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Certain Quadruple Integral Equations

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Abstract

In this paper, we have obtained the solution of quadruple integral equations involving I-functions. The method used is that of fractional integration. The given quadruple integral equations have been transformed by the application of fractional Erdelyi-Kober operators to four others integral equations with a common Kernel. Here for the sake of generality the I-function is assumed as unsymmetrical Fourier kernel. Here with the help of Mellin transform, the solution of Quadruple Integral equations is obtained.

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1. Introduction

V.P. Saxena [1] defined the I-function, which is more general hypergeometric function than the Fox's H-Function. Saxena's I-function is defined as

$$I^{m,n}_{pi;qi;r}[z] = I^{m,n}_{pi;qi;r}\left[z\,\middle|\,\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,pi}\\ \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,qi}\end{matrix}\right] = \frac{1}{2\Pi i}\int_L t(s)z^s ds \qquad (1.1)$$

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Where

$$t(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}} \tag{1.2}$$

Here p_i and q_i are positive integers and m, n are integers satisfying $0 \leq n \leq p_i, 0 \leq m \leq q_i, (i = 1, 2, \ldots, r)$, r is finite. $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers such that $\alpha_j(b_n + \nu) \neq \beta_h(\alpha_j - 1 - \nu)$ for $\nu = 0, 1, 2, \ldots$; $h = 1, 2, \ldots, m$; $j = 1, 2, \ldots, r$.

L is a suitable Contour of Barnes type which runs from $\sigma - i\infty$ to $\sigma + i\infty$, (σ is real) in the complex s-plane such that the points $s = \dfrac{\alpha_j - 1 - \nu}{\alpha_j}$: $j = 1, 2, \ldots, n$; $\nu = 0, 1, 2, \ldots$ and $s = \dfrac{b_j + \nu}{\beta_j}$: $j = 1, 2, \ldots, m$; $\nu = 0, 1, 2, \ldots$ lie to the left hand side and right hand side of the contour L respectively. The conditions under which (1.1) converges are given as follows:

$$A_i=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_j+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_j : (i=1,2,\ldots,r) \tag{1.3}$$

$$B_i=\frac{1}{2}(p_i-q_i)+\sum_{j=1}^{q_i}b_j-\sum_{j=1}^{n}a_j : (i=1,2,\ldots,r) \tag{1.4}$$

$$A_i>0, |argz|<\frac{\Pi A_i}{2} \quad \text{and} \quad B_i\geq 0 : (i=1,2,\ldots,r) \tag{1.5}$$

The aim of the present paper is to obtain the solution of the quadruple integral equations involving I-functions. The method followed is that of fractional integral operators. By the application of fractional integral operators, given equations are transformed into a equation with common Kernel.

2. Results Used

Mellin transform

$$M\{f(x)\}=F(s)=\int_0^\infty f(x)x^{s-1}dx \tag{2.1}$$

Inverse Mellin transform $C + i\infty$

$$M^{-1}\{F(s)\}=f(x)=\frac{1}{2\Pi i}\int_{c-i\infty}^{c+i\infty}F(s)x^{-s}ds \tag{2.2}$$

For $s = c + in, x > 0$.

Persaval theorem for mellin transforms
Let $M\{f(u)\} = F(s)$ and $M\{a(u)\} = A(s)$ then

$$M\{a(ux)\} = x^{-s}A(s) \quad and \quad \int_0^\infty f(ux)a(u)du = \frac{1}{2\Pi i}\int_L x^{-s}F(s)A(1-s)ds \tag{2.3}$$

Fractional integral formulae
Fractional Integral formulae have been defined by Fox as follows:

$$\int_0^x (x^{\frac{1}{c}} - \nu^{\frac{1}{c}})^{d-e-1} \cdot \nu^{\frac{e}{c}-s-1}d\nu = \frac{e\Gamma(d-e)\Gamma(e-cs)}{\Gamma(d-cs)} \cdot x^{\frac{d}{c}-\frac{1}{c}-s} \tag{2.4}$$

provided $d > e$ and $\frac{e}{c} > \sigma$ where $s = \sigma + it$ and $0 < x < 1$ and

$$\int_x^\infty (\nu^{\frac{1}{c}} - x^{\frac{1}{c}})^{d-e-1} \cdot \nu^{\frac{1}{c}-\frac{d}{c}-s-1}d\nu = \frac{c\Gamma(d-e)\Gamma(e+cs)}{\Gamma(d+cs)} \cdot x^{-\frac{e}{c}-s} \tag{2.5}$$

provided $d > e$ and $\frac{e}{c} > \sigma$ where $s = \sigma + it$ and $x > 1$.

Fractional Eerdelyi−Kober operators
Fox [9] used the following generalized Erdelyi−Kober Operators:

$$T[\gamma, \epsilon : m]\{f(x)\} = \frac{m}{\Gamma\gamma} \cdot x^{-\gamma m-\epsilon+m-1}\int_0^x (x^m - \nu^m)^{\gamma-1} \cdot \nu^\epsilon f(\nu)d\nu, \tag{2.6}$$

where $0 < x < 1$, and

$$R[\gamma, \epsilon : m]\{f(x)\} = \frac{m}{\Gamma\gamma} \cdot x^\epsilon \int_x^\infty (\nu^m - x^m)^{\gamma-1} \cdot \nu^{-\epsilon-\gamma m+m-1}f(\nu)d\nu, \tag{2.7}$$

where $x > 1$.

The operator T exists if $f(x) \in L_p(0, \infty)$, $p > 1$, $\gamma > 0$, and $\epsilon > \frac{1-p}{p}$ and if $f(x)$ can be differentiated sufficient number of times then the operator T exists for both negative and positive value of γ.

The operator R exists if $f(x) \in L_p(0, \infty)$, $p \geq 1$ and if $f(x)$ can be differentiated sufficient number of times then the operator R exists. If $m > \epsilon > \frac{-1}{p}$ while γ can take any negative or positive value.

A theorem for Mellin transforms
If $M\{f(u)\} = F(s)$ and $M\{g(u)\} = G(s)$ then

$$\int_0^\infty g(u)f(u)du = \frac{1}{2\Pi i}\lim_{\substack{T\to\infty \\ \&\sigma_o=R_e(s)}}\int_{\sigma_o-iT}^{\sigma_o+iT} G(s)F(1-s)ds \tag{2.8}$$

Thus if $f(ux)$ is considered to be a function of u with x as a parameter, where $x > 0$ then

$$M\{f(ux)\} = x^{-s} M\{f(u)\} \tag{2.9}$$

From (2.8) and (2.9) we have

$$\int_0^\infty g(ux) f(u) du = \frac{1}{2\Pi i} \lim_{\substack{T\to\infty \\ \&\sigma_o = R_e(s)}} \int_{\sigma_o - iT}^{\sigma_o + iT} x^{-s} G(s) F(1-s) ds \tag{2.10}$$

Additional conditions for the validity of (2.10) are that $F(s) \in L_p(\sigma_o - i\infty, \sigma_o + i\infty)$ and $x^{1-\sigma_o} g(x) \in L_p(0, \infty)$, $p \geq 1$ where L_p denotes the class of functions $g(x)$ such that

$$\int_0^\infty |g(x)|^p \cdot \frac{dx}{x} < \infty \tag{2.11}$$

3. Quadruple Integral Equations

In this paper, we consider the following quadruple integral equations

$$\int_0^\infty I_{pi;qi;r}^{m,n}\left[ux \left| \begin{matrix} (a_j,\alpha_j)_{1,n}, (a_{ji},\alpha_{ji})_{n+1,pi} \\ \\ (b_j,\beta_j)_{1,m}, (b_{ji},\beta_{ji})_{m+1,qi} \end{matrix} \right.\right] f(u) du = \phi_1(x); \quad \text{where} \quad x \in (0, a) \tag{3.1}$$

$$\int_0^\infty I_{pi;qi;r}^{m,n}\left[ux \left| \begin{matrix} (c_j,\alpha_j)_{1,n}, (a_{ji},\alpha_{ji})_{n+1,pi} \\ \\ (b_j,\beta_j)_{1,m}, (b_{ji},\beta_{ji})_{m+1,qi} \end{matrix} \right.\right] f(u) du = \phi_2(x); \quad \text{where} \quad x \in (a, b) \tag{3.2}$$

$$\int_0^\infty I_{pi;qi;r}^{m,n}\left[ux \left| \begin{matrix} (c_j,\alpha_j)_{1,n}, (a_{ji},\alpha_{ji})_{n+1,pi} \\ \\ (b_j,\beta_j)_{1,m}, (b_{ji},\beta_{ji})_{m+1,qi} \end{matrix} \right.\right] f(u) du = \phi_3(x); \quad \text{where} \quad x \in (b, c) \tag{3.3}$$

$$\int_0^\infty I_{pi;qi;r}^{m,n}\left[ux \left| \begin{matrix} (c_j,\alpha_j)_{1,n}, (a_{ji},\alpha_{ji})_{n+1,pi} \\ \\ (d_j,\beta_j)_{1,m}, (b_{ji},\beta_{ji})_{m+1,qi} \end{matrix} \right.\right] f(u) du = \phi_4(x); \quad \text{where} \quad x \in (c, \infty) \tag{3.4}$$

$0 < a < 1$ and $1 < c < \infty$. $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, and $\phi_4(x)$ are prescribed functions and $f(x)$ is unknown function which is to be determined. Applying Persaval's theorem in the integral equations (3.1), (3.2) ,(3.3) and (3.4) under the conditions.

$$\begin{aligned} &- Min_{1\leq j\leq m} R_e(b_j/\beta_j) < R_e(s) < 1/\alpha_j \\ &- Max_{1\leq j\leq n} R_e(a_j/\alpha_j) \end{aligned} \tag{3.5}$$

$$\begin{aligned} &- Min_{1\leq j\leq m} R_e(b_j/\beta_j) < R_e(s) < 1/\alpha_j \\ &- Max_{1\leq j\leq n} R_e(c_j/\alpha_j) \end{aligned} \tag{3.6}$$

$$\begin{aligned} &- Min_{1\leq j\leq m} R_e(d_j/\beta_j) < R_e(s) < 1/\alpha_j \\ &- Max_{1\leq j\leq n} R_e(c_j/\alpha_j) \end{aligned} \tag{3.7}$$

$$|argx| < \frac{\Pi D_i}{2}, \quad i = 1, 2, \ldots, r. \tag{3.8}$$

where,

$$D_i = \sum_{j=1}^{m} \beta_j - \sum_{j=n+1}^{p_i} \alpha_j + \sum_{j=m+1}^{q_i} \beta_j - \sum_{j=1}^{m} \alpha_j \tag{3.9}$$

where $i = 1, 2, \ldots, r.$, then the integral equations (3.1), (3.2) ,(3.3) and (3.4) are reduced to the following forms

$$\frac{1}{2\Pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s \right\}}, \tag{3.10}$$

$$x^{-s} F(1 - s) ds = \phi_1(x).$$

where $x \in (0, a)$.

$$\frac{1}{2\Pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - c_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s \right\}}, \tag{3.11}$$

$$x^{-s} F(1 - s) ds = \phi_2(x).$$

where $x \in (a, b)$.

$$\frac{1}{2\Pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - c_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s \right\}}, \tag{3.12}$$

$$x^{-s} F(1 - s) ds = \phi_3(x).$$

where $x \in (b, c)$.

$$\frac{1}{2\Pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(d_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - c_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s \right\}}, \tag{3.13}$$

$$x^{-s} F(1 - s) ds = \phi_4(x).$$

where $x \in (c, \infty)$.

Here $\quad M\{f(u)\} = F(s)$ (3.14)

Now in integral equation (3.10) replacing x by ν and multiplying both sides of the equation (3.10) by $(x^{\frac{1}{\alpha_n}} - \nu^{\frac{1}{\alpha_n}})^{c_n - a_n - 1} \cdot \nu^{\frac{(1-c_n)}{(\alpha_n - 1)}}$ and integrating both sides of integral equation (3.10) with respect to ν from 0 to x where $x \in (0, a)$ with $0 < a < 1$ and applying well known fractional integral formula (2.4) in equation (3.10), we find

$$\frac{1}{2\Pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n-1} \Gamma(1 - a_j - \alpha_j s)\Gamma(1 - c_n - \alpha_n s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s \right\}},$$

$$x^{-s} F(1 - s)ds = \frac{1}{\alpha_n \Gamma(c_n - a_n)} \cdot x^{\frac{a_n}{\alpha_n}}, \tag{3.15}$$

$$\int_0^x \left(x^{\frac{1}{\alpha_n}} - \nu^{\frac{1}{\alpha_n}} \right)^{c_n - a_n - 1} \cdot \nu^{\frac{1-c_n}{\alpha_n - 1}} \phi_1(\nu) d\nu$$

where $0 < x < 1$.

Using the Erdelyi-Kober operator T from (2.6) in equation (3.15),For brevity we write,

$$T\left[c_j - a_j, \frac{1 - c_j}{\alpha_j - 1} : \frac{1}{\alpha_j} \right] \{\phi_1(x)\} = T_j\{\phi_1(x)\}, \quad \text{where} \quad x \in (0, a). \tag{3.16}$$

then

$$T\left[c_n - a_n, \frac{1 - c_n}{\alpha_n - 1} : \frac{1}{\alpha_n} \right] \{\phi_1(x)\} = T_n\{\phi_1(x)\}, \quad \text{where} \quad x \in (0, a). \tag{3.17}$$

Hence from (3.17), the integral equation (3.15) can be written as

$$\frac{1}{2\Pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n-1} \Gamma(1 - a_j - \alpha_j s)\Gamma(1 - c_n - \alpha_n s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s \right\}}, \tag{3.18}$$

$$x^{-s} F(1 - s)ds = T_n\{\phi_1(x)\}, \quad \text{where} \quad x \in (0, a).$$

Now repeating the same process in integral equation (3.18) for $j = n-1, n-2, \ldots, 3, 2, 1$ then the integral equation (3.18) takes the form

$$\frac{1}{2\Pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - c_j - \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} + \alpha_{ji} s \right\}}, \tag{3.19}$$

$$x^{-s} F(1 - s)ds = \prod_{j=1}^{n} T_j\{\phi_1(x)\}, \quad \text{where} \quad x \in (0, a).$$

Now in integral equation (3.13) replacing x by ν and multiplying both sides of the equation (3.13) by $\left(\nu^{\frac{1}{\beta_m}} - x^{\frac{1}{\beta_m}}\right)^{d_m - b_m - 1} \cdot \nu^{\frac{1-d_m}{\beta_m - 1}}$ and integrating both sides of integral equation (3.13) with respect to ν from x to ∞, where $x \in (c, \infty)$ with $c > 1$ and applying well known fractional integral formula [2.5], We find,

$$\frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m-1}\Gamma(d_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-c_j-\alpha_j s)\Gamma(b_m+\beta_m s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s\right\}} x^{-s},$$

$$= \frac{1}{\beta_m\Gamma(d_m-b_m)} \cdot x^{\frac{b_m}{\beta_m}} \int_x^\infty \left(\nu^{\frac{1}{\beta_m}} - x^{\frac{1}{\beta_m}}\right)^{d_m-b_m-1} \cdot \nu^{\frac{1-d_m}{\beta_m-1}} \phi_4(\nu)d\nu, \tag{3.20}$$

where $x > 1$. Using the Erdely-Kober operator R from [2.7] in equation [3.20],For brevity we write,

$$R\left[d_j - b_j, \frac{b_j}{\beta_j} : \frac{1}{\beta_j}\right]\{\phi_4(x)\} = R_j\{\phi_4(x)\}, \quad \text{where} \quad x \in (c, \infty). \tag{3.21}$$

then

$$R\left[d_m - b_m, \frac{b_m}{\beta_m} : \frac{1}{\beta_m}\right]\{\phi_4(x)\} = R_m\{\phi_4(x)\}, \quad \text{where} \quad x \in (c, \infty). \tag{3.22}$$

Hence from [3.22], the integral equation [3.20] can be written as

$$\frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m-1}\Gamma(d_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-c_j-\alpha_j s)\Gamma(b_m+\beta_m s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right\}},$$

$$x^{-s}F(1-s)ds = R_m\{\phi_4(x)\}, \quad \text{where} \quad x \in (c, \infty). \tag{3.23}$$

Now repeating the same process in integral equation [3.23] for $j = m-1, m-2, \ldots, 3, 2, 1$ then the integral equation [3.23] takes the form

$$\frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-c_j-\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s\right\}},$$

$$x^{-s}F(1-s)ds = \prod_{j=1}^{m} R_j\{\phi_4(x)\}, \quad \text{where} \quad x \in (c, \infty). \tag{3.24}$$

Now if we set

$$P(X) = \begin{cases} \prod_{j=1}^{n} T_j\{\phi_1(x)\} & ; x \in (0, a), \\ \phi_2(x) & ; x \in (b, c), \\ \phi_3(x) & ; x \in (b, c), \\ \prod_{j=1}^{m} R_j\{\phi_4(x)\} & ; x \in (c, \infty) \end{cases} \tag{3.25}$$

then integral equations [3.19], [3.11], [3.12] and [3.24] having common kernel can be put into the compact form as

$$\frac{1}{2\Pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-c_j-\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s\right\}}, \tag{3.26}$$

$$x^{-s}F(1-s)ds = p(x), \quad \text{where} \quad x \in (0, \infty).$$

In order to solve the integral equation [3.26], the additional condition required is that the I-function is symmetrical or unsymmetrical Fourier kernel and $f(u)$ is continuous at $u = x$. For the sake of generality we assume that I-function is unsymmetrical Fourier kernel, In this case the parameters have to satisfy the following set of conditions:

$$\left.\begin{array}{ll} \text{(i)} & \sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p_i}\alpha_j = \sum_{j=m+1}^{q_i}\beta_j - \sum_{j=1}^{n}\alpha_j : \quad (i = 1, 2, \ldots, r) \\ \text{(ii)} & \sum_{j=1}^{m} b_j - \sum_{j=n+1}^{p_i} a_j = \sum_{j=m+1}^{q_i} b_j - \sum_{j=1}^{n} c_j : \quad (i = 1, 2, \ldots, r). \end{array}\right\} \tag{3.27}$$

$$\left.\begin{array}{ll} \text{(iii)} & R_e\left[(1-c_j) - \frac{\alpha_j}{2}\right] > \frac{\alpha_j}{2D_i}; \quad (j = 1, 2, \ldots, n); \quad (i = 1, 2, \ldots, r) \\ \text{(iv)} & R_e\left[\left(a_j + \frac{\alpha_j}{2}\right)\right] > \frac{\alpha_j}{2D_i}; \quad (j = n+1, 2, \ldots, p_i); \quad (i = 1, 2, \ldots, r) \\ \text{(v)} & R_e\left[\left(d_j + \frac{\beta_j}{2}\right)\right] > \frac{\beta_j}{2D_i}; \quad (j = 1, 2, \ldots, m); \quad (i = 1, 2, \ldots, r) \\ \text{(vi)} & R_e\left[(1-b_j) - \frac{\beta_j}{2}\right] > \frac{\beta_j}{2D_i}; \quad (j = m+1, 2, \ldots, q_i); \quad (i = 1, 2, \ldots, r) \end{array}\right\} \tag{3.28}$$

where

$$D_i = \sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p_i}\alpha_j + \sum_{j=m+1}^{q_i}\beta_j - \sum_{j=1}^{n}\alpha_j, \quad \text{where} \quad i = 1, 2, \ldots, r. \tag{3.29}$$

Now with the help of theorem V. P Saxena [1], we find its respective reciprocal kernel

$$I^{m,n}_{pi;qi;r}\left[x\left|\begin{matrix}(a_{ji}+\alpha_{ji},\alpha_{ji})_{1,n},(c_j-\alpha_j,\alpha_j)_{n+1,pi}\\ \\ (b_{ji}-\beta_{ji},\beta_{ji})_{1,m},(b_j+\beta_j,\beta_j)_{m+1,qi}\end{matrix}\right.\right] \tag{3.30}$$

Taking Mellin transform and making use of the theorem of V. P. Saxena [1] in integral equation (3.26), We have the following solution:

$$f(x)=\frac{1}{2\Pi i}\lim_{T\to\infty}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{\prod_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_j-\beta_j+\beta_j s)\prod_{j=n+1}^{p_i}\Gamma(c_j-\alpha_j+\alpha_j s)\right\}}{\prod_{j=1}^{m}\Gamma(b_{ji}-\beta_{ji}+\beta_{ji}s)\prod_{j=1}^{n}\Gamma(1-a_{ji}-\alpha_{ji}-\alpha_{ji}s)}$$
$$\cdot\, x^{-s}P(1-s)ds$$

where $M\{p(x)\}=P(s)$.

Now using Parseval's theorem in (3.27), we find the solution finally

$$f(x)=\sum_{i=1}^{r}\int_0^{\infty}H_i[ux]p(u)du \tag{3.31}$$

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