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On the maximum number of limit cycles for a generalization of polynomial Liénard differential systems via averaging theory

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Abstract

We apply the averaging theory of first and second order to a class of polynomial differential systems of the form

$$\dot{x} = y - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x, y)y,$$

where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x)$, $f_2(x, y) = \varepsilon f_{21}(x, y) + \varepsilon^2 f_{22}(x, y)$ and $g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x)$ where f_{1i}, f_{2i} and g_{2i} have degree l, n and m respectively for each $i = 1, 2$, and ε is a small parameter.

We study the maximum number of limit cycles that this class of systems can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$.

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1. Introduction

These years hundreds of papers have studied the limit cycles of planar polynomial differential systems, their existence and their number. The second part of the 16^{th} Hilbert's problem is related with the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. In this paper using the averaging theory we study the maximum number of limit cycles that the following system

$$\begin{aligned} \dot{x} &= y - f_1(x)y, \\ \dot{y} &= -x - g_2(x) - f_2(x, y)y, \end{aligned} \tag{1}$$

can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x)$, $f_2(x, y) = \varepsilon f_{21}(x, y) + \varepsilon^2 f_{22}(x, y)$ and $g_2(x) = \varepsilon g_{21}(x)+\varepsilon^2 g_{22}(x)$ where f_{1i}, f_{2i} and g_{2i} have degree l, n and m respectively for each $i = 1, 2$, and ε is a small parameter. Note that when $f_1(x) = 0$ these systems coincide with the generalized polynomial Liénard differential systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x) - f(x)y, \end{aligned} \tag{2}$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively.

In 1977 [11] studied the classical polynomial Liénard differential system (2) obtained when $g(x) = x$ and stated the following conjecture: If $f(x)$ has degree $n \geq 1$ and $g(x) = x$, then (2) has at most $\left[\frac{n}{2}\right]$ limit cycles. They prove this conjecture for $n = 1, 2$. The conjecture for $n = 3$ has been proved recently by Chengzi and Llibre in [12]. For more information see [15].

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point, that are called small amplitude limit cycles, see for instance [16]. We denote by $H(m, n)$ the number of limit cycles that systems (2) can have (This number is usually called the Hilbert number). we shall describe briefly the main results about the limit cycles of system (2).

1. In 1928, Liénard [12] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then (2) has a unique limit cycle.
2. In 1973 Rychkov [18] proved that if $m = 1$ and $F(x)$ is an odd polynomial of degree five, then (2) has at most two limit cycles.
3. In 1977 Lins, de Melo and Pugh [11] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$.
4. In 1998 Coppel [5] proved that $H(2, 1) = 1$.
5. Dumortier and Li in ([6], [8]) proved that $H(3, 1) = 1$.

6. In 1997 Dumortier and Li [7] proved that $H(2,2) = 1$.

7. In 2011 Chengzi and Llibre [4] proved that $H(1,3) = 1$.

The maximum number of small amplitude limit cycles for systems (2) is denoted by $\hat{H}(m,n)$. Blows and Lloyd [2], Lloyd and Lynch [16] and Lynch [17] have used inductive arguments in order to prove the following results.

1. If g is odd then $\hat{H}(m,n) = n/2$.

2. If f is even then $\hat{H}(m,n) = n$, whatever g is.

3. If f is odd then $\hat{H}(m, 2n+1) = [(m-2)/2] + n$.

4. If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m, 2) = m$.

In 1998 Gasull and Torregrosa [9] obtained upper bounds for $\hat{H}(7,6)$, $\hat{H}(6,7)$, $\hat{H}(7,7)$ and $\hat{H}(4,20)$. In 2006, Yu and Han [21] proved that $\hat{H}(m,n) = \hat{H}(n,m)$ for $n = 4,\ \ m = 10, 11, 12, 13$;
$n = 5,\ \ m = 6, 7, 8, 9$; $\ n = 6,\ \ m = 5, 6$, see also [14] for a table with all the specific values.

In 2010 Llibre and Mereu [14] compute the maximum number of limit cycles $\tilde{H}_k(m,n)$ of systems (2) which bifurcate from the periodic orbits of the linear center $\dot{x} = y,\ \ \dot{y} = -x$, using the averaging theory of order k, for $k = 1, 2, 3$.

In 2013 Badi and Makhlouf [1] using the averaging theory studied the maximum number of limit cycles $\tilde{H}(l,m,n)$ which can bifurcate from the periodic orbits of a linear center perturbed inside the class of generalized polynomial Liénard differential system of the form

$$\begin{aligned} \dot{x} &= y + \sum_{k\geq 1} \varepsilon^k h_l^k(x), \\ \dot{y} &= -x - \sum_{k\geq 1} \varepsilon^k (f_n^k(x,y)y + g_m^k(x)), \end{aligned} \tag{3}$$

where for every k the polynomials $h_l^k(x),\ \ g_n^k(x)$ and $f_n^k(x,y)$ have degree $l,\ \ m$ and n respectively and ε is a small parameter. More precisely the maximum number of medium amplitude limit cycles which can bifurcate from the periodic orbits of the linear center $\dot{x} = y,\ \ \dot{y} = -x$ perturbed as in (3).

In [15] Llibre and Valls studied the polynomial differential systems

$$\begin{aligned} \dot{x} &= y - f_1(x)y, \\ \dot{y} &= -x - g_2(x) - f_2(x)y, \end{aligned} \tag{4}$$

where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x) + \varepsilon^3 f_{13}(x),\ \ g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x) + \varepsilon^3 g_{23}(x)$ and $f_2(x) = \varepsilon f_{21}(x) + \varepsilon^2 f_{22}(x) + \varepsilon^3 f_{23}(x)$, where $f_{1i},\ \ f_{2i},\ \ g_{2i}$ have degree l, n and

m respectively for each $i = 1, 2, 3$ and ε is a small parameter. They proved an accurate upper bound of the maximum number of limit cycle that (4) can have bifurcating from the periodic orbits of the linear center $\dot{x} = y, \ \ \dot{y} = -x$ using the averaging theory of third order.

In this work, we study the polynomial differential system

$$\begin{aligned} \dot{x} &= y - f_1(x)y, \\ \dot{y} &= -x - g_2(x) - f_2(x, y)y, \end{aligned} \tag{5}$$

where $f_1(x) = \varepsilon f_{11}(x) + \varepsilon^2 f_{12}(x), \ \ g_2(x) = \varepsilon g_{21}(x) + \varepsilon^2 g_{22}(x)$ and $f_2(x, y) = \varepsilon f_{21}(x, y) + \varepsilon^2 f_{22}(x, y)$ where $f_{1i}, \ \ f_{2i}, \ \ g_{2i}$ have degree l, n and m respectively for each $i = 1, 2$ and ε is a small parameter. This system is more general than the one studied in [15]. We find the maximum number of limit cycle that (5) can have bifurcating from the periodic orbits of the linear center $\dot{x} = y, \ \ \dot{y} = -x$ using the averaging theory of first and second order. The main results of this paper is the following theorem:

Theorem 1.1. For ε sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems (1) bifurcating from the periodic orbits of the linear center $\dot{x} = y, \ \ \dot{y} = -x$ is

1. $\left[\dfrac{n}{2}\right]$ using the averaging theory of first order.
2. $\dfrac{1}{2}\max\{2O(n) - 2; \ \ O(n) + E(m) - 1; \ \ O(n) + O(l) - 2; E(n)\}$ using the averaging theory of second order, where $O(i)$ is the largest odd integer less than or equal to i, and $E(i)$ is the largest even integer less than or equal to i.

The proof of statement (1) of Theorem 1.1 is given in section 3. The proof of statement (2) of Theorem 1.1 is given in section 4. The results that we shall use from the averaging theory of second order for computing limit cycles are presented in Section 2.

2. The Averaging Theory of first and Second Orders

We use the averaging theory of first and second order for studying specifically limit cycles. It is summarized as follows.

Consider the differential system

$$\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \tag{6}$$

where $F_1, F_2 : \mathbb{R} \times D \longrightarrow \mathbb{R}, \ \ R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \longrightarrow \mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following conditions hold.

(i) $F_1(t, .) \subset C^2(D), \ \ F_2(t, .) \subset C^1(D)$, for all $t \in \mathbb{R}$, F_1, F_2, R, are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0} : D \longrightarrow \mathbb{R}$ for $k = 1, 2$ as

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s, z)ds,$$

and

$$F_{20}(z) = \frac{1}{T}\int_0^T [D_z F_1(s, z) y_1(s, z) + F_2(s, z)]ds,$$

where

$$y_1(s, z) = \int_0^\theta F_1(t, z)dt.$$

(ii) For $V \subset D$, an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f)$ 0, there exists $a_\varepsilon \in V$ such that

$$F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$$

and

$$d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0.$$

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T-periodic solution $\phi(., \varepsilon)$ of the system such that $\phi(0, a_\varepsilon) \longrightarrow a_\varepsilon$ when $\varepsilon \longrightarrow 0$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \longrightarrow \mathbb{R}_n$ at the fixed point a_ε is not zero. A sufficient condition in order that this inequality holds is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case, the previous result provides the *averaging theory of first order*.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case, the previous result provides the *averaging theory of second order*.

For more information about the averaging theory see ([19], [20]).

3. Proof of statement (1) of Theorem 1.1

In this proof, we use the first order averaging theory. In order to write system (1) in the standard form (5) for applying the averaging method, we set $x = r\cos\theta$, $y = r\cos\theta$, $r > 0$, with (r, θ) being the polar coordinates. If we write

$$f_{11}(x) = \sum_{i=0}^{l} a_{i,1}x^i, \quad f_{21}(x, y) = \sum_{i+j=0}^{n} a_{ij,2}x^i y^j, \quad g_{21} = \sum_{i=0}^{m} b_{i,2}x^i,$$

then system (1) becomes

$$
\begin{aligned}
\dot{r} &= -\varepsilon\left(\sum_{i+j=0}^{n} a_{ij,2} r^{i+j+1}\cos^{i}(\theta)\sin^{j+2}(\theta) + \sum_{i=0}^{m} b_{i,2} r^{i}\cos^{i}(\theta)\sin(\theta)\right.\\
&\quad \left. + \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i+1}(\theta)\sin(\theta)\right),\\
\dot{\theta} &= -1 - \frac{\varepsilon}{r}\left(\sum_{i+j=0}^{n} a_{ij,2} r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta) + \sum_{i=0}^{m} b_{i,2} r^{i}\cos^{i+1}(\theta)\right.\\
&\quad \left. - \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i}(\theta)\sin^{2}(\theta)\right).
\end{aligned}
\tag{7}
$$

Taking θ as the new independent variable, system (7) becomes

$$
\frac{dr}{d\theta} = \varepsilon F(r,\theta) + O(\varepsilon^2),
$$

where

$$
\begin{aligned}
F(r,\theta) = &\sum_{i+j=0}^{n} a_{ij,2} r^{i+j+1}\cos^{i}(\theta)\sin^{j+2}(\theta)\\
&+ \sum_{i=0}^{m} b_{i,2} r^{i}\cos^{i}(\theta)\sin(\theta) + \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i+1}(\theta)\sin(\theta).
\end{aligned}
$$

Now to apply the theorem of section 2, we calculate

$$
F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta,
$$

since

$$
\int_0^{2\pi}\cos^{i}(\theta)\sin^{j+2}(\theta)d\theta = \begin{cases} 0, & \text{if } i \text{ is odd or } j \text{ is odd,}\\ \pi\alpha_{ij}, & \text{if } i \text{ is even and } j \text{ is even,}\end{cases}
$$

where α_{ij} is a constant.

Finally, we obtain

$$
F_{10}(r) = \frac{1}{2}\sum_{i+j=0}^{n} a_{ij,2}\alpha_{ij} r^{i+j+1}, \tag{8}
$$

where i and j are even.

Since $F_{10}(r)$ has at most $\left[\frac{n}{2}\right]$ simple positive roots, according to the theorem of section 2, we get that for $|\varepsilon|$ sufficiently small, system (1) has at most $\left[\frac{n}{2}\right]$ limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y,\quad \dot{y} = -x$. Hence statement (1) of Theorem 1.1 is proved.

Example 3.1. We consider the system

$$\begin{aligned} \dot{x} &= y - \varepsilon(\sqrt{2} + 5x^4)y, \\ \dot{y} &= -x - \varepsilon\left(1 + x + \left(3 + xy^2 - \frac{1}{2}y^2\right)y\right), \end{aligned} \tag{9}$$

in polar coordinates (r, θ) where $x = r\cos\theta,\quad y = r\sin\theta,\quad r > 0$. System (8) becomes

$$\begin{aligned} \dot{r} &= -\varepsilon\sin(\theta)\Big(1 + ((\sqrt{2}+1)\cos(\theta) + 3\sin(\theta))r \qquad (10)\\ &\qquad -\frac{1}{2}\sin^3(\theta)r^3 + \cos(\theta)\sin^3(\theta)r^4 + 5\cos^5(\theta)r^5\Big), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r}\Big(\cos(\theta) + (\cos^2(\theta) + 3\cos(\theta)\sin(\theta) - \sqrt{2}\sin^2(\theta))r \\ &\qquad -\frac{1}{2}\cos(\theta)sin^3(\theta)r^3 + 5\cos^2(\theta)\sin^3(\theta)r^4 - 5\cos^4(\theta)\sin^2(\theta)r^5\Big), \qquad (11) \end{aligned}$$

To look for limit cycles, we must solve the equation

$$F_{10}(r) = \frac{1}{2\pi}\left(3\pi r - \frac{3}{8}\pi r^3\right) = 0.$$

This equation has one positive root $r = 2\sqrt{2}$. According with Theorem 1.1, system (8) has exactly one limit cycle bifurcating from the periodic orbits of the linear differential system with $\varepsilon = 0$.

4. Proof of statement (2) of Theorem 1.1

In this section we use the second-order averaging theory. We take f_{11}, f_{21} and g_{21} as in the section below and

$$f_{12}(x) = \sum_{i=0}^{l} c_{i,1}x^i,\quad f_{22}(x, y) = \sum_{i+j=0}^{n} c_{ij,2}x^i y^j,\quad g_{22}(x) = \sum_{i=0}^{m} d_{i,2}x^i.$$

Then system (1) in polar coordinates (r, θ) with $r > 0$ becomes

$$\begin{aligned} \dot{r} &= -I(r, \theta)\varepsilon - II(r, \theta)\varepsilon^2, \\ \dot{\theta} &= -1 - \frac{1}{r}\ [I_1(r, \theta)\varepsilon + II_1(r, \theta)\varepsilon^2], \end{aligned} \tag{12}$$

where

$$\begin{aligned} I(r,\theta) &= \sum_{i+j=0}^{n} a_{ij,2} r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta) + \sum_{i=0}^{m} b_{i,2} r^i \cos^i(\theta)\sin(\theta) \\ &\quad + \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i+1}(\theta)\sin(\theta), \\ II(r,\theta) &= \sum_{i+j=0}^{n} c_{ij,2} r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta) + \sum_{i=0}^{m} d_{i,2} r^i \cos^i(\theta)\sin(\theta) \\ &\quad + \sum_{i=0}^{l} c_{i,1} r^{i+1}\cos^{i+1}(\theta)\sin(\theta), \end{aligned}$$

and

$$\begin{aligned} I_1(r,\theta) &= \sum_{i+j=0}^{n} a_{ij,2} r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta) + \sum_{i=0}^{m} b_{i,2} r^i \cos^{i+1}(\theta) \\ &\quad - \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i}(\theta)\sin^2(\theta), \\ II_1(r,\theta) &= \sum_{i+j=0}^{n} c_{ij,2} r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta) + \sum_{i=0}^{m} d_{i,2} r^i \cos^{i+1}(\theta) \\ &\quad - \sum_{i=0}^{l} c_{i,1} r^{i+1}\cos^{i}(\theta)\sin^2(\theta). \end{aligned}$$

Taking θ as the new independent variable in the system (10), it becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r,\theta) + \varepsilon^2 F_2(r,\theta) + O(\varepsilon^3),$$

where

$$\begin{aligned} F_1(r,\theta) &= I(r,\theta), \\ F_2(r,\theta) &= II(r,\theta) - \frac{1}{r}\ I(r,\theta) I_1(r,\theta). \end{aligned}$$

We determine the corresponding function

$$F_{20}(r) = \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{d}{dr}F_1(r,\theta) y_1(r,\theta) + F_2(r,\theta)\right] d\theta,$$

where $y_1(r,\theta) = \int_0^\theta F_1(r,s)ds$. For this we need that F_{10} be identically zero, which is equivalent to $a_{ij,2} = 0$ for i even and j even. Now we compute

$$\begin{aligned}\frac{d}{dr}F_1(r,\theta) &= \sum_{\substack{i+j=1\\ i \text{ odd or } j \text{ odd}}}^{n} (i+j+1)a_{ij,2}r^{i+j}cos^i(\theta)\sin^{j+2}(\theta)\\ &\quad + \sum_{i=0}^{m} ib_{i,2}r^{i-1}\cos^i(\theta)\sin(\theta) + \sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}(\theta)\sin(\theta),\end{aligned}$$

and

$$\begin{aligned} y_1(r,\theta) &= a_{10,2}r^2(\alpha_{110}\sin(\theta)+\alpha_{210}\sin(3\theta)) + \cdots + a_{ce,2}r^{c+e+1}(\alpha_{1ce}\sin(\theta)\\ &\quad +\alpha_{2ce}\sin(3\theta)+\cdots+\alpha_{\frac{(c+e+2)+1}{2}ce}\sin((c+e+2)\theta))\\ &\quad +a_{01,2}r^2(\alpha_{101}+\alpha_{201}\cos(\theta)+\alpha_{301}\cos(3\theta))\\ &\quad +\cdots+a_{pq,2}r^{p+q+1}(\alpha_{1pq}+\alpha_{2pq}\cos(\theta)+\alpha_{3pq}\cos(3\theta)\\ &\quad +\cdots+\alpha_{\frac{(p+q+2)+3}{2}pq}\cos((p+q+2)\theta))\\ &\quad +a_{11,2}r^3(\alpha_{111}+\alpha_{211}\cos(2\theta)+\alpha_{311}\cos(4\theta))+\cdots\\ &\quad +a_{cq,2}r^{c+q+1}(\alpha_{1cq}+\alpha_{2cq}\cos(2\theta)+\alpha_{3cq}\cos(4\theta)+\cdots\\ &\quad +\alpha_{\frac{(c+q+2)+2}{2}cq}\cos((c+q+2)\theta)) + \sum_{i=0}^{m}\frac{1}{i+1}b_{i,2}r^i(1-\cos^{i+1}(\theta))\\ &\quad +\sum_{i=0}^{l}\frac{1}{i+2}a_{i,1}r^{i+1}(1-\cos^{i+2}(\theta)),\end{aligned}$$

where c is the greatest odd number and e is the greatest even number so that $c+e$ is less than or equal to n.
p is the greatest even number and q is the greatest odd number so that $p+q$ is less than or equal to n.
α_{ijk} are real constants exhibited during the computation of $\int_0^\theta \cos^i(s)\sin^{j+2}(s)ds$ for all i and j.

We know from (8) that F_{10} is identically zero if and only if $a_{ij} = 0$ for all i even and j even.

From the products of $\frac{d}{dr}F_1(r,\theta)y_1(r,\theta)$, only the following integrals are not zero when we integrate them between 0 and 2π.

$$\int_0^{2\pi} \cos^i(\theta)\sin^{j+2}(\theta)\sin((2h+1)\theta)d\theta = \begin{cases} 0 & \text{if } i \text{ is odd and } j\in\mathbb{N}, \\ \pi A_{ij}^{2h+1} & \text{if } i \text{ is even and } j \text{ is odd}, \\ & h=0,1,\dots \end{cases}$$

$$\int_0^{2\pi} \cos^i(\theta)\sin^{j+2}(\theta)\cos((2h+1)\theta)d\theta = \begin{cases} 0 & \text{if } j \text{ is odd and } i\in\mathbb{N}, \\ \pi B_{ij}^{2h+1} & \text{if } i \text{ is odd and } j \text{ is even}, \\ & h=0,1,\dots \end{cases}$$

$$\int_0^{2\pi} \cos^{i+k+1}(\theta)\sin^{j+2}(\theta)d\theta = \begin{cases} 0 & \text{if } j \text{ is odd and } i,k\in\mathbb{N}, \\ \pi C_{ijk} & \text{if } i \text{ is odd, } j \text{ is even and } k \text{ is even,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+k+2}(\theta)\sin^{j+2}(\theta)d\theta = \begin{cases} 0 & \text{if } j \text{ is odd and } i,k\in\mathbb{N}, \\ \pi D_{ijk} & \text{if } i \text{ is odd, } j \text{ is even and } k \text{ is odd} \end{cases}$$

$$\int_0^{2\pi} \cos^{i}(\theta)\sin(\theta)\sin((2h+1)\theta)d\theta = \begin{cases} 0 & \text{if } i \text{ is odd, } h=0,1,\cdots \\ \pi E_i^{2h+1} & \text{if } i \text{ is even, } h=0,1,\cdots \end{cases}$$

$$\int_0^{2\pi} \cos^{i+1}(\theta)\sin(\theta)\sin((2h+1)\theta)d\theta = \begin{cases} 0 & \text{if } i \text{ is even, } h=0,1,\cdots \\ \pi E_{i+1}^{2h+1} & \text{if } i \text{ is odd, } h=0,1,\cdots \end{cases}$$

where A_{ij}^{2h+1}, B_{ij}^{2h+1}, C_{ijk}, D_{ijk}, E_i^{2h+1} and E_{i+1}^{2h+1} are non-zero constants. Hence, using these integrals we obtain that

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{d}{dr}F_1(r,\theta)y_1(r,\theta)\right]d\theta = \frac{1}{2}rH_1(r), \tag{13}$$

where

$$\begin{aligned} H_1(r) = & \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{n} (i+j+1)a_{ij,2}r^{i+j-1}\Big[a_{10,2}r^2\left(\alpha_{110}A_{ij}^1+\alpha_{210}A_{ij}^3\right)+\cdots \\ & +a_{ce,2}r^{c+e+1}(\alpha_{1ce}A_{ij}^1+\alpha_{2ce}A_{ij}^3+\cdots+\alpha_{\frac{(c+e+2)+1}{2}ce}A_{ij}^{c+e+2})\Big] \\ & + \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{n} (i+j+1)a_{ij,2}r^{i+j-1}[a_{01,2}r^2(\alpha_{201}B_{ij}^1 \\ & +\alpha_{301}B_{ij}^3)+\cdots+a_{pq,2}r^{p+q+1}(\alpha_{2pq}B_{ij}^1+\alpha_{3pq}B_{ij}^3+\cdots \\ & +\alpha_{\frac{(p+q+2)+3}{2}pq}B_{ij}^{p+q+2})] \\ & - \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{n}\ \sum_{\substack{k=0\\ k \text{ even}}}^{m} \frac{i+j+1}{k+1}a_{ij,2}b_{k,2}r^{i+j+k-1}C_{ijk} \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{i+j=1 \\ i \text{ odd},\ j \text{ even}}}^{n} \sum_{\substack{k=1 \\ k \text{ odd}}}^{l} \frac{i+j+1}{k+2} a_{ij,2} a_{k,1} r^{i+j+k} D_{ijk} \\
&+ \sum_{\substack{i=2 \\ i \text{ even}}}^{m} i b_{i,2} r^{i-2} [a_{10,2} r^2 (\alpha_{110} E_i^1 + \alpha_{210} E_i^3) + \cdots + a_{ce,2} r^{c+e+1} (\alpha_{1ce} E_i^1 + \alpha_{2ce} E_i^3 \\
&+ \cdots + \alpha_{\frac{(c+e+2)+1}{2} ce} E_i^{c+e+2})] + \sum_{\substack{i=1 \\ i \text{ odd}}}^{l} (i+1) a_{i,1} r^{i-1} [a_{10,2} r^2 (\alpha_{110} E_{i+1}^1 \\
&+ \alpha_{210} E_{i+1}^3) + \cdots + a_{ce,2} r^{c+e+1} (\alpha_{1ce} E_{i+1}^1 + \alpha_{2ce} E_{i+1}^3 + \cdots + \alpha_{\frac{(c+e+2)+1}{2} ce} E_{i+1}^{c+e+2})].
\end{aligned}
$$

We have also

$$
\begin{aligned}
F_2(r,\theta) \;=\; & \sum_{i+j=0}^{n} c_{ij,2} r^{i+j+1} \cos^i(\theta) \sin^{j+2}(\theta) + \sum_{i=0}^{m} d_{i,2} r^i \cos^i(\theta) \sin(\theta) \\
& + \sum_{i=0}^{l} c_{i,1} r^{i+1} \cos^{i+1}(\theta) \sin(\theta) \\
& - \sum_{\substack{i+j=0 \\ i \text{ odd or } j \text{ odd}}}^{n} \sum_{\substack{k+h=0 \\ k \text{ odd or } h \text{ odd}}}^{n} a_{ij,2} a_{kh,2} r^{i+j+k+h+1} \\
& \times \cos^{i+k+1}(\theta) \sin^{i+h+3}(\theta) \\
& -2 \sum_{\substack{i+j=0 \\ i \text{ odd or } j \text{ odd}}}^{n} \sum_{k=0}^{m} a_{ij,2} b_{k,2} r^{i+j+k} \cos^{i+k+1}(\theta) \sin^{j+2}(\theta) \\
& + \sum_{\substack{i+j=0 \\ i \text{ odd or } j \text{ odd}}}^{n} \sum_{k=0}^{l} a_{ij,2} a_{k,1} r^{i+j+k+1} \cos^{i+k}(\theta) \sin^{j+4}(\theta)
\end{aligned}
$$

$$-\sum_{i=0}^{m}\sum_{j=0}^{m} b_{i,2}b_{j,2}r^{i+j-1}\cos^{i+j+1}(\theta)\sin(\theta)+\sum_{i=0}^{m}\sum_{j=0}^{l} b_{i,2}a_{j,1}r^{i+j}\cos^{i+j}(\theta)\sin^{3}(\theta)$$

$$-\sum_{\substack{i+j=0\\ i \text{ odd or } j \text{ odd}}}^{n}\sum_{k=0}^{l} a_{ij,2}a_{k,1}r^{i+j+k+1}\cos^{i+k+2}(\theta)\sin^{j+2}(\theta)$$

$$-\sum_{i=0}^{l}\sum_{j=0}^{m} a_{i,1}b_{j,2}r^{i+j}\cos^{i+j+2}(\theta)\sin(\theta)+\sum_{i=0}^{l}\sum_{j=0}^{l} a_{i,1}a_{j,1}r^{i+j+1}\cos^{i+j+1}(\theta)\sin^{3}(\theta).$$

Again, we take only the non-zero integrals when we integrate $F_2(r,\theta)$ between 0 and 2π.

$$\int_0^{2\pi}\cos^{i}(\theta)\sin^{j+2}(\theta)d\theta = \begin{cases} 0 & \text{If } i \text{ is odd or } j \text{ is odd,}\\ \pi F_{ij} & \text{If } i \text{ is even and } j \text{ is even,}\end{cases}$$

$$\int_0^{2\pi}\cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta)d\theta = \begin{cases} 0 & \text{if } i+k \text{ is even or } j+h \text{ is even,}\\ \pi G_{ijkh}, & \text{if } i \text{ is odd and } j \text{ is even,}\\ & k \text{ is even and } h \text{ is odd,}\\ \pi \bar{G}_{ijkh}, & \text{if } i \text{ is even and } j \text{ is odd,}\\ & k \text{ is odd and } h \text{ is even,}\end{cases}$$

$$\int_0^{2\pi}\cos^{i+k}(\theta)\sin^{j+4}(\theta)d\theta = \begin{cases} 0 & \text{if } j \text{ is odd, } i,k\in\mathbb{N}\\ \pi H_{ijk} & \text{if } i \text{ is odd, } j \text{ is even and } k \text{ is odd,}\end{cases}$$

where F_{ij}, G_{ijkh}, $\bar{G}_{ijkh}$ and H_{ijk} are non-zero constants and we get that

$$\frac{1}{2\pi}\int_0^{2\pi}F_2(r,\theta)d\theta = \frac{1}{2}rH_2(r), \tag{14}$$

where

$$
\begin{aligned}
H_2(r) = & \sum_{\substack{i+j=0\\ i \text{ even}, \ j \text{ even}}}^{n} c_{ij,2} r^{i+j} F_{ij} \\
& - \sum_{\substack{i+j=1\\ i \text{ odd}, \ j \text{ even}}}^{n} \sum_{\substack{k+h=1\\ k \text{ even}, \ h \text{ odd}}}^{n} a_{ij,2} a_{kh,2} r^{i+j+k+h} G_{ijkh} \\
& - \sum_{\substack{i+j=1\\ i \text{ even}, \ j \text{ odd}}}^{n} \sum_{\substack{k+h=1\\ k \text{ odd}, \ h \text{ even}}}^{n} a_{ij,2} a_{kh,2} r^{i+j+k+h} \bar{G}_{ijkh} \\
& -2 \sum_{\substack{i+j=1\\ i \text{ odd}, \ j \text{ even}}}^{n} \sum_{\substack{k=0\\ k \text{ even}}}^{m} a_{ij,2} b_{k,2} r^{i+j+k-1} C_{ijk} \\
& + \sum_{\substack{i+j=1\\ i \text{ odd}, \ j \text{ even}}}^{n} \sum_{\substack{k=1\\ k \text{ odd}}}^{l} a_{ij,2} a_{k,1} r^{i+j+k} H_{ijk} \\
& - \sum_{\substack{i+j=1\\ i \text{ odd}, \ j \text{ even}}}^{n} \sum_{\substack{k=1\\ k \text{ odd}}}^{l} a_{ij,2} a_{k,1} r^{i+j+k} D_{ijk}.
\end{aligned}
$$

Now to find the positive roots of F_{20} we must find the zeros of the polynomial in r^2 : $H_1(r) + H_2(r)$.

We conclude that F_{20} has at most

$$\frac{1}{2}\max\{2O(n) - 2,\ O(n) + E(m) - 1,\ O(n) + O(l) - 2,\ E(n)\}$$

positive roots. Hence statement (b) of Theorem 1.1 follows.

Example 4.1. We consider the system

$$
\begin{aligned}
\dot{x} &= y - \varepsilon(x^3 - 2x)y - \varepsilon^2 \frac{1}{2} x^3 y, \\
\dot{y} &= -x - \varepsilon(4x^2 + (x - xy^2 + 2x^2 y - xy^3 + x^3)y) - \varepsilon^2(x^2 - x \\
&\quad +(-xy + 2x^2 + 3y + y^4)y).
\end{aligned} \tag{15}
$$

in polar coordinates (r,θ) where $x = r\cos\theta,\quad y = r\sin\theta,\quad r > 0$, system (14) becomes

$$\begin{aligned}
\dot{r} &= -[\varepsilon((2\cos^2(\theta)\sin(\theta) + cos(\theta)\sin^2(\theta))r^2 + (\cos^4(\theta)\sin(\theta) - \cos(\theta)\sin^4(\theta) \\
&\quad +2\cos^2(\theta)\sin^3(\theta) + \cos^3(\theta)\sin^2(\theta))r^4 - \cos(\theta)\sin^5(\theta)r^5) + \varepsilon^2(-\cos(\theta)\sin(\theta)r \\
&\quad +(\cos^2(\theta)\sin(\theta) + 3\sin^3(\theta))r^2 + (2\cos^2(\theta)\sin^2(\theta) - \cos(\theta)\sin^3(\theta))r^3 \\
&\quad +\frac{1}{2}\cos^4(\theta)\sin(\theta)r^4 + \sin^6 r^5)], \\
\dot{\theta} &= -1 - [\varepsilon((4\cos^3(\theta) + \cos^2(\theta)\sin(\theta) + 2\cos(\theta)\sin^2(\theta))r + (\cos^4(\theta)\sin(\theta) \\
&\quad -\cos^2(\theta)\sin^3(\theta) + \cos^3(\theta)\sin^2(\theta))r^3 - \cos^2(\theta)\sin^4(\theta)r^4) + \varepsilon^2(-\cos^2(\theta) \\
&\quad +(\cos^3(\theta) + 3\cos(\theta)\sin^2(\theta))r + (2\cos^3(\theta)\sin(\theta) - \cos^2(\theta)\sin^2(\theta))r^2 \\
&\quad -\frac{1}{2}\cos^3(\theta)\sin^2(\theta)r^3 + \cos(\theta)\sin^5(\theta)r^4)].
\end{aligned}$$

To look for the limit cycles, we must solve the equation

$$F_{20}(r) = r^3\left(-\frac{1}{4} + \frac{3}{16}r^2 - \frac{1}{64}r^4\right) = 0.$$

This equation has two positive roots $r_1 = -1 + \sqrt{5}$ and $r_2 = 1 + \sqrt{5}$. So system (14) has exactly two limit cycles bifurcating from the periodic orbits of the linear differential system with $\varepsilon = 0$.

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