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Symmetric properties for the degenerate q-tangent polynomials associated with p-adic integral on $\mathbb{Z}_p$

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Abstract

In [5], we studied the degenerate q-tangent numbers and polynomials associated with p-adic integral on $\mathbb{Z}_p$. In this paper, by using the symmetry of p-adic integral on $\mathbb{Z}_p$, we give recurrence identities the degenerate q-tangent polynomials and the generalized factorial sums.

AMS subject classification:
Keywords: Degenerate tangent numbers and polynomials, degenerate q-tangent numbers and polynomials, generalized factorial sums.

1. Introduction

L. Carlitz introduced the degenerate Bernoulli polynomials(see [1]). Feng Qi *et al.* [2] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. T. Kim studied the Barnes' type multiple degenerate Bernoulli and Euler polynomials (see [3]), Recently, Ryoo introduced the degenerate q-tangent numbers $\mathcal{T}_{n,q}(\lambda)$ and polynomials $\mathcal{T}_{n,q}(x, \lambda)$ (see [5]). In this paper, by using these numbers and polynomials, we give some interesting relations between the generalized factorial sums and the degenerate q-tangent polynomials.

Let p be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that

$|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \quad \text{(see [2, 3])}. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \text{ (see [2, 3])}. \tag{1.2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x - \lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}. \tag{1.5}$$

For $t, \lambda \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$, if we take $g(x) = q^x(1+\lambda t)^{2x/\lambda}$ in (1.2), then we easily see that

$$\int_{\mathbb{Z}_p} q^x (1+\lambda t)^{2x/\lambda} d\mu_{-1}(x) = \frac{2}{q(1+\lambda t)^{2/\lambda}+1}.$$

Let us define the degenerate q-tangent numbers $\mathcal{T}_{n,q}(\lambda)$ and polynomials $\mathcal{T}_{n,q}(x,\lambda)$ as follows:

$$\int_{\mathbb{Z}_p} q^y (1+\lambda t)^{2y/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(\lambda)\frac{t^n}{n!}, \tag{1.6}$$

$$\int_{\mathbb{Z}_p} q^y (1+\lambda t)^{(2y+x)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,\lambda)\frac{t^n}{n!}. \tag{1.7}$$

By (1.6) and (1.7), we obtain the following Witt's formula.

Theorem 1.1. For $n \geq 0$, we have

$$\mathcal{T}_{n,q}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{T}_{l,q}(\lambda)(x|\lambda)_{n-l}.$$

Theorem 1.2. For $n \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} q^x (2x|\lambda)_n d\mu_{-1}(x) = \mathcal{T}_{n,q}(\lambda),$$

$$\int_{\mathbb{Z}_p} q^y (x+2y|\lambda)_n d\mu_{-1}(y) = \mathcal{T}_{n,q}(x,\lambda).$$

Recently, many mathematicians have studied in the area of the q-analogues of the degenerate Bernoulli umbers and polynomials, Euler numbers and polynomials, tangent numbers and polynomials (see [2, 3, 5, 7]). Our aim in this paper is to obtain symmetric properties for the degenerate q-tangent numbers and polynomials. We investigate some properties which are related to degenerate q-tangent polynomials $\mathcal{T}_{n,q}(x,\lambda)$ and the generalized factorial sums.

2. The alternating generalized factorial sums and q-tangent polynomials

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. By using (1.6), we give the alternating generalized factorial sums as follows:

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}(\lambda)\frac{t^n}{n!} = \frac{2}{q(1+\lambda t)^{2/\lambda}+1} = 2\sum_{n=0}^{\infty}(-1)^n q^n (1+\lambda t)^{2n/\lambda}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty}(-1)^n q^n (1+\lambda t)^{(2n+2k)/\lambda} + \sum_{n=0}^{\infty}(-1)^{n-k} q^{(n-k)} (1+\lambda t)^{2n/\lambda}$$
$$= \sum_{n=0}^{k-1}(-1)^{n-k} q^{(n-k)} (1+\lambda t)^{2n/\lambda}.$$

By using (1.6) and (1.7), we obtain

$$-\frac{1}{2}\sum_{j=0}^{\infty}T_{j,q}(2k)\frac{t^j}{j!}+\frac{1}{2}(-1)^{-k}q^{-k}\sum_{j=0}^{\infty}T_{j,q}\frac{t^j}{j!}$$
$$=\sum_{j=0}^{\infty}\left((-1)^{-k}q^{-k}\sum_{n=0}^{k-1}(-1)^n q^n (2n|\lambda)_j\right)\frac{t^j}{j!}.$$

By comparing coefficients of $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1}(-1)^n q^n (2n|\lambda)_j = \frac{(-1)^{k+1}q^k\mathcal{T}_{j,q}(2k)+\mathcal{T}_{j,q}}{2}.$$

By using the above equation we arrive at the following theorem:

Theorem 2.1. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain

$$S_{j,q}(k-1,\lambda)=\sum_{n=0}^{k-1}(-1)^n q^n (2n|\lambda)_j = \frac{(-1)^{k+1}q^k\mathcal{T}_{j,q}(2k)+\mathcal{T}_{j,q}}{2}. \tag{2.1}$$

Remark 2.2. For the alternating generalized factorial sums, we have

$$\lim_{q\to 1} S_{j,q}(k-1)=\sum_{n=0}^{k-1}(-1)^n (2n|\lambda)_j = \frac{(-1)^{k+1}\mathcal{T}_j(2k)+\mathcal{T}_j}{2},$$

where $\mathcal{T}_j(x)$ and $\mathcal{T}_j$ denote the tangent polynomials and the tangent numbers, respectively (see [6]).

3. Symmetry properties of the q-deformed fermionic integral on $\mathbb{Z}_p$

In this section, we assume that $q \in \mathbb{C}_p$. In this section, we obtain recurrence identities the degenerate q-tangent polynomials and the alternating generalized factorial sums. By using (1.1), we have

$$I_{-1}(g_n)+(-1)^{n-1}I_{-1}(g)=2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k),$$

where $n \in \mathbb{N}$, $g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$I_{-1}(g_n)+I_{-1}(g)=2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k) \text{ (see [2], [3], [4], [5]).} \tag{3.1}$$

It will be more convenient to write (3.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^{n-1-k}g(k). \tag{3.2}$$

Substituting $g(x) = q^x(1+\lambda t)^{2x/\lambda}$ into the above, we obtain

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^{(x+n)}(1+\lambda t)^{(2x+2n)/\lambda}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x(1+\lambda t)^{2x/\lambda}d\mu_{-1}(x)\\&= 2\sum_{j=0}^{n-1}(-1)^j q^j(1+\lambda t)^{2j/\lambda}.\end{aligned} \tag{3.3}$$

After some calculations, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^x(1+\lambda t)^{2x/\lambda}d\mu_{-1}(x) = \frac{2}{q(1+\lambda t)^{2/\lambda}+1},\\&\int_{\mathbb{Z}_p} q^{(x+n)}(1+\lambda t)^{(2x+2n)/\lambda}d\mu_{-1}(x) = q^n(1+\lambda t)^{2n/\lambda}\frac{2}{q(1+\lambda t)^{2/\lambda}+1}.\end{aligned} \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^{(x+n)}(1+\lambda t)^{(2x+2n)/\lambda}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x(1+\lambda t)^{2x/\lambda}d\mu_{-1}(x)\\&= \frac{2(1+q^n(1+\lambda t)^{2n/\lambda})}{q(1+\lambda t)^{2/\lambda}+1}.\end{aligned}$$

From the above, we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} q^{(x+n)}(1+\lambda t)^{(2x+2n)/\lambda}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x(1+\lambda t)^{2x/\lambda}d\mu_{-1}(x)\\&= \frac{2\int_{\mathbb{Z}_p} q^x(1+\lambda t)^{2x/\lambda}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} q^{nx}(1+\lambda t)^{2nx/\lambda}d\mu_{-1}(x)}.\end{aligned} \tag{3.5}$$

By (3.3), we obtain

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{(x+n)}(2x+2n|\lambda)_m d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x(2x|\lambda)_m d\mu_{-1}(x)\right)\frac{t^m}{m!}\\&= \sum_{m=0}^{\infty}\left(2\sum_{j=0}^{n-1}(-1)^j q^j(2j|\lambda)_m\right)\frac{t^m}{m!}\end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$q^n \sum_{k=0}^{m} \binom{m}{k} (2n|\lambda)_{m-k} \int_{\mathbb{Z}_p} q^x (2x|\lambda)_k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x (2x|\lambda)_m d\mu_{-1}(x)$$
$$= 2\sum_{j=0}^{n-1} (-1)^j q^j (2j|\lambda)_m$$

By using (2.1), we have

$$\begin{aligned} &q^n \sum_{k=0}^{m} \binom{m}{k} (2n)^{m-k} \int_{\mathbb{Z}_p} q^x (2x)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^x (2x)^m d\mu_{-1}(x) \\ &= 2S_{m,q}(n-1,\lambda). \end{aligned} \tag{3.6}$$

By using (3.5) and (3.6), we arrive at the following theorem:

Theorem 3.1. Let n be odd positive integer. Then we obtain

$$\frac{2\int_{\mathbb{Z}_p} q^x (1+\lambda t)^{2x/\lambda} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} q^{nx} (1+\lambda t)^{2nx/\lambda} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} \left(2S_{m,q}(n-1,\lambda)\right) \frac{t^m}{m!}. \tag{3.7}$$

Let w_1 and w_2 be odd positive integers. By using (3.7), we have

$$\begin{aligned} &\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} q^{(w_1x_1+w_2x_2)} (1+\lambda t)^{(w_1 2x_1 + w_2 2x_2 + w_1 w_2 x)/\lambda} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} q^{w_1w_2x} (1+\lambda t)^{2w_1w_2x/\lambda} d\mu_{-1}(x)} \\ &= \frac{2(1+\lambda t)^{w_1w_2x/\lambda} \left(q^{w_1w_2}(1+\lambda t)^{2w_1w_2/\lambda}+1\right)}{(q^{w_1}(1+\lambda t)^{2w_1/\lambda}+1)(q^{w_2}(1+\lambda t)^{2w_2/\lambda}+1)} \end{aligned} \tag{3.8}$$

By using (3.7) and (3.8), after elementary calculations, we obtain

$$\begin{aligned} a &= \left(\frac{1}{2}\int_{\mathbb{Z}_p} q^{w_1x_1} (1+\lambda t)^{(w_1 2x_1 + w_1 w_2 x)/\lambda} d\mu_{-1}(x_1)\right) \\ &\quad \times \left(\frac{2\int_{\mathbb{Z}_p} q^{w_2x_2} (1+\lambda t)^{2x_2w_2/\lambda} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} q^{w_1w_2x} (1+\lambda t)^{2w_1w_2x/\lambda} d\mu_{-1}(x)}\right) \\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty} \mathcal{T}_{m,q^{w_1}}\left(w_2x, \frac{\lambda}{w_1}\right) w_1^m \frac{t^m}{m!}\right) \left(2\sum_{m=0}^{\infty} S_{m,q^{w_2}}\left(w_1-1, \frac{\lambda}{w_2}\right) w_2^m \frac{t^m}{m!}\right). \end{aligned} \tag{3.9}$$

By using Cauchy product in the above, we have

$$a = \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \binom{m}{j} \mathcal{T}_{j,q^{w_1}}\left(w_2x, \frac{\lambda}{w_1}\right) w_1^j S_{m-j,q^{w_2}}\left(w_1-1, \frac{\lambda}{w_2}\right) w_2^{m-j} \right) \frac{t^m}{m!}. \tag{3.10}$$

By using the symmetry in (3.9), we have

$$\begin{aligned}a &= \left(\frac{1}{2}\int_{\mathbb{Z}_p} q^{w_2x_2}(1+\lambda t)^{(w_2 2x_2+w_1w_2x)/\lambda} d\mu_{-1}(x_2)\right)\\ &\quad\times\left(\frac{2\int_{\mathbb{Z}_p} q^{w_1x_1}(1+\lambda t)^{2x_1w_1/\lambda} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} q^{w_1w_2x}(1+\lambda t)^{2w_1w_2x/\lambda} d\mu_{-1}(x)}\right)\\ &= \left(\frac{1}{2}\sum_{m=0}^{\infty}\mathcal{T}_{m,q^{w_2}}\left(w_1x,\frac{\lambda}{w_2}\right)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}S_{m,q^{w_1}}\left(w_2-1,\frac{\lambda}{w_1}\right)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$a=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}\mathcal{T}_{j,q^{w_2}}\left(w_1x,\frac{\lambda}{w_2}\right)w_2^j S_{m-j,q^{w_1}}\left(w_2-1,\frac{\lambda}{w_1}\right)w_1^{m-j}\right)\frac{t^m}{m!} \tag{3.11}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (3.10) and (3.11), we arrive at the following theorem:

Theorem 3.2. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}\mathcal{T}_{j,q^{w_2}}\left(w_1x,\frac{\lambda}{w_2}\right)S_{m-j,q^{w_1}}\left(w_2-1,\frac{\lambda}{w_1}\right)w_2^j w_1^{m-j}\\ &=\sum_{j=0}^{m}\binom{m}{j}\mathcal{T}_{j,q^{w_1}}\left(w_2x,\frac{\lambda}{w_1}\right)S_{m-j,q^{w_2}}\left(w_1-1,\frac{\lambda}{w_2}\right)w_1^j w_2^{m-j},\end{aligned}$$

where $\mathcal{T}_{k,q}(x)$ and $\mathcal{T}_{m,q}(k)$ denote the degenerate q-tangent polynomials and the alternating generalized factorial sums, respectively (see [5]).

By using Theorem 2, we have the following corollary:

Corollary 3.3. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-j}w_2^j\left(w_1x\Big|\frac{\lambda}{w_2}\right)_{j-k}T_{k,q^{w_2}}\left(\frac{\lambda}{w_2}\right)S_{m-j,q^{w_1}}(w_2-1)\\ &=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^j w_2^{m-j}\left(w_2x\Big|\frac{\lambda}{w_1}\right)_{j-k}T_{k,q^{w_1}}\left(\frac{\lambda}{w_1}\right)S_{m-j,q^{w_2}}(w_1-1).\end{aligned}$$

By using (3.8), we have

$$\begin{aligned}
a &= \left(\frac{1}{2}(1+\lambda t)^{w_1 w_2 x/\lambda}\int_{\mathbb{Z}_p} q^{w_1 x_1}(1+\lambda t)^{2x_1 w_1/\lambda} d\mu_{-1}(x_1)\right) \\
&\quad \times \left(\frac{2\int_{\mathbb{Z}_p} q^{w_2 x_2}(1+\lambda t)^{2x_2 w_2/\lambda} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 x}(1+\lambda t)^{2w_1 w_2 x/\lambda} d\mu_{-1}(x)}\right) \\
&= \left(\frac{1}{2}(1+\lambda t)^{w_1 w_2 x/\lambda}\int_{\mathbb{Z}_p} q^{w_1 x_1}(1+\lambda t)^{2x_1 w_1/\lambda} d\mu_{-1}(x_1)\right) \\
&\quad \times \left(2\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}(1+\lambda t)^{2jw_2/\lambda}\right) \\
&= \sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\int_{\mathbb{Z}_p} q^{w_1 x_1}(1+\lambda t)^{\left(2x_1+w_2 x+\frac{2jw_2}{w_1}\right)(w_1)/\lambda} d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{T}_{n,q^{w_1}}\left(w_2 x+\frac{2jw_2}{w_1},\frac{\lambda}{w_1}\right)w_1^n\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.12}$$

By using the symmetry property in (3.12), we also have

$$\begin{aligned}
a &= \left(\frac{1}{2}(1+\lambda t)^{w_1 w_2 x/\lambda}\int_{\mathbb{Z}_p} q^{w_2 x_2}(1+\lambda t)^{2x_2 w_2/\lambda} d\mu_{-1}(x_2)\right) \\
&\quad \times \left(\frac{2\int_{\mathbb{Z}_p} q^{w_1 x_1}(1+\lambda t)^{2x_1 w_1/\lambda} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} q^{w_1 w_2 x}(1+\lambda t)^{2w_1 w_2 x/\lambda} d\mu_{-1}(x)}\right) \\
&= \left(\frac{1}{2}(1+\lambda t)^{w_1 w_2 x/\lambda}\int_{\mathbb{Z}_p} q^{w_2 x_2}(1+\lambda t)^{2x_2 w_2/\lambda} d\mu_{-1}(x_2)\right) \\
&\quad \times \left(2\sum_{j=0}^{w_2-1}(-1)^j q^{w_1 j}(1+\lambda t)^{2jw_1/\lambda}\right) \\
&= \sum_{j=0}^{w_2-1}(-1)^j q^{w_1 j}\int_{\mathbb{Z}_p} q^{w_2 x_2}(1+\lambda t)^{\left(2x_2+w_1 x+\frac{2jw_1}{w_2}\right)(w_2)/\lambda} d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j q^{w_1 j}\mathcal{T}_{n,q^{w_2}}\left(w_1 x+\frac{2jw_1}{w_2},\frac{\lambda}{w_2}\right)w_2^n\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.13}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (3.12) and (3.13), we have the following theorem.

Theorem 3.4. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{T}_{n,q^{w_1}}\left(w_2 x+\frac{2jw_2}{w_1},\frac{\lambda}{w_1}\right)w_1^n$$
$$=\sum_{j=0}^{w_2-1}(-1)^j q^{w_1 j}\mathcal{T}_{n,q^{w_2}}\left(w_1 x+\frac{2jw_1}{w_2},\frac{\lambda}{w_2}\right)w_2^n.$$

Observe that if $\lambda \to 0$, then Theorem 9 reduces to Theorem 3.4 in [4]. Substituting $w_1 = 1$ into Theorem 9, we have the following corollary.

Corollary 3.5. Let w_2 be odd positive integer. Then we obtain

$$\mathcal{T}_{n,q}(x)=w_2^n\sum_{j=0}^{w_2-1}(-1)^j q^j\mathcal{T}_{n,q^{w_2}}\left(\frac{x+2j}{w_2},\frac{\lambda}{w_2}\right).$$

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