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B-spline Approach for Solving Boundary Value Problems

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Abstract

Numerical methods for boundary value problems have been an important area for research in Mathematics for last few decades. Spline based methods have been shown to be very efficient and easy to use tools for the same. The present paper surveys the research articles published during last ten years related to B-spline based computational techniques for numerical solution of various types of boundary value problems. The paper depicts the development and chronological advancement of various B-spline methods for BVPs.

Key Words: Boundary Value Problems (BVPs), B-spline, Nodal points, Singular point, Singular Perturbation Problem.

Introduction

Many real world phenomena give rise to boundary value problems which are differential equations with given boundary conditions. Sometimes it is possible to find an exact formula for the solution of a differential equation. For example, the solution might be expressed in terms of the data as a combination of elementary functions or as a trigonometric or power series. However, this approach only works on a relatively small class of differential equations. In more realistic models, solutions of differential equations cannot be found explicitly in terms of known functions, and the alternative is to determine an approximate solution for given data through numerical computations. Spline (piecewise polynomials) based numerical methods provide an effective tool for solution of various types of boundary value problems. Out of these spline methods, techniques based on B-splines have been shown to be particularly very significant by many authors. The present paper compiles the précis of research papers from last ten years in which B-spline has been used for boundary value problems including singular and singularly perturbed BVPs.

Anthology of Research Articles Based on B-Splines For BVPS

Kadalbajoo and Kumar in the paper [1] in 2005 have proposed spline solution of singular boundary value problems. They consider finding the solution of singular BVP having regular singularity given by

$$y''(x)+P_1(x)y'(x)+P_2(x)y(x)=0,\ 0\le x\le 1 \tag{1}$$

s.t. $y'(0)=\alpha \ \&\ y(1)=\beta$

Functions P_1 and P_2 are not analytic at $x=0$. It gives singularity at $x=0$. To remove the singularity authors used Chebychev economization near the singularity. For finding the numerical solution, B-spline method is used on resulting regular BVP.

Remarks

The results obtained by this method are better than using the finite difference method with the same number of knots. The method gives $O(h^4)$ accuracy.

In the paper [2] in 2006, the authors N. Caglar and H. Caglar considered a class of singular two-point boundary value problems of the form

$$y''(x)+\frac{k}{x}y'(x)+b(x)y(x)=c(x), 0<x<1,\ k\ge 1 \tag{2}$$

subject to the boundary conditions

$$y'(0)=0 \ \&\ y(1)=\beta \tag{3}$$

Since $x=0$ is singular point, they modify the equation (2) at $x=0$ by L'Hospital rule. Then boundary value problem (2) is transformed into

$$y''(x)+p(x)y'(x)+q(x)y(x)=r(x) \tag{4}$$

$$y'(0)=0 \tag{5}$$

$$y(1)=\beta \tag{6}$$

Where

$$p(x)=\begin{cases}0 & ,x=0\\ \dfrac{k}{x} & ,x\neq 0\end{cases},\quad q(x)=\begin{cases}\dfrac{b_0}{k+1} & ,x=0\\ b(x) & ,x\neq 0\end{cases} \text{ and}$$

$$r(x)=\begin{cases}\dfrac{c_0}{k+1} & ,x=0\\ c(x) & ,x\neq 0\end{cases}$$

Now authors give a spline method for solving (4). Let

$$S(x)=\sum_{i=1}^{n+2} C_i B_{3,i}(x) \tag{7}$$

be an approximate solution of Eq. (4), where C_j are unknown real coefficients and $B_{3,j}(x)$ are third-degree B-spline functions. Let $x_0, x_1, \ldots, x_n$ be $n+1$ grid points in the interval $[a,b]$, so that $x_i = a+ih; i=0,1,\ldots,n$, $h=\frac{b-a}{n}$

It is required that the approximate solution (7) satisfies the differential equation at the points $x=x_i$. Putting (7) in (4), it follows that

$$S''(x)+p(x)S'(x)+q(x)S(x)=r(x) \tag{8}$$

This leads to the linear system given by

$$\sum_{i=1}^{n+2} C_i B_i'(x)=y'(0) \text{ for } x=0 \tag{9}$$

$$\sum_{i=1}^{n+2} C_i B_i(x)=y(1) \text{ for } x=1 \tag{10}$$

$$\sum_{i=1}^{n+2} C_i B_i''(x)+p(x)\sum_{i=1}^{n+2} C_i B_i'(x)+q(x)\sum_{i=1}^{n+2} C_i B_i(x)=r(x) \tag{11}$$

(for $x=0,h,2h,\ldots,1$)

The approximate solution is obtained by solving above system.

Remarks

A B-spline method has been considered for the numerical solution of singular boundary value problems. The third-degree B-spline was tested on one homogeneous singular boundary value problem and three non-homogenous singular boundary value problems. Numerical results showed that the present method approximates the exact solution very well.

In the paper [3] in 2007, the authors Kadalbajoo and Kumar have proposed B-spline solution for singular boundary value problems given as

$$u''(x)+\frac{k}{x}u'(x)=f(x,u(x)), \text{ where } 0<x\le 1 \tag{12}$$

with boundary conditions

$$y'(0)=0 \tag{13}$$

$$y(1)=\beta \tag{14}$$

If the range of the independent variable x is $[0,1]$, choose equidistant points represented as $\pi = \{x_0, x_1, ..., x_N\}$ such that $x_0 = 0$ and $x_N = 1$ and $h = \frac{1}{N}$ is the uniform spacing. Now, define the basis function for $i = 0,1,2,...,N$

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3 & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3 & x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3 & x \in [x_i, x_{i+1}] \\ (x_{i+2} - x)^3 & , x \in [x_{i+1}, x_{i+2}] \\ 0 & , otherwise \end{cases} \tag{15}$$

Let $S(x)$ be the B-spline interpolating function $u(x)$ at the nodal points. Then

$S(x)$ can be written as $S(x) = \sum_{k=-1}^{N+1} c_k B_k(x)$ (16)

Now putting Eq. (16) in given BVP and satisfying interpolation conditions, authors get a system of equations.

At the singular point $x_0 = 0$, modification of problem is done with L'Hospital rule and then B-spline method is applied.

Remarks

Authors show that results obtained by this method are better than using the finite difference method. They have solved linear as well as nonlinear problems. Also, this method produces a spline function which may be used to obtain the solution at any point in the range, whereas the finite-difference methods and the invariant imbedding methods give the solution only at the chosen knots.

Kadalbajoo and Kumar in their paper [4] constructed a fitted mesh B-spline collocation method for a singularly perturbed boundary value problem for a differential-difference equation containing negative shift in the differentiated term

$$\varepsilon y''(x) + p(x)y'(x - \delta) + q(x)y(x) = r(x) \tag{17}$$

with the boundary conditions

$$y(x) = \varphi(x) \text{ for } -\delta \le x \le 0, \ y(1) = \gamma \tag{18}$$

They showed that the method has almost second-order parameter-uniform convergence. Kadalbajoo again with Gupta and Awasthi [5] in the same year proposed a numerical method for solving singularly perturbed parabolic convection-diffusion problem. The method comprises a standard implicit finite difference scheme to discretize in temporal direction on uniform mesh by means of Rothe's method and B-spline collocation method in spatial direction on piecewise uniform mesh of Shishkin type. Authors showed that the method is unconditionally stable.

Dag and Sahin [6] considered the solution of the singularly perturbed problems of the form

$$-\varepsilon u'' + p(x)u' + q(x)u = f(x),\ 0 \le x \le 1 \tag{19}$$

with the boundary conditions

$$u(0) = \lambda,\ u(1) = \beta \tag{20}$$

In this article, they used the finite element method with the quadratic and the cubic B-splines. After giving the expressions of the mentioned B-splines over the geometrically graded mesh authors applied the collocation method. To be able to use the quadratic B-splines in the collocation method, the setting $-u' = v$ gives a first order system of equations for equation (19). This system can be solved by employing the quadratic B-spline collocation method. Numerical results are illustrated for some test problems. According to authors, in getting the numerical solution of the differential equations having boundary layers, B-spline collocation methods over the geometrically graded mesh are advisable.

In the paper [7] in 2010, the authors Viswanadham, Krishna and Koneru have proposed a quintic B-spline solution of a general fourth order linear boundary value problem given by

$$a_0(x)y^4 + a_1(x)y''' + a_2(x)y'' + a_3(x)y' + a_4(x)y = b(x)\ ,\ a < x < b \tag{21}$$

with the boundary conditions

$$y(a) = A_0, y(b) = B_0, \tag{22}$$

$$y'(a) = A_1, y'(b) = B_1 \tag{23}$$

or $y(a) = A_0, y(b) = B_0,$ (24)

$$y''(a) = A_2, y''(b) = B_2 \tag{25}$$

where $A_0, A_1, A_2, B_0, B_1, B_2$ are real finite constant and $a_0, a_1, a_2, a_3, a_4, b$ are all continuous functions defined on the interval $[a,b]$.

To solve the boundary value problem (21) by the Galerkin method with quintic B-splines as basis functions, authors take the approximation for $y(x)$ as

$$y(x) = \sum_{j=-2}^{n+2} \alpha_j B_j(x) \tag{26}$$

The existence of the sixth degree spline interpolate $S(x)$ to a function in a closed interval $[a,b]$ for spaced knots $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ is established by constructing it. The construction of $S(x)$ is done with the help of the sixth degree B-splines.

Introduce twelve additional knots $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}$ such that $x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$ and $x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6}$. Now the sixth degree B-splines $B_j(x)$ are defined as

$$B_i(x) = \begin{cases} \displaystyle\sum_{r=i-3}^{i+4} \frac{(x_r - x)_+^6}{\pi'(x_r)} & , x \in [x_{i-3}, x_{i+4}] \\ 0 & , otherwise \end{cases} \tag{27}$$

where

$$(x_r - x)_+^5 = \begin{cases} (x_r - x)^5 & , x_r \ge x \\ 0 & , x_r \le x \end{cases} \text{ and } \pi(x) = \mathop{\Pi}_{r=i-3}^{i+3} (x - x_r)$$

Using the quintic B-splines, they get a system of equations in matrix form as

$$A\alpha = b$$

where $a = [\alpha_{ij}]$, $b = [b_i]$ and $\alpha = [\alpha_{-1}\alpha_0\alpha_1 \dots \alpha_{n+1}]^T$

Remarks

The proposed method is applied to solve a several number of linear and nonlinear problems to test the efficiency of the method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of their paper is to present a simple and accurate method to solve a fourth order boundary value problem.

The same authors in the same year in [8] have developed a collocation method with sixth degree B-splines as basis functions for getting the numerical solution of fifth order special case boundary value problems, which are in the form

$$y^{(5)}(x) + f(x)y(x) = g(x) \text{ , } a < x < b \tag{28}$$

subject to

$$y(a) = a_0, y(b) = b_0, y'(a) = a_1, y'(b) = b_1, y''(a) = a_2 \tag{29}$$

where a_0, a_1, a_2, b_0, b_1 are finite real constants and $f(x), g(x)$ are continuous functions on $[a,b]$.

The existence of the sixth degree spline interpolate $S(x)$ to a function in a closed interval $[a,b]$ for spaced knots $a = x_0 < x_1 < x_2 < \dots < x_n = b$ is established by constructing it. The construction of $S(x)$ is done with the help of the sixth degree B-splines.

Introduce twelve additional knots $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}, x_{n+6}$ such that $x_{-6} < x_{-5} < x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0$ and $x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4} < x_{n+5} < x_{n+6}$.

Now the sixth degree B-splines $B_i(x)$ are defined by

$$B_i(x)=\begin{cases}\displaystyle\sum_{r=i-3}^{i+4}\frac{(x_r-x)_+^6}{\pi'(x_r)} & ,x\in[x_{i-3},x_{i+4}]\\ 0 & ,otherwise\end{cases} \tag{30}$$

where

$$(x_r-x)_+^6=\begin{cases}(x_r-x)^6 & ,x_r\geq x\\ 0 & ,x_r\leq x\end{cases} \text{ and } \pi(x)=\mathop{\Pi}_{r=i-3}^{i+4}(x-x_r)$$

Authors defined the approximation for $y(x)$ as

$$y(x)=\sum_{j=-3}^{n+2}\alpha_j B_j(x) \tag{31}$$

where α_j 's are the nodal parameters to be determined.

Remarks

In this paper, authors have developed a collocation method with sixth order B-splines as basis functions to solve fifth order special case boundary value problems. In the collocation method, they have selected the mesh points as collocation points. The method is tested for solving both linear and nonlinear boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions.

In the paper [9] in 2010, the authors Viswanadham and Krishna have proposed a sixth degree B-spline solution of a sixth order boundary value problem of the type

$$y^{(6)}(x)+f(x)y(x)=g(x) \tag{32}$$

subject to

$$y(a)=a_0,y(b)=b_0,y'(a)=a_1,y'(b)=b_1,y''(a)=a_2,y''(b)=b_2 \tag{33}$$

where a_0,a_1,a_2,b_0,b_1,b_2 are finite real constants and $f(x),g(x)$ are continuous functions on $[a,b]$. They used the seventh degree B-splines $B_i(x)$'s defined by

$$B_i(x)=\begin{cases}\displaystyle\sum_{r=i-4}^{i+4}\frac{(x_r-x)_+^7}{\pi'(x_r)} & ,x\in[x_{i-4},x_{i+4}]\\ 0 & ,otherwise\end{cases} \tag{34}$$

where $(x_r-x)_+^7=\begin{cases}(x_r-x)^7 & ,x_r\geq x\\ 0 & ,x_r\leq x\end{cases}$ and $\pi(x)=\mathop{\Pi}_{r=i-4}^{i+4}(x-x_r)$

Remarks:
In this paper, authors have developed a collocation method with septic B-splines as basis functions to solve a sixth order special case boundary value problem. The proposed method is applied to solve three linear problems and two non-linear problems to test the efficiency of the proposed method. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature.

In the paper [10] in 2011, the authors Feng-Gong Lang and Xiao-Ping Xu have proposed numerical solution of linear fifth order boundary value problems by using cubic B-splines. They consider a fifth-order boundary value problem of the form

$$y^5(x) + f(x)y(x) = g(x)\ ,\ a < x < b \tag{35}$$

subject to

$$y(a) = a_0, y(b) = b_0\, , y'(a) = a_1, y'(b) = b_1\, , y''(a) = a_2.$$

For a given function *y(x)* there exists a unique cubic spline

$$s(x) = \sum_{i=-1}^{n+1} c_i B_i(x)$$

Where $B_i(x)$'s define cubic B-splines. Now, they derive y and its derivatives upto fifth order in terms of this spline to get tables below.

	$y(x_j)$	$y'(x_j)$	$y''(x_j)$	$y'''(x_j)$
Approximate value	$s(x_j)$	$s'(x_j) = m_j$	$s''(x_j) = M_j$	$s'''(x_j) = \frac{M_{j+1} - M_{j-1}}{2h}$
Representation in c_j	$\frac{c_{j-1} + 4c_j + c_{j+1}}{6}$	$\frac{c_{j+1} - c_{j-1}}{2h}$	$\frac{c_{j-1} - 2c_j + c_{j+1}}{h^2}$	$\frac{c_{j+2} - c_{j+1} + 2c_{j-1} - c_{j-2}}{2h^3}$
Error order	$O(h^4)$	$O(h^4)$	$O(h^2)$	$O(h^2)$

	$y^4(x_j)$	$y^5(x_j)$
Approximate value	$s^4(x_j) = \frac{M_{j+1} - 2M_j + M_{j-1}}{h^2}$	$s^5(x_j) = \frac{M_{j+2} - 2M_{j+1} + 2M_{j-1} - M_{j-2}}{2h^3}$
Representation in c_j	$\frac{c_{j+2} - 4c_{j+1} + 6c_j - 4c_{j-1} - c_{j-2}}{h^4}$	$\frac{c_{j+3} - 4c_{j+2} + 5c_{j+1} - 5c_{j-1} + 4c_{j-2} - c_{j-3}}{h^4}$
Error order	$O(h^4)$	$O(h^2)$

Using these in given BVP, the resulting linear system can be written in matrix notations as

$$(A + h^5 FB)\tilde{C} = D, \tag{36}$$

Remarks

In this paper, authors develop a cubic B-spline method for solving fifth order boundary value problems. Their method is very encouraging with second order convergence. The given numerical results show cubic spline is effective in approximating the analytic solution and its derivatives. Authors believe that cubic spline can also be applied to study other higher order boundary value problems. In the paper [11] in 2011, they also proposed a quartic B-spline collocation method for linear and nonlinear fifth order boundary value problems.

In the paper [12] in 2011 by the authors Korkmaz, Aksoy and Dag a quartic B-spline differential quadrature method is constructed to obtain numerical solution of the nonlinear Burgers' Equation. The weighting coefficients of the derivative approximations are determined by solving some linear algebraic equation systems with 4-banded coefficient matrix. Once the weighting coefficients are determined, the Burgers' equation is discretized in space by using the differential quadrature derivative approximations. Resultant ordinary differential equation system is integrated in time by using the Runge-Kutta method of order four. In order to show the validity of the method, the shock wave and the sinusoidal disturbance solutions of the Burgers' equation are selected as test problems.

In the paper [13] in 2014, the authors Viswanadham and Raju developed a collocation method with sextic B-splines as basis functions for getting the numerical solution of a general linear eighth order boundary value problem

$$\begin{aligned}&a_0(x)y^{(8)}(x)+a_1(x)y^{(7)}(x)+a_2(x)y^{(6)}(x)+a_3(x)y^{(5)}(x)+a_4(x)y^{(4)}(x)+\\&a_5(x)y'''(x)+a_6(x)y''(x)+a_7(x)y'(x)+a_8(x)y(x)=b(x)\end{aligned} \tag{37}$$

where $c<x<d$

subject to the boundary conditions

$$y(c)=A_0,y(d)=C_0,y'(c)=A_1,y'(d)=C_1, \tag{38}$$

$$y''(c)=A_2,y''(d)=C_2,y'''(c)=A_3,y'''(d)=C_3, \tag{39}$$

where $A_0,C_{0,}A_1,C_{1,}A_2,C_{2,}A_3,C_3$ are finite real constants and $a_0(x),a_1(x),a_2(x),a_3(x),$

$a_4(x),a_5(x),a_6(x),a_7(x),a_8(x)$ and $b(x)$ are all continuous functions defined on the interval $[a,b]$.

In the paper [14] in 2015, the authors Sonali Saini and H. K. Mishra have proposed a reliable algorithm for third-order singularly perturbed boundary value problem of the form:

$$\begin{aligned}&Ly(x)=-\varepsilon y'''(x)+u(x)y(x)=f(x),\\&y(0)=\alpha,y(1)=\beta,y'(0)=\gamma.\end{aligned} \tag{40}$$

Let $y(x)=r(x)=\sum_{i=-4}^{n-1} c_i B_i(x)$ be be the Quartic B-spline function at the nodal points.

To solve the third-order boundary value problems $B_i, B_i^{'}, B_i^{''}, B_i^{'''}$ are evaluated at nodal points. Now, differentiating *r(x)* they get the values of $r^{'}, r^{''}, r^{'''}$ and putting these values in equation (40) with boundary conditions they get a system of $N+4$ equations with $N+4$ unknowns.

Remarks

In this paper, Quartic B-spline method for solving third-order self-adjoint singularly perturbed boundary value problem is proposed which is very efficient and its implementation is also very easy.

Conclusion

B-spline functions give simple and practical methods to solve boundary value problems. It is more advantageous than other available computational techniques. In comparison with the finite difference methods, B-spline solution has its own advantages. For example, once the solution has been computed the information required for spline interpolation between mesh points is available. This survey paper presented the development and chronological advancement of various B-spline methods for different classes of boundary value problems. This paper would be helpful to researchers working in this area to design their new improved numerical methods for the solution of boundary value problems.

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