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On New Subclasses of Analytic Functions with Respect to Conjugate and Symmetric Conjugate Points

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Abstract

Let $G_c(\alpha, \beta)$ and $G_{sc}(\alpha, \beta)$ be the generalized classes of analytic functions with respect to conjugate and symmetric conjugate points of order δ where functions in both classes satisfy the conditions

$$\mathrm{Re}\left(\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z) + \overline{f(\overline{z})}\right) + \alpha z\left(f(z) + \overline{f(\overline{z})}\right)'}\right) > \delta,$$

$z \in E$ and

$$\mathrm{Re}\left(\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z) - \overline{f(-\overline{z})}\right) + \alpha z\left(f(z) - \overline{f(-\overline{z})}\right)'}\right) > \delta,$$

$z \in E$ respectively for $0 \leq \delta < 1$ and $0 \leq \alpha < 1$. In this paper, the coefficient estimates and the upper bounds for second Hankel determinant for both classes are determined.

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1. Introduction

Let S be the class of normalized, analytic univalent functions in the unit disk E, $|z| < 1$ written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Set P is the set of all functions that can be represented in the form

$$p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n. \tag{1.2}$$

that are regular in E, such that for z in E, $\mathrm{Re}(f(z) > 0)$.

Let S_s^* be the subclass of S consisting of the functions given by (1.1) such that

$$Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0, z \in E.$$

This class was defined by Sakaguchi in [1], called the class of starlike functions with respect to symmetric points.

El-Ashwah and Thomas [2], introduced functions $f \in S^*$ to be starlike with respect to conjugate points denoted by $f \in S_c^*$ and starlike with respect to symmetric conjugate points denoted by $f \in S_c^*$ if

$$Re\left(\frac{zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > 0, \quad z \in E,$$

and

$$Re\left(\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right) > 0, \quad z \in E$$

respectively.

Halim [3] then defined the class of functions starlike with respect to conjugate points of order δ denoted by $S_c^*(\delta)$ and the class of functions starlike with respect to symmetric conjugate points of order δ denoted by $S_{sc}^*(\delta)$, where functions for both classes satisfy

$$Re\left(\frac{2zf'(z)}{f(z) + \overline{f(\overline{z})}}\right) > \delta, \quad z \in E,$$

and

$$Re\left(\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right) > \delta, \quad z \in E$$

for $0 \le \delta < 1$ respectively.

In terms of subordination, Goel and Mehrok in [4] generalized the Sakaguchi class by introducing the class $S_s^*(A, B)$ as the class of starlike functions with respect to symmetric points, such that the functions in this class satisfy

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in E,$$

where $-1 \leq B < A \leq 1$; A, B, are the arbitrary fixed numbers.

Subsequently, Dahhar and Janteng [5], generalized both El-Ashwah and Thomas [2] and Goel and Mehrok [4] class by introducing $S_c^*(A, B)$, the class of starlike functions with respect to conjugate points.

The class $S_{sc}^*(A, B)$ is the class of starlike functions with respect to symmetric conjugate points which was defined by Ping and Janteng [6].

We define $G_c(\alpha, \delta)$ as the class of analytic functions with respect to conjugate points of order δ and $G_{sc}(\alpha, \delta)$ as the class of analytic functions with respect to symmetric conjugate points of order δ. The function $f \in S$ given by (1.1) in this class must satisfy the conditions

$$Re\left(\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z) + \overline{f(\overline{z})}\right) + \alpha z\left(f(z) + \overline{f(\overline{z})}\right)'}\right) > \delta, \quad z \in E \tag{1.3}$$

and

$$Re\left(\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)\left(f(z) - \overline{f(-\overline{z})}\right) + \alpha z\left(f(z) - \overline{f(-\overline{z})}\right)'}\right) > \delta, z \in E \tag{1.4}$$

respectively for $0 \leq \delta < 1$ and $0 \leq \alpha < 1$. These classes are inspired by Selvaraj and Vasanthi [7].

2. Coefficient Estimates

We shall require the following lemma in order to prove our results.

Lemma 2.1. (Pommerenke [8]) If $P \in p(z)$ is given by (1.2), then $|p_n| \leq 2$ for $n = 1, 2, 3, \ldots$.
We give the coefficient inequalities for the class $G_c(\alpha, \delta)$ and $G_{sc}(\alpha, \delta)$.

Theorem 2.2. Let $f \in G_c(\alpha, \delta)$, then for $n \geq 1$, $0 \leq \delta < 1$ *and* $0 \leq \alpha < 1$,

$$|a_{2n}| \leq \frac{2}{(2n-1-\delta)(1+(2n-1)\alpha)} \prod_{j=1}^{2n-2} \frac{2+j-\delta}{j-\delta}, \tag{2.5}$$

$$|a_{2n+1}| \leq \frac{2}{(2n-\delta)(1+2n\alpha)} \prod_{j=1}^{2n-1} \frac{2+j-\delta}{j-\delta} \tag{2.6}$$

Proof. From (1.3) and (1.2), we have

$$\frac{2\left(z+2\alpha_2 z^2+3a_3 z^3+\cdots\right)+2\alpha\left(2a_2 z^2+6a_3 z^3+\cdots\right)}{(1-\alpha)\left(2z+2\alpha_2 z^2+2a_3 z^3+\cdots\right)+\alpha\left(2z+4a_2 z^2+6a_3 z^3+\cdots\right)}-\delta$$
$$=1+p_1 z+p_2 z^2+\cdots$$

Equating the coefficients of like powers of z we have

$$\begin{aligned} a_2 &= \frac{p_1}{(1-\delta)(1+\alpha)}, \\ a_3 &= \frac{p_2(1-\delta)+p_1^2}{(2-\delta)(1+2\alpha)(1-\delta)}, \\ a_4 &= \frac{(p_3(1-\delta)+p_1 p_2)(2-\delta)+p_1\left(p_2(1-\delta)+p_1^2\right)}{(3-\delta)(1+3\alpha)(2-\delta)(1-\delta)}, \\ a_5 &= \frac{p_4(2-\delta)(1-\delta)+p_1 p_3(2-\delta)+p_2^2(1-\delta)+p_1^2 p_2}{(4-\delta)(1+4\alpha)(2-\delta)(1-\delta)} \\ &\quad +\frac{p_1(p_3(1-\delta)+p_1 p_2)(2-\delta)+p_1^2\left(p_2(1-\delta)+p_1^2\right)}{(4-\delta)(1+4\alpha)(3-\delta)(2-\delta)(1-\delta)}. \end{aligned} \tag{2.7}$$

Thus, we can conclude,

$$\begin{aligned} (2n-1-\delta)(1+(2n-1)\alpha)a_{2n} &= p_{2n-1}+(1+\alpha)a_2 p_{2n-2}+\cdots \\ &\quad +(1+(2n-2)\alpha)a_{2n-1}p_1 \end{aligned} \tag{2.8}$$

and

$$(2n-\delta)(1+2n\alpha)a_{2n+1}=p_{2n}+(1+\alpha)a_2 p_1+\cdots+(1+(2n-1)\alpha)a_{2n}p_1 \tag{2.9}$$

Utilizing Lemma 2.1, we have $|p_1|=|p_2|=|p_3|=|p_4|\leq 2$, thus, from (2.7),

$$\begin{aligned} |a_2| &\leq \frac{2}{(1-\delta)(1+\alpha)},\ |a_3|\leq\frac{2(3-\delta)}{(2-\delta)(1+2\alpha)(1-\delta)}, \\ |a_4| &\leq \frac{2(4-\delta)}{(1+3\alpha)(2-\delta)(1-\delta)},\ |a_5|\leq\frac{2(5-\delta)}{(1+4\alpha)(2-\delta)(1-\delta)}. \end{aligned} \tag{2.10}$$

It follows that (2.5) and (2.6) hold for $n=1,2$.

We now prove (2.5) using induction. Equation (2.8) in conjunction with Lemma 2.1 yield

$$\begin{aligned} \left|a_{2n}\right| &\leq \frac{2}{(2n-1-\delta)(1+(2n-1)\alpha)} \\ &\quad\times\left[1+\sum_{k=1}^{n-1}(1+(2k-1)\alpha)|a_{2k}|+\sum_{k=1}^{n-1}(1+2k\alpha)|a_{2k+1}|\right]. \end{aligned} \tag{2.11}$$

We assume that (2.5) holds for $k = 3, 4, \ldots, (n-1)$. Then from (2.11), we obtain

$$\left|a_{2n}\right| \leq \frac{2}{(2n-1-\delta)\left(1+(2n-1)\alpha\right)}$$

$$\left[1+\sum_{k=1}^{n-1}\frac{2}{2k-1-\delta}\prod_{j=1}^{2k-2}\frac{2+j-\delta}{j-\delta}+\sum_{k=1}^{n-1}\frac{2}{2k-\delta}\prod_{j=1}^{2k-1}\frac{2+j-\delta}{j-\delta}\right]. \tag{2.12}$$

In order to complete the proof, it is sufficient to show that

$$\begin{aligned}&\frac{2}{(2m-1-\delta)\left(1+(2m-1)\alpha\right)}\\ &\quad\times\left[1+\sum_{k=1}^{m-1}\frac{2}{2k-1-\delta}\prod_{j=1}^{2k-2}\frac{2+j-\delta}{j-\delta}+\sum_{k=1}^{n-1}\frac{2}{2k-\delta}\prod_{j=1}^{2k-1}\frac{2+j-\delta}{j-\delta}\right]\\ &=\frac{2}{(2m-1-\delta)\left(1+(2m-1)\alpha\right)}\prod_{j=1}^{2m-2}\frac{2+j-\delta}{j-\delta}.\end{aligned} \tag{2.13}$$

(2.13) is valid for $m = 3$.

Let us suppose that (2.13) is true for all m, $3 < m \leq (n-1)$. Then from (2.12)

$$\begin{aligned}&\frac{2}{(2n-1-\delta)\left(1+(2n-1)\alpha\right)}\\ &\quad\times\left[1+\sum_{k=1}^{n-1}\frac{2}{2k-1-\delta}\prod_{j=1}^{2k-2}\frac{2+j-\delta}{j-\delta}+\sum_{k=1}^{n-1}\frac{2}{2k-\delta}\prod_{j=1}^{2k-1}\frac{2+j-\delta}{j-\delta}\right]\\ &=\left(\frac{2(n-1)-1-\delta}{(2n-1-\delta)}\right)\frac{2}{(2(n-1)-1-\delta)\left(1+(2n-1)\alpha\right)}\\ &\quad\times\left[1+\sum_{k=1}^{n-2}\frac{2}{2k-1-\delta}\prod_{j=1}^{2k-2}\frac{2+j-\delta}{j-\delta}+\sum_{k=1}^{n-2}\frac{2}{2k-\delta}\prod_{j=1}^{2k-1}\frac{2+j-\delta}{j-\delta}\right]\\ &\quad+\frac{2}{(2n-1-\delta)\left(1+(2n-1)\alpha\right)}\cdot\frac{2}{(2(n-1)-1-\delta)}\prod_{j=1}^{2n-4}\frac{2+j-\delta}{j-\delta}\\ &\quad+\frac{2}{(2n-1-\delta)\left(1+(2n-1)\alpha\right)}\cdot\frac{2}{(2(n-1)-\delta)}\prod_{j=1}^{2n-3}\frac{2+j-\delta}{j-\delta}\end{aligned}$$

$$= \frac{2\,(2\,(n-1)-1-\delta)}{(2n-1-\delta)\,(1+(2n-1)\,\alpha)\,(2\,(n-1)-1-\delta)} \prod_{j=1}^{2n-4} \frac{2+j-\delta}{j-\delta}$$
$$+ \frac{2}{(2n-1-\delta)\,(1+(2n-1)\,\alpha)} \cdot \frac{2}{(2\,(n-1)-1-\delta)} \prod_{j=1}^{2n-4} \frac{2+j-\delta}{j-\delta}$$
$$+ \frac{2}{(2n-1-\delta)\,(1+(2n-1)\,\alpha)} \cdot \frac{2}{(2\,(n-1)-\delta)} \prod_{j=1}^{2n-3} \frac{2+j-\delta}{j-\delta}$$
$$= \frac{2}{(2n-1-\delta)\,(1+(2n-1)\,\alpha)\,(2\,(n-1)-1-\delta)}$$
$$\times \prod_{j=1}^{2n-4} \frac{(2+j-\delta)\,(2+(2\,(n-1)-1-\delta))}{j-\delta}$$
$$+ \frac{2}{(2n-1-\delta)\,(1+(2n-1)\,\alpha)} \cdot \frac{2}{(2\,(n-1)-\delta)} \prod_{j=1}^{2n-3} \frac{2+j-\delta}{j-\delta}$$
$$= \frac{2}{(2n-1-\delta)\,(1+(2n-1)\,\alpha)} \prod_{j=1}^{2n-2} \frac{2+j-\delta}{j-\delta}.$$

Thus (2.13) holds for $m = n$ and hence (2.5) follows. Similarly we can prove (2.6). ∎

Remark 2.3. From Theorem 2.1, if we let $\alpha = 0$ and $\delta = 0$, we obtain the results of Dahhar and Janteng [5], by setting $A = 1$ and $B = -1$ in their results. Then, if we set $\delta = 0$, we obtain the results of Selvaraj and Vasanthi [7] by setting $A = 1$ and $B = -1$ in their results.

Theorem 2.4. Let $f \in G_{sc}(\alpha, \delta)$, then for $n \geq 1$, $0 \leq \delta < 1$ and $0 \leq \alpha < 1$,

$$|a_{2n}| \leq \frac{2}{2n\,(1+(2n-1)\,\alpha)} \prod_{j=1}^{n-1} \frac{2+2j-\delta}{2j-\delta} \tag{2.14}$$

$$|a_{2n+1}| \leq \frac{2}{(2n-\delta)\,(1+2n\alpha)} \prod_{j=1}^{n-1} \frac{2+2j-\delta}{2j-\delta} \tag{2.15}$$

Proof. From (1.4) and (1.2) we have

$$\frac{2z\left(1+2\alpha_2 z+3a_3z^2+\cdots\right)+2\alpha z^2\,(2a_2+6a_3z+\cdots)}{(1-\alpha)\left(2z+2a_3z^3+\cdots\right)+\alpha z\left(2+6a_3z^2+10a_5z^4+\cdots\right)} - \delta = 1+p_1z+p_2z^2+\cdots$$

Equating the coefficients of like powers of z, we have

$$a_2 = \frac{p_1}{2(1+\alpha)}, a_3 = \frac{p_2}{(2-\delta)(1+2\alpha)},$$
$$a_4 = \frac{p_3(2-\delta)+p_1p_2}{4(1+3\alpha)(2-\delta)}, a_5 = \frac{p_4(2-\delta)+p_2^2}{(4-\delta)(1+4\alpha)(2-\delta)}. \tag{2.16}$$

Thus, we can conclude,

$$2n(1+(2n-1)\alpha)a_{2n} = p_{2n-1} + (1+2\alpha)a_3p_{2n-3} + \cdots + (1+(2n-2)\alpha)a_{2n-1}p_1 \tag{2.17}$$

and

$$(2n-\delta)(1+2n\alpha)a_{2n+1} = p_{2n} + (1+2\alpha)a_3p_{2n-2} + \cdots + (1+(2n-2)\alpha)a_{2n-1}p_2. \tag{2.18}$$

Utilizing Lemma 2.1 we have$|p_1| = |p_2| = |p_3| = |p_4| \le 2$, thus from (2.16) we have that

$$|a_2| \le \frac{1}{(1+\alpha)}, |a_3| \le \frac{2}{(2-\delta)(1+2\alpha)}, |a_4| \le \frac{(4-\delta)}{2(1+3\alpha)(2-\delta)},$$
$$|a_5| \le \frac{2}{(1+4\alpha)(2-\delta)}. \tag{2.19}$$

It follows that (2.14) and (2.15) hold for $n = 1, 2$. We prove (2.14) using induction.

Equation (2.17) in conjunction with Lemma 2.1 yield

$$|a_{2n}| \le \frac{2}{2n(1+(2n-1)\alpha)}\left[1 + \sum_{k=1}^{n-1}(1+2k\alpha)|a_{2k+1}|\right]. \tag{2.20}$$

We assume that (2.14) holds for $k = 3, 4, \ldots, (n-1)$. Then from (2.20), we obtain

$$|a_{2n}| \le \frac{2}{2n(1+(2n-1)\alpha)}\left[1 + \sum_{k=1}^{n-1}\frac{2}{2k-\delta}\prod_{j=1}^{k-1}\frac{2+2j-\delta}{2j-\delta}\right]. \tag{2.21}$$

In order to complete the proof, it is sufficient to show that

$$\frac{2}{2m(1+(2m-1)\alpha)}\left[1 + \sum_{k=1}^{m-1}\frac{2}{2k-\delta}\prod_{j=1}^{k-1}\frac{2+2j-\delta}{2j-\delta}\right]$$
$$= \frac{2}{2m(1+(2m-1)\alpha)}\prod_{j=1}^{m-1}\frac{2+2j-\delta}{2j-\delta}. \tag{2.22}$$

(2.22) is valid for $m = 3$.

Let us suppose that (2.22) is true for all m, $3 < m \le (n-1)$. Then from (2.21)

$$\begin{aligned}
&\frac{2}{2n\left(1+(2n-1)\alpha\right)}\left[1+\sum_{k=1}^{n-1}\frac{2}{2k-\delta}\prod_{j=1}^{k-1}\frac{2+2j-\delta}{2j-\delta}\right]\\
&\quad=\left(\frac{2\left(n-1\right)-\delta}{(2n)}\right)\frac{2}{\left(2\left(n-1\right)-\delta\right)\left(1+(2n-1)\alpha\right)}\\
&\qquad\times\left[1+\sum_{k=1}^{n-2}\frac{2}{2k-\delta}\prod_{j=1}^{k-1}\frac{2+2j-\delta}{2j-\delta}\right]\\
&\qquad+\frac{2}{2n\left(1+(2n-1)\alpha\right)}\cdot\frac{2}{\left(2\left(n-1\right)-\delta\right)}\prod_{j=1}^{n-2}\frac{2+2j-\delta}{2j-\delta}\\
&\quad=\frac{2\left(2\left(n-1\right)-\delta\right)}{2n\left(2\left(n-1\right)-\delta\right)\left(1+(2n-1)\alpha\right)}\prod_{j=1}^{n-2}\frac{2+2j-\delta}{2j-\delta}\\
&\qquad+\frac{2}{2n\left(1+(2n-1)\alpha\right)}\cdot\frac{2}{\left(2\left(n-1\right)-\delta\right)}\prod_{j=1}^{n-2}\frac{2+2j-\delta}{2j-\delta}\\
&\quad=\frac{2}{2n\left(1+(2n-1)\alpha\right)\left(2\left(n-1\right)-\delta\right)}\prod_{j=1}^{n-2}\frac{\left(2+2j-\delta\right)\left(2+\left(2\left(n-1\right)-\delta\right)\right)}{2j-\delta}\\
&\quad=\frac{2}{2n\left(1+(2n-1)\alpha\right)}\prod_{j=1}^{n-1}\frac{\left(2+2j-\delta\right)}{2j-\delta}.
\end{aligned}$$

Thus (2.22) holds for $m = n$ and hence (2.14) follows. Similarly, we can prove (2.15). ■

3. Second Hankel Determinant

The qth Hankel determinant for $q \ge 1$ and $n \ge 1$ was first introduced by Noonan and Thomas [9],

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+q+1} \\ a_{n+1} & a_{n+2} & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & a_{n+2q-2} \end{vmatrix}.$$

The bounds have been investigated by many researchers for different classes. Such as in 1983, Noor [10] studied the Hankel determinant problem for the class of functions with bounded boundary rotation and Ehrenborg [11], considered the Hankel determinant of exponential polynomials.

Second Hankel determinant is obtained when setting $q = 2$and$n = 2$, where

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

The finding of the upper bounds of $\left|a_2a_4 - a_3^2\right|$ for the Hankel determinant started in the year 2000. Janteng et al. [12] studied the second Hankel determinant for the classes of starlike functions and convex functions. Second Hankel determinant for starlike and convex functions with respect to symmetric points were studied by Janteng et al. in [13].

In order to seek the upper bounds for $\left|a_2a_4 - a_3^2\right|$ for the class $G_c(\alpha, \delta)$ and $G_{sc}(\alpha, \delta)$, we shall require the following Lemmas.

Lemma 3.1. (Pommerenke [8]) If $p \in P$, then $|p_k| \leq 2$ for $k = 1, 2, 3, \ldots$

Lemma 3.2. (Libera and Zlotkiewicz [14,15]) Let the function $p(z) \in P$ be given by (1.2). Assume without restriction that $p_1 \geq 0$. By rewriting Lemma 3.1 for the cases $n = 2$and $n = 3$,

$$D_2 = \begin{vmatrix} 2 & p_1 & p_2 \\ p_1 & 2 & p_1 \\ p_2 & p_1 & 2 \end{vmatrix} = 8 + 2Re\left(p_1^2 p_2\right) - 2\left|p_2\right|^2 - 4p_1^2 \geq 0$$

which is equivalent to $2p_2 = p_1^2 + x\left(4 - p_1^2\right)$ for some x, $|x| \leq 1$. Then $D_3 \leq 0$ is equivalent to

$$\left|\left(4p_3 - 4p_1p_2 + p_1^3\right)\left(4 - p_1^2\right) + p_1\left(2p_2 - p_1^2\right)^2\right| \leq 2\left(4 - p_1^2\right)^2 - 2\left|2p_2 - p_1^2\right|^2$$

and provides the relation

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z$$

for some value of z, $|z| \leq 1$.

We give the upper bounds for the second Hankel determinant for the class $G_c(\alpha, \delta)$and$G_{sc}(\alpha, \delta)$.

Theorem 3.3. If $f(z) \in G_c(\alpha, \delta)$, then

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{(1 + 2\alpha)^2(2 - \delta)^2} \tag{3.23}$$

Proof. Since $f(z) \in G_c(\alpha, \delta)$, so from (2.7) we have

$$\begin{aligned} a_2 &= \frac{p_1}{(1 - \delta)(1 + \alpha)}, \\ a_3 &= \frac{p_2(1 - \delta) + p_1^2}{(2 - \delta)(1 + 2\alpha)(1 - \delta)}, \\ a_4 &= \frac{p_1^3 + p_1p_2(3 - 2\delta)}{(3 - \delta)(1 + 3\alpha)(2 - \delta)(1 - \delta)} + \frac{p_3}{(3 - \delta)(1 + 3\alpha)}. \end{aligned} \tag{3.24}$$

Then,

$$a_2a_4 = \frac{p_1^4 + p_1^2 p_2 (3-2\delta)}{(3-\delta)(1+3\alpha)(1+\alpha)(2-\delta)(1-\delta)^2} + \frac{p_1 p_3}{(3-\delta)(1+3\alpha)(1-\delta)(1+\alpha)}$$

and

$$a_3^2 = \frac{p_1^4 + 2p_1^2 p_2 (1-\delta) + p_2^2 (1-\delta)^2}{(1+2\alpha)^2 (2-\delta)^2 (1-\delta)^2}.$$

Therefore,

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{p_1^4 + p_1^2 p_2 (3-2\delta)}{(3-\delta)(1+3\alpha)(1+\alpha)(2-\delta)(1-\delta)^2} + \frac{p_1 p_3}{(3-\delta)(1+3\alpha)(1-\delta)(1+\alpha)} \\ &\quad - \frac{p_1^4 + 2p_1^2 p_2 (1-\delta) + p_2^2 (1-\delta)^2}{(1+2\alpha)^2 (2-\delta)^2 (1-\delta)^2} \\ &= \frac{1}{C(\alpha)} \left(\begin{array}{l} \left[(1+2\alpha)^2 (2-\delta) - (1+3\alpha)(1+\alpha)(3-\delta)\right] p_1^4 \\ + \left[(3-2\delta)(1+2\alpha)^2 (2-\delta) - 2(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)\right] p_1^2 p_2 \\ + \left[(1+2\alpha)^2 (2-\delta)^2 (1-\delta)\right] p_1 p_3 - \left[(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2\right] p_2^2 \end{array} \right) \end{aligned}$$

where

$$C(\alpha) = (1+3\alpha)(1+2\alpha)^2 (1+\alpha)(3-\delta)(2-\delta)^2 (1-\delta)^2. \tag{3.25}$$

Then,

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{C(\alpha)} \left| \begin{array}{l} \left[(1+2\alpha)^2 (2-\delta) - (1+3\alpha)(1+\alpha)(3-\delta)\right] p_1^4 \\ + \left[(3-2\delta)(1+2\alpha)^2 (2-\delta) \right. \\ \left. -2(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)\right] p_1^2 p_2 \\ + \left[(1+2\alpha)^2 (2-\delta)^2 (1-\delta)\right] p_1 p_3 \\ - \left[(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2\right] p_2^2 \end{array} \right| \tag{3.26}$$

Using Lemma 3.2 and Lemma 3.3 in (3.26) we obtain

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{C(\alpha)} \left| \begin{array}{l} \left[(1+2\alpha)^2 (2-\delta) - (1+3\alpha)(1+\alpha)(3-\delta)\right] p_1^4 \\ +[(3-2\delta)(1+2\alpha)^2 (2-\delta) \\ -2(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)] p_1^2 \left(\frac{p_1^2 + \left(4-p_1^2\right) x}{2}\right) \\ +[(1+2\alpha)^2 (2-\delta)^2 (1-\delta)] \\ \times p_1 \frac{p_1^3 + 2\left(4-p_1^2\right) p_1 x - p_1 \left(4-p_1^2\right) x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right) z}{4} \\ -\left[(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2\right] \left(\frac{p_1^2 + \left(4-p_1^2\right) x}{2}\right)^2 \end{array} \right|$$

$$= \frac{1}{4C(\alpha)} \left| \begin{array}{l} \left[(1+2\alpha)^2(2-\delta)\left(12-\delta+\delta^2\right)-(1+3\alpha)(1+\alpha)(3-\delta)^3\right]p_1^4 \\ +[2(2-\delta)(1+2\alpha)^2(5-5\delta+\delta^2) \\ -2(1+3\alpha)(1+\alpha)(3-\delta)^2(1-\delta)]p_1^2\left(4-p_1^2\right)x \\ -[p_1^2(1+2\alpha)^2(2-\delta)^2(1-\delta) \\ +(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2(4-p_1^2)]\left(4-p_1^2\right)x^2 \\ +2\left[(1+2\alpha)^2(2-\delta)^2(1-\delta)\right]p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z \end{array} \right|. \tag{3.27}$$

We now assume $p_1 = p$ and $p \in [0, 2]$. Therefore, from (3.27) by using triangular inequality and $|z| \leq 1$, we have

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4C(\alpha)} \left\{ \begin{array}{l} \left[(1+2\alpha)^2(2-\delta)\left(12-\delta+\delta^2\right)-(1+3\alpha)(1+\alpha)(3-\delta)^3\right]p^4 \\ +2\left[(2-\delta)(1+2\alpha)^2\left(5-5\delta+\delta^2\right)\right. \\ \left.-(1+3\alpha)(1+\alpha)(3-\delta)^2(1-\delta)\right]p^2\left(4-p^2\right)\beta \\ +\left[p^2(1+2\alpha)^2(2-\delta)^2(1-\delta)\right. \\ \left.+(1+3\alpha)(1+\alpha)(3-\delta)\left(1-\delta^2\right)\left(4-p^2\right)\right]\left(4-p^2\right)\beta^2 \\ +2\left[(1+2\alpha)^2(2-\delta)^2(1-\delta)\right]p\left(4-p^2\right)\left(1-\beta^2\right) \end{array} \right\}$$

Thus,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4C(\alpha)}F(\beta)$$

where $\beta = |x| \leq 1$ and

$$F(\beta) = \left\{ \begin{array}{l} \left[(1+2\alpha)^2(2-\delta)\left(12-\delta+\delta^2\right)-(1+3\alpha)(1+\alpha)(3-\delta)^3\right]p^4 \\ +2\left[(2-\delta)^2(1+2\alpha)^2(1-\delta)\right]p\left(4-p^2\right) \\ +2\left[(2-\delta)(1+2\alpha)^2\left(5-5\delta+\delta^2\right)\right. \\ \left.-(1+3\alpha)(1+\alpha)(3-\delta)^2(1-\delta)\right]p^2\left(4-p^2\right)\beta \\ +\left[p(1+2\alpha)^2(2-\delta)^2(1-\delta)(p-2)\right. \\ \left.+(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2\left(4-p^2\right)\right]\left(4-p^2\right)\beta^2 \end{array} \right\}$$

Differentiating $F(\beta)$ with respect to β, we have

$$\begin{aligned} F'(\beta) = 2&\left[(2-\delta)(1+2\alpha)^2\left(5-5\delta+\delta^2\right)\right. \\ &\left.-(1+3\alpha)(1+\alpha)(3-\delta)^2(1-\delta)\right]p^2\left(4-p^2\right) \\ &+2\left[p(1+2\alpha)^2(2-\delta)^2(1-\delta)(p-2)\right. \\ &\left.+(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2\left(4-p^2\right)\right]\left(4-p^2\right)\beta. \end{aligned}$$

Since $F'(\beta) > 0$, then $F(\beta)$ is an increasing. Hence, $\text{Max.}F(\beta) = F(1)$.

Consequently,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4C(\alpha)}G(p) \tag{3.28}$$

Now we let $G(p) = F(1)$.

$$G(p) = \left\{ \begin{array}{l} \left[(1+2\alpha)^2(2-\delta)\left(12-\delta+\delta^2\right)-(1+3\alpha)(1+\alpha)(3-\delta)^3\right]p^4 \\ +2\left[(2-\delta)^2(1+2\alpha)^2(1-\delta)\right]p\left(4-p^2\right) \\ +2\left[(2-\delta)(1+2\alpha)^2\left(5-5\delta+\delta^2\right)\right. \\ \left.-(1+3\alpha)(1+\alpha)(3-\delta)^2(1-\delta)\right]p^2\left(4-p^2\right) \\ +\left[p(1+2\alpha)^2(2-\delta)^2(1-\delta)(p-2)\right. \\ \left.+(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2\left(4-p^2\right)\right]\left(4-p^2\right) \end{array} \right\}.$$

After simplifying, we have

$$\begin{aligned} G(p) = &\left[(1+2\alpha)^2(2-\delta)(4+\delta)+(1+3\alpha)(1+\alpha)(3-\delta)\left(15-13\delta+3\delta^2\right)\right]p^4 \\ &-\left[4(2-\delta)^2(1+2\alpha)^2(3-\delta)(3-2\delta)+16(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)\right]p^2 \\ &+16(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2. \end{aligned}$$

Then,

$$G(p) = A(\alpha)p^4 - B(\alpha)p^2 + 16(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)^2$$

where

$$A(\alpha) = (1+2\alpha)^2(2-\delta)(4+\delta)+(1+3\alpha)(1+\alpha)(3-\delta)\left(15-13\delta+3\delta^2\right)$$

and

$$B(\alpha) = 4(2-\delta)^2(1+2\alpha)^2(3-\delta)(3-2\delta)+16(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta).$$

By differentiating $G(p)$ with respect to p, we have

$$G'(p) = 4A(\alpha)p^3 - 2B(\alpha)p$$

$$G''(p) = 12A(\alpha)p^2 - 2B(\alpha).$$

Letting $G'(p) = 0$, we obtain $p = 0$ and

$$p = \sqrt{\frac{4(2-\delta)^2(1+2\alpha)^2(3-\delta)(3-2\delta)+16(1+3\alpha)(1+\alpha)(3-\delta)(1-\delta)}{2(1+2\alpha)^2(2-\delta)(4+\delta)+(1+3\alpha)(1+\alpha)(3-\delta)\left(15-13\delta+3\delta^2\right)}}.$$

Clearly, it shows that $G(p)$ attains its maximum value at $p = 0$.

So, Max. $G(p) = G(0)$. Therefore, by substituting (3.25) and $G(p)$ such that $p = 0$ into (3.28), we obtain

$$\left|a_2a_4 - a_3^2\right| \le \frac{4}{(1+2\alpha)^2(2-\delta)^2}.$$

Hence, by substituting $p = 0$ into Lemma 3.3, we have $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.

The results for $G_c(\alpha, \delta)$ is reduced to the result of Singh [16] when $\alpha = 0$ and $\delta = 0$. ■

Theorem 3.4. If $f(z) \in G_{sc}(\alpha, \delta)$, then

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{(1+2\alpha)^2(2-\delta)^2} \tag{3.29}$$

Proof. Since $f(z) \in G_{sc}(\alpha, \delta)$, so from (2.16) we have

$$\begin{aligned} a_2 &= \frac{p_1}{2(1+\alpha)}, \\ a_3 &= \frac{p_2}{(2-\delta)(1+2\alpha)}, \\ a_4 &= \frac{p_1p_2}{4(1+3\alpha)(2-\delta)} + \frac{p_3}{4(1+3\alpha)}. \end{aligned} \tag{3.30}$$

Then,

$$a_2a_4 = \frac{p_1^2p_2}{8(1+3\alpha)(1+\alpha)(2-\delta)} + \frac{p_1p_3}{8(1+\alpha)(1+3\alpha)}$$

and

$$a_3^2 = \frac{p_2^2}{(1+2\alpha)^2(2-\delta)^2}.$$

Therefore,

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{p_1^2p_2}{8(1+3\alpha)(1+\alpha)(2-\delta)} + \frac{p_1p_3}{8(1+\alpha)(1+3\alpha)} - \frac{p_2^2}{(1+2\alpha)^2(2-\delta)^2} \\ &= \frac{1}{C(\alpha)}\big((2-\delta)(1+2\alpha)^2 p_1^2p_2 + (2-\delta)^2(1+2\alpha)^2 p_1p_3 \\ &\quad -8(1+\alpha)(1+3\alpha)p_2^2\big) \end{aligned}$$

where

$$C(\alpha) = 8(1+3\alpha)(1+2\alpha)^2(1+\alpha)(2-\delta)^2. \tag{3.31}$$

Then,

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \frac{1}{C(\alpha)}\big|&(2-\delta)(1+2\alpha)^2 p_1^2p_2 + (2-\delta)^2(1+2\alpha)^2 p_1p_3 \\ &-8(1+\alpha)(1+3\alpha)p_2^2\big| \end{aligned} \tag{3.32}$$

Using Lemma 3.2 and Lemma 3.3 in (3.32) we obtain

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{C(\alpha)}\left|\begin{array}{l} (2-\delta)(1+2\alpha)^2 p_1^2\left(\frac{p_1^2+\left(4-p_1^2\right)x}{2}\right) \\ +(2-\delta)^2(1+2\alpha)^2 p_1 \frac{p_1^3+2\left(4-p_1^2\right)p_1x - p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-|x|^2\right)z}{4} \\ -8(1+\alpha)(1+3\alpha)\left(\frac{p_1^2+\left(4-p_1^2\right)x}{2}\right)^2 \end{array}\right|$$

$$= \frac{1}{4C(\alpha)}\left|\begin{array}{l} \left[(1+2\alpha)^2(2-\delta)(4-\delta) - 8(1+3\alpha)(1+\alpha)\right]p_1^4 \\ +\left[2(2-\delta)(1+2\alpha)^2(3-\delta) - 16(1+3\alpha)(1+\alpha)\right]p_1^2\left(4-p_1^2\right)x \\ -\left[p_1^2(1+2\alpha)^2(2-\delta)^2 + 8(1+3\alpha)(1+\alpha)\left(4-p_1^2\right)\right]\left(4-p_1^2\right)x^2 \\ +2\left[(1+2\alpha)^2(2-\delta)^2\right]p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z \end{array}\right|. \tag{3.33}$$

We now assume $p_1 = p$ and $p \in [0,2]$. Therefore, from (3.33) by using triangular inequality and $|z| \leq 1$, we have

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4C(\alpha)}\left\{\begin{array}{l} \left[(1+2\alpha)^2(2-\delta)(4-\delta) - 8(1+3\alpha)(1+\alpha)\right]p_4 \\ +\left[2(2-\delta)(1+2\alpha)^2(3-\delta) - 16(1+3\alpha)(1+\alpha)\right]p^2\left(4-p^2\right)\beta \\ +\left[p^2(1+2\alpha)^2(2-\delta)^2 + 8(1+3\alpha)(1+\alpha)\left(4-p^2\right)\right]\left(4-p^2\right)\beta^2 \\ +2\left[(1+2\alpha)^2(2-\delta)^2\right]p\left(4-p^2\right)\left(1-\beta^2\right) \end{array}\right\}$$

$$\leq \frac{1}{4C(\alpha)}\left\{\begin{array}{l} \left[(1+2\alpha)^2(2-\delta)(4-\delta) - 8(1+3\alpha)(1+\alpha)\right]p_4 \\ +8p(2-\delta)^2(1+2\alpha)^2 - 2p^3(2-\delta)^2(1+2\alpha)^2 \\ +\left[2(2-\delta)(1+2\alpha)^2(3-\delta) - 16(1+3\alpha)(1+\alpha)\right]p^2\left(4-p^2\right)\beta \\ +\left[(1+2\alpha)^2(2-\delta)^2\left(p^2-2p\right) + 8(1+3\alpha)(1+\alpha)\left(4-p^2\right)\right]\left(4-p^2\right)\beta^2 \end{array}\right\}$$

Thus,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4C(\alpha)}F(\beta)$$

where

$$\beta = |x| \leq 1$$

and

$$F(\beta) = \left[(1+2\alpha)^2(2-\delta)(4-\delta) - 8(1+3\alpha)(1+\alpha)\right]p_4$$

$$\begin{array}{l} +8p(2-\delta)^2(1+2\alpha)^2 - 2p^3(2-\delta)^2(1+2\alpha)^2 \\ +\left[2(2-\delta)(1+2\alpha)^2(3-\delta) - 16(1+3\alpha)(1+\alpha)\right]p^2\left(4-p^2\right)\beta \\ +\left[(1+2\alpha)^2(2-\delta)^2\left(p^2-2p\right) + 8(1+3\alpha)(1+\alpha)\left(4-p^2\right)\right] \end{array}$$

Differentiating $F(\beta)$ with respect to β, we have

$$\begin{aligned} F^{'}(\beta) &= 2\left[(2-\delta)(1+2\alpha)^2(3-\delta) - 16(1+3\alpha)(1+\alpha)\right]p^2\left(4-p^2\right) \\ &\quad + 2\left[(1+2\alpha)^2(2-\delta)^2\left(p^2-2p\right) + 8(1+3\alpha)(1+\alpha)\left(4-p^2\right)\right]\left(4-p^2\right)\beta. \end{aligned}$$

Since $F^{'}(\beta) > 0$,then $F(\beta)$ is an increasing. Hence, Max. $F(\beta) = F(1)$.

Consequently,

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{4C(\alpha)}G(p) \tag{3.34}$$

Now we let $G(p) = F(1)$.

$$G(p) = \left\{ \begin{array}{l} \left[2(1+2\alpha)^2(2-\delta)\right]p^4 - \left[64(1+\alpha)(1+3\alpha) - 4(1+2\alpha)^2(2-\delta)^2\right]p^2 \\ +128(1+\alpha)(1+3\alpha) \end{array} \right\}.$$

Then,

$$G(p) = A(\alpha)p^4 - B(\alpha)p^2 + 128(1+\alpha)(1+3\alpha)$$

where

$$A(\alpha) = 2(2-\delta)(1+2\alpha)^2$$

and

$$B(\alpha) = 64(1+3\alpha)(1+\alpha) - 4(2-\delta)^2(1+2\alpha)^2.$$

By differentiating $G(p)$ with respect to p, we have

$$G^{'}(p) = 4A(\alpha)p^3 - 2B(\alpha)p,$$

$$G^{''}(p) = 12A(\alpha)p^2 - 2B(\alpha).$$

Letting $G^{'}(p) = 0$ we obtain $p = 0$ and

$$p = \sqrt{\frac{16(1+3\alpha)(1+\alpha) - (2-\delta)^2(1+2\alpha)^2}{(2-\delta)(1+2\alpha)^2}}.$$

Clearly, it shows that $G(p)$ attains its maximum value at $p = 0$.

So, Max. $G(p) = G(0)$.

Therefore, by substituting (3.31) and $G(p)$ such that $p = 0$ into (3.34), we obtain

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{(1+2\alpha)^2(2-\delta)^2}.$$

Hence, by substituting $p = 0$ into Lemma 3.3, we have $p_1 = 0$, $p_2 = -2$ and $p_3 = 0$.

When $\alpha = 0$, $\delta = 0$ and $\alpha = 1$, $\delta = 0$ the results for $G_{sc}(\alpha, \delta)$ is reduced to the results of Singh [16]. ■

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