increases.

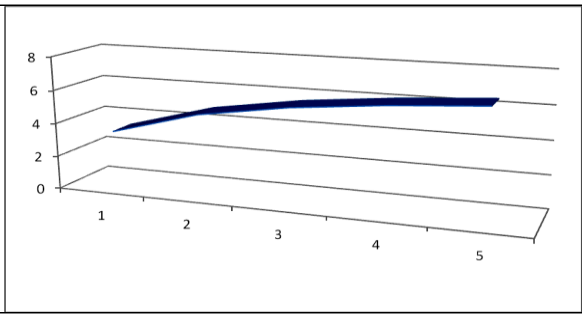
Table and Graph of $E\left[\hat{S}\right]$

δ	$E\left[\hat{S}\right]$	Graph of $E\left[\hat{S}\right]$
10	41.68421	
20	41.20301	
30	40.93567	
40	40.76555	
50	40.64777	



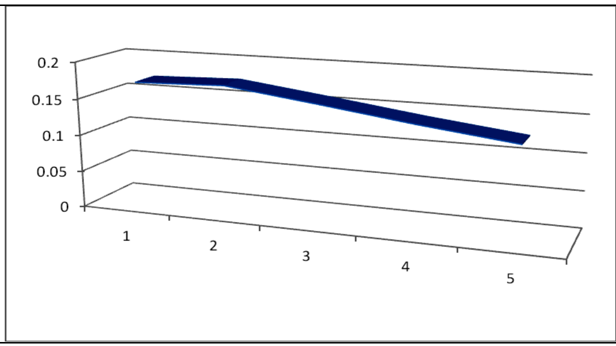
As the value of δ increases, the value of $E\left[\hat{S}\right]$ decreases.

Table and Graph of $E\left[\hat{V}\right]$

δ	$E\left[\hat{V}\right]$	Graph of $E\left[\hat{V}\right]$
10	3.368421	
20	4.81203	
30	5.614035	
40	6.124402	
50	6.477733	

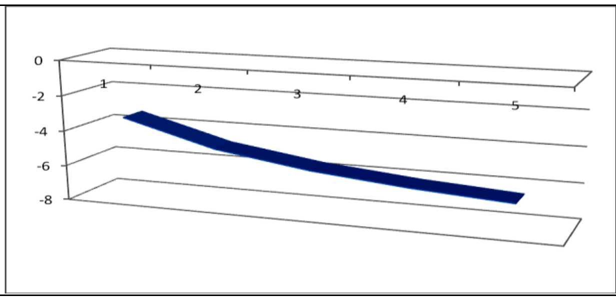
As the value of δ increases, the value of $E\left[\hat{V}\right]$ increases .

Table and Graph of $E\left[X\hat{V}\right]$

δ	$E\left[X\hat{V}\right]$	Graph of $E\left[X\hat{V}\right]$
10	0.170194	
20	0.173667	
30	0.157587	
40	0.140656	
50	0.125883	

As the value of δ increases, the value of $E\left[X\hat{V}\right]$ decreases.

Table and Graph of $Cov\left[X\hat{V}\right]$

δ	$Cov\left[X\hat{V}\right]$	Graph of $Cov\left[X\hat{V}\right]$
10	-3.34006	
20	-4.78309	
30	-5.58777	
40	-6.10096	
50	-6.45675	

As the value of δ increases, the value of the co variance between operation time and the recruitment decreases.

Model-II

Assumptions

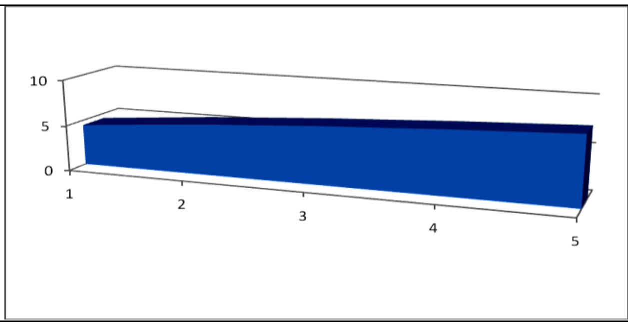
We assume fixed values for

$a=2$, b=3, $E(R_1)=10$, $E(R_2)=07$, $E(R_3)=13$, $E(G)=8$, $E(V)=15$, p=0.2, q=0.8, $\theta=25$

$\lambda_1=10$, $\lambda_2=15$, $\mu_1=10$, $\mu_2=15$

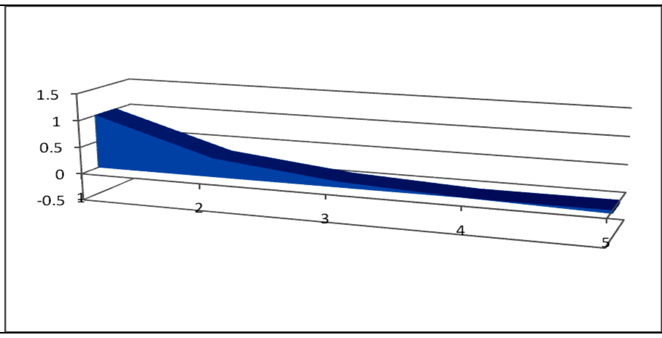
We provide the different values for the parameter $\delta=10,\ 20, 30,\ 40,\ 50$

Table and Graph of $E\left[\hat{V}\right]$

δ	$E\left[\hat{V}\right]$	Graph of $E\left[\hat{V}\right]$
10	4.470486	
20	5.484694	
30	6.274414	
40	6.905864	
50	7.421875	

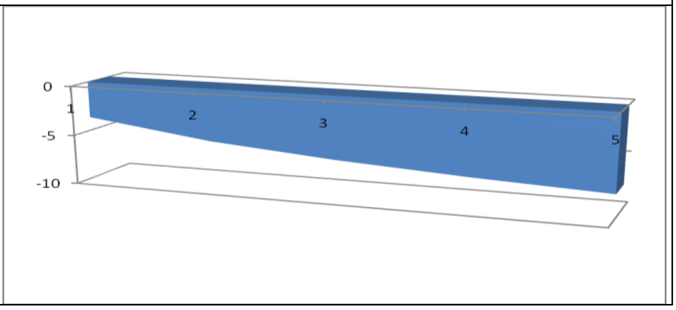
As the value of δ increases, the value of $E\left[\hat{V}\right]$ increases .

Table and Graph of $E\left[X\hat{V}\right]$

δ	$E\left[X\hat{V}\right]$	Graph of $E\left[X\hat{V}\right]$
10	1.0141092	
20	0.352084	
30	0.108426	
40	-0.00287	
50	-0.06192	

As the value of δ increases, the value of $E\left[X\hat{V}\right]$ decreases .

Table and Graph of $Cov\left[X\hat{V}\right]$

<table>
<tr><th>δ</th><th>$Cov\left[X\hat{V}\right]$</th><th>Graph of $Cov\left[X\hat{V}\right]$</th></tr>
<tr><td>10</td><td>-3.64463</td><td rowspan="5">
</td></tr>
<tr><td>20</td><td>-5.29756</td></tr>
<tr><td>30</td><td>-6.31276</td></tr>
<tr><td>40</td><td>-7.04091</td></tr>
<tr><td>50</td><td>-7.60399</td></tr>
</table>

As the value of δ increases, the value of $Cov\left[X\hat{V}\right]$ decreases.

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