

Chebyshev-Halley's Method without Second Derivative of Eight-Order Convergence

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Abstract

In this paper, we present a family of modified Chebyshev-Halley method without second derivative for solving nonlinear equation. Based on the convergence analysis, it is obtained that the proposed method has eighth-order convergence for $\theta = 1$ and $\beta = 3/2$. This new method requires evaluation of three functions and one of first derivative per iteration with efficiency index $8^{1/4} \approx 1,6281$. The numerical simulation is given to illustrate the method and to compare with other method so that one can see the efficiency and performance of the proposed method.

AMS subject classification: 65H04, 65G50.

Keywords: Chebyshev-Halley's Method, Convergence order.

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1. Introduction

The non-linear equations is one of the most important problems in science and engineering. There are many the nonlinear equation can not be solved analytically, so the numerical technique is an alternative to solve it using the iterative computation.

The Newton method is one of a widely iterative method used to find a root α of nonlinear equation $f(x) = 0$ by using algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1.1)$$

that converges quadratically in a neighborhood of α .

To improve a local convergence order, some modifications have been proposed by many researchers. One of the developed iterative method is a classical Chebyshev-Halley method[1, 2]:

$$x_{n+1} = x_n - \left(1 + \frac{f''(x_n)f(x_n)}{2[f'(x_n)^2 - \beta f''(x_n)f(x_n)]} \right) \frac{f(x_n)}{f'(x_n)}. \quad (1.2)$$

This method is known to converges cubically which requires three evaluation of functions with efficiency index $3^{1/3} \approx 1,4224$ by including some famous iterative method depend on β as particular cases, namely, the classical Chebyshev's method ($\beta = 0$), Halley's method ($\beta = 1/2$), and Super-Halley method ($\beta = 1$)[8, 9].

In some cases, second derivative in (1.2) became to be a serious problem. So, many variant of Chebyshev-Halley method without second derivative have been studied using various approaches in some literature, see[3, 4, 5, 7, 11, 10, 14, 15, 16, 17, 20].

In order to improve order-convergence, the Chebyshev-Halley method has been experience various modification. This modification is done by adding the third step in Newton form in which f' is approximated by Taylor series [13]. Moreover, Cordero, *et al.* [6] and Sharma [18] added the third step in Newton form in which f' is approximated by a third order polynomial interpolation.

In this paper we study the Chebyshev-Halley's method with avoid second derivative by using equality of Potra-Ptak and Halley method [16] in which f' in third step is approximated by Gauss' quadrature. A numerical simulation is given to compare the efficiency of the proposed method with other methods.

2. Modification of Chebyshev-Halley's Method

To derive an approximation to $f''(x_n)$, let us consider Potra-Ptak and Halley method [16],

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} \quad (2.3)$$

and

$$x_{n+1} = x_n - \frac{2f'(x_n)f(x_n)}{2f'(x_n)^2 - \theta f''(x_n)f(x_n)}. \quad (2.4)$$

In order to improve efficiency index, we approximate $f''(x_n)$ by arranging again two equations of (2.3) and (2.4) into the form of the following equality:

$$\frac{f(x_n) + f(y_n)}{f'(x_n)} = \frac{2f'(x_n)f(x_n)}{2f'(x_n)^2 - \theta f''(x_n)f(x_n)} \tag{2.5}$$

such that an explicit form of $f''(x_n)$ is given by

$$f''(x_n) = \frac{2f(y_n)f'(x_n)^2}{\theta f(x_n)(f(x_n) + f(y_n))}. \tag{2.6}$$

By substituting (2.6) into (1.2), we obtain a variant of Chebyshev-Halley method without second derivative as follows:

$$x_{n+1} = x_n - \left(1 + \frac{\frac{2f(y_n)f'(x_n)^2}{\theta f(x_n)(f(x_n)+f(y_n))}}{2 \left[f'(x_n)^2 - \beta \left(\frac{2f(y_n)f'(x_n)^2}{\theta f(x_n)(f(x_n)+f(y_n))} \right) f(x_n) \right]} \right) \frac{f(x_n)}{f'(x_n)}, \tag{2.7}$$

and henceforth we find

$$x_{n+1} = x_n - \left(1 + \frac{f(y_n)}{\theta f(x_n) + (\theta - 2\beta)f(y_n)} \right) \frac{f(x_n)}{f'(x_n)}, \tag{2.8}$$

where y_n is given by (1.1).

The family iterative method in (2.8) is variant of second steps Chebyshev-Halley method which involving evaluation of three functions, i.e. $f(x_n)$, $f(y_n)$ and $f'(x_n)$. Furthermore, to improve the order of convergence of (2.8), we add the third step Newton,

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \tag{2.9}$$

in which z_n is given by (2.8). One can see that (2.9) involving three functions evaluation. Since the three steps iterative method use at least four function evaluation [19], we must reduce $f'(z_n)$ using Gauss' quadrature approximation [12].

In order to derive Gauss' quadrature approximation, let us consider the following Newton's formula:

$$f'(z_n) = f'(x_n) + \int_{x_n}^{z_n} f''(t)dt, \tag{2.10}$$

where second derivative in (2.10) is approximated by weight function:

$$\int_{x_n}^{z_n} f''(t)dt = a_1f(x_n) + a_2f(y_n) + a_3f(z_n) + a_4f'(x_n). \tag{2.11}$$

In order to find parameters a_1, a_2, a_3 and a_4 , we use four functions $f(t) = 1, f(t) = t, f(t) = t^2$ and $f(t) = t^3$ such that we obtained the following four equations:

$$\left. \begin{aligned} a_1 + a_2 + a_3 &= 0 \\ a_1x_n + a_2y_n + a_3z_n + a_4 &= 0 \\ a_1x_n^2 + a_2y_n^2 + a_3z_n^2 + 2a_4x_n &= 2(z_n - x_n) \\ a_1x_n^3 + a_2y_n^3 + a_3z_n^3 + 3a_4x_n^2 &= 3(z_n^2 - x_n^2) \end{aligned} \right\} \tag{2.12}$$

The solution of the system (2.12) gives four constants a_1, a_2, a_3 and a_4 , and henceforth by substituting it into (2.11), one obtain for $f'(z_n)$ as follows:

$$f'(z_n) = -\frac{(y_n - z_n)(3x_n - 2y_n - z_n)}{(x_n - z_n)(x_n - y_n)^2} f(x_n) + \frac{(x_n - z_n)^2}{(y_n - z_n)(x_n - y_n)^2} f(y_n) - \frac{x_n + 2y_n - 3z_n}{(x_n - z_n)(y_n - z_n)} f(z_n) + \left(1 - \frac{-x_n - 2y_n + z_n}{x_n - y_n}\right) f'(x_n) \quad (2.13)$$

Simplifying of (2.13) and substitute into (2.9), we obtained:

$$x_{n+1} = z_n - f(z_n) \frac{(x_n - z_n)(x_n - y_n)^2(y_n - z_n)}{Af(x_n) + Bf(y_n) + Cf(z_n) + Df'(x_n)}, \quad (2.14)$$

where

$$A = -(y_n - z_n)^2(3x_n - 2y_n - z_n), \quad (2.15)$$

$$B = (x_n - z_n)^3, \quad (2.16)$$

$$C = -(x_n - y_n)^2(x_n + 2y_n - 3z_n), \quad (2.17)$$

$$D = (y_n - z_n)^2(x_n - z_n)(x_n - y_n). \quad (2.18)$$

Therefore, the completely three steps of Chebyshev-Halley iterative method can be written in the following form:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (2.19)$$

$$z_n = x_n - \left(1 + \frac{f(y_n)}{\theta f(x_n) + (\theta - 2\beta)f(y_n)}\right) \frac{f(x_n)}{f'(x_n)}, \quad (2.20)$$

$$x_{n+1} = z_n - f(z_n) \frac{(x_n - z_n)(x_n - y_n)^2(y_n - z_n)}{Af(x_n) + Bf(y_n) + Cf(z_n) + Df'(x_n)}. \quad (2.21)$$

The family of (2.19)–(2.21) is known as iterative method without second derivative with evaluation of four functions, i.e. $f(x_n), f(y_n), f(z_n)$ and $f'(x_n)$.

3. Order of Convergence

The following Theorem assert that convergence order of the method defined by (2.19)–(2.21) is eight.

Theorem 3.1. Let $D \subset \mathbb{R}$ is an open interval and the function $f : D \rightarrow \mathbb{R}$ has a simple root $\alpha \in D$. Let $\theta = 1, \beta = 3/2$ and $f(x)$ is sufficiently smooth in the neighborhood of the root α , then the order of convergence of the proposed method defined by (2.19)–(2.21) is eight, with error:

$$e_{n+1} = -c_2^3(c_2^2 - c_3)^2 e_n^8 + O(e_n^9). \quad (3.22)$$

Proof. Let $e_n = x_n - \alpha$ and $c_j = \frac{1}{j} \frac{f^{(j)}(\alpha)}{f'(\alpha)}$. Expanding f near α using Taylor series, we find

$$f(x_n) = f'(\alpha) \left(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5) \right) \quad (3.23)$$

and

$$f'(x_n) = f'(\alpha) \left(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4) \right). \quad (3.24)$$

From (3.23)–(3.24), we obtain

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + (7c_2 c_3 + 4c_2^2 - 3c_4) e_n^4 + O(e_n^5), \quad (3.25)$$

and henceforth substituting (3.25) into (2.19), we find

$$y_n = \alpha + c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 - (7c_2 c_3 + 4c_2^2 - 3c_4) e_n^4 + O(e_n^5). \quad (3.26)$$

Hence, Taylor expansion of $f(y_n)$ around α is given by

$$f(y_n) = f'(\alpha) (c_2 e_n^2 - 2(c_2^2 - c_3) e_n^3 - (7c_2 c_3 + 4c_2^2 - 3c_4) e_n^4 + O(e_n^5)). \quad (3.27)$$

Furthermore, we find from (3.23) and (3.27)

$$\begin{aligned} \frac{f(y_n)}{\theta f(x_n) + (\theta - 2\beta) f(y_n)} &= \frac{c_2}{\theta} e_n + \left(\frac{2c_3 - 4c_2^2}{\theta} + \frac{2c_2^2 \beta}{\theta^2} \right) e_n^2 \\ &+ \left(\frac{7c_2^3 - 14c_2 c_3 + 3c_4}{\theta} + \frac{8\beta(c_2 c_3 - 2c_2^3)}{\theta^2} \right) e_n^3 \\ &+ O(e_n^4), \end{aligned} \quad (3.28)$$

and henceforth by using (3.25) and (3.28) we get

$$\begin{aligned} \frac{f(y_n)}{\theta f(x_n) + (\theta - 2\beta) f(y_n)} \frac{f(x_n)}{f'(x_n)} &= \left(\frac{c_2}{\theta} \right) e_n^2 + \left(\frac{-5c_2^2 + 2c_3}{\theta} + \frac{2c_2^2 \beta}{\theta^2} \right) e_n^3 \\ &+ \frac{13c_2^3 - 18c_2 c_3 + 3c_4}{\theta} + \frac{(8c_2 c_3 - 18c_2^3) \beta}{\theta^2} \\ &+ \frac{4c_2^3 \beta}{\theta^3} e_n^4 + O(e_n^5). \end{aligned} \quad (3.29)$$

Substituting (3.29) into (2.20) and by using $x_n = \alpha + e_n$, then it is found

$$\begin{aligned} z_n &= \alpha + \left(c_2 - \frac{c_2}{\theta} \right) e_n^2 + \left(2c_3 - 2c_2^2 + \frac{5c_2^2 - 2c_3}{\theta} - \frac{2c_2^2 \beta}{\theta^2} \right) e_n^3 \\ &+ \left(-4c_2^3 - 7c_2 c_3 + 3c_4 + \frac{-13c_2^3 + 18c_2 c_3 - 3c_4}{\theta} \right. \\ &\left. + \frac{(18c_2^3 - 8c_2 c_3) \beta}{\theta^2} - \frac{4c_2^3 \beta^2}{\theta^3} \right) e_n^4 + O(e_n^5). \end{aligned} \quad (3.30)$$

Likewise, the Taylor expansion of $f(x)$ around α for $x = z_n$ is given

$$\begin{aligned} f(z_n) &= \frac{1}{\theta}(\theta - 1)c_2e_n^2 + \left(\frac{1}{\theta^2}(-2\theta^2 + 5\theta - 2\beta)c_2^2 + \frac{1}{\theta}(2\theta - 2)c_3 \right) e_n^3 \\ &\quad \left(\frac{1}{\theta^3}(-15\theta^2 + \theta(18\beta + 1) - 4\beta^2)c_2^3 - 3c_2^2 + \frac{1}{\theta^2}(-7\theta^2 + \theta - 8\beta)c_3c_2 \right. \\ &\quad \left. + \frac{1}{\theta}(3\theta - 3)c_4 \right) e_n^4 + O(e_n^5) \end{aligned} \quad (3.31)$$

By using (3.23), (3.24), (3.26), (3.27), (3.30), (3.31), then it is obtained

$$\begin{aligned} (x_n - y_n)^2(y_n - z_n)(x_n - z_n) &= \left(\frac{c_2}{\theta} \right) e_n^5 + \left(\frac{(-8\theta + 2\theta\beta + 1)c_2^2 + 2c_3\theta}{\theta^2} \right) e_n^6 \\ &\quad + O(e_n^7) \end{aligned} \quad (3.32)$$

and

$$Af(x_n) = \frac{1}{\theta^2}(-3c_2^2)e_n^6 + O(e_n^7), \quad (3.33)$$

$$Bf(y_n) = c_2e_n^5 + \frac{1}{\theta}((5\theta - 3)c_2^2 - 2c_3\theta)e_n^6 + O(e_n^7), \quad (3.34)$$

$$\begin{aligned} Cf(z_n) &= \frac{1}{\theta}(\theta - 1)c_2e_n^5 + \frac{1}{\theta^2}((2\theta - 5 + 2\beta)c_2^2 + 3(\theta - 1)c_2 - 2(\theta - 1)c_3)e_n^6 \\ &\quad + O(e_n^7), \end{aligned} \quad (3.35)$$

$$Df'(x_n) = \left(\frac{c_2^2}{\theta^2} \right) e_n^6 + O(e_n^7), \quad (3.36)$$

where A , B , C and D are given by (2.15)–(2.18), respectively. Furthermore, by using (3.32)–(3.36) we find the approximation of $f'(z_n)$ as follows:

$$\frac{1}{f'(z_n)} = 1 + \frac{2}{\theta}(1 - \theta)c_2^2e_n^2 + \frac{1}{\theta^2}((-10\theta + 4\theta^2 + 4\beta)c_2^3 + (4\theta - 4\theta^2)c_3c_2)e_n^3 + O(e_n^4), \quad (3.37)$$

and henceforth from (2.21) this implies

$$\begin{aligned}
 x_{n+1} = & \alpha + \frac{1}{\theta^2}(\theta - 1)^2 c_2^3 e_n^4 \\
 & + \frac{1}{\theta^3} \left(2c_2^2(\theta - 1) (2c_3\theta(\theta - 1) + c_2^2(2\theta^2 - 5\theta + 2\beta)) \right) e_n^5 \\
 & + \frac{1}{\theta^4} \left((10\theta^4 + 62\theta^3 + (-80\beta - 83)\theta^2 + (16\beta^2 + 92\beta + 2)\theta - 20\beta^2) c_2^5 \right. \\
 & \left. (35\theta^4 - 125\theta^3 + (40\beta + 89)\theta^2 + (-40\beta + 1)\theta) c_3 c_2^3 - 12\theta^2(\theta - 1)^2 c_4 c_2^2 \right. \\
 & \left. - 4\theta^2(\theta^2 - 1) c_3^2 c_2 \right) e_n^6 \\
 & + O(e_n^7)
 \end{aligned} \tag{3.38}$$

The coefficients of e_n^4 and e_n^5 in equation (3.38) contain a factor of $(\theta - 1)$, consequently the order of convergence of (3.38) is at least six for $\theta = 1$, and given by

$$\begin{aligned}
 x_{n+1} = & -c_2^5(2\beta - 3)^2 e_n^6 - 2(2\beta - 3) \left((4\beta^2 - 18\beta + 17) c_2^2 + (8\beta - 11) c_3 \right) c_2^4 e_n^7 \\
 & + O(e_n^8).
 \end{aligned} \tag{3.39}$$

Ultimately, by taking $\beta = \frac{3}{2}$, we have

$$x_{n+1} = -c_2^3(-c_3 + c_2^2)^2 e_n^8 + O(e_n^9) \tag{3.40}$$

that shows the order of convergence is eight. ■

The proposed method require evaluation of three functions and one first derivative per iteration. Based on definition of efficiency index, i.e. $IE = p^{1/m}$ [19], where p is the order of the method and m is the number of function evaluations per iteration, we have the efficiency index equals to $8^{1/4} \approx 1.6818$, which is better than the Newton's method $2^{1/2} \approx 1.4142$, classical Chebyshev-Halley ($\beta = 1/2$) $3^{1/3} \approx 1.4422$ and variant of Chebyshev-Halley $6^{1/4} \approx 1.5784$ [20].

4. Numerical Simulation

In this section we present some numerical simulations by using several functions to show the performance of the proposed method (M-8) in (2.21), and compare it with Newton's method (N2), classical Chebyshev-Halley method with $\beta = 1/2$ (CH3) [9], variant of Chebyshev-Halley with fourth order of convergence (VCH4) [20] and variant of Chebyshev-Halley with sixth order of convergence (VCH6) [13]. We used several following test functions and displayed the computed approximate zeros α round up to

20th decimal places.

$$\begin{aligned}
 f_1(x) &= e^{x^2+7x-30} - 1, \alpha = 3.00000000000000000000 \\
 f_2(x) &= e^x - 4x^2, \alpha = 4.30658472822069929833 \\
 f_3(x) &= \cos(x) - 1, \alpha = 0.73908513321516064165 \\
 f_4(x) &= (x - 1)^3 - 1, \alpha = 2.00000000000000000000 \\
 f_5(x) &= x^3 + 4x^2 - 10, \alpha = 1.36523003414096845760 \\
 f_6(x) &= e^{-x^2+x+2} - \cos(x + 1) + x^3 + 1, \alpha = -1.00000000000000000000
 \end{aligned}$$

All computations are performed by using Maple 13.0 with 850 digits floating point arithmetics.

Table 1 shows the number of iteration (IT) required such that $|x_{n+1} - x_n| < \epsilon$ where $\epsilon = 10^{-95}$ and the computational order of convergence (COC) in the parentheses by using as following formula

$$\rho \approx \frac{\ln |(x_{n+2} - \alpha)/(x_{n+1} - \alpha)|}{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}. \tag{4.41}$$

Table 1: The number of iteration and COC

$f(x)$	x_0	N2	CH3 $\beta = 1/2$	VCH4 $\beta = 1/2$	VCH6 $\beta = 1$	M-8 $\theta = 1, \beta = 3/2$
$f_1(x)$	2.9	10 (1.9999)	6 (2.9999)	5 (3.9999)	4 (5.9999)	3 (7.9995)
	3.2	10 (1.9999)	6 (3.0000)	5 (3.9999)	4 (5.9998)	4 (7.9999)
$f_2(x)$	4.0	8 (1.9999)	5 (3.0000)	4 (3.9999)	3 (5.9999)	3 (7.9999)
	4.5	7 (1.9999)	5 (2.9999)	4 (3.9999)	3 (5.9999)	3 (7.9999)
$f_3(x)$	-0.5	8 (1.9999)	6 (3.0000)	5 (3.9999)	4 (5.9999)	3 (7.9997)
	1.5	7 (1.9999)	5 (2.9999)	4 (3.9999)	3 (5.9999)	3 (7.9999)
$f_4(x)$	1.5	10 (1.9999)	6 (3.0000)	5 (3.9999)	4 (6.0000)	3 (7.9971)
	3.0	9 (1.9999)	6 (2.9999)	5 (3.9999)	4 (5.9999)	3 (7.9989)
$f_5(x)$	1.0	8 (1.9999)	5 (3.0000)	4 (3.9999)	3 (6.0000)	3 (7.9999)
	2.0	8 (1.9999)	5 (2.9999)	4 (3.9999)	3 (6.0000)	3 (7.9999)
$f_6(x)$	-1.5	7 (1.9999)	5 (2.9999)	4 (3.9999)	3 (5.9999)	3 (8.0000)
	0.0	7 (1.9999)	5 (2.9999)	4 (3.9999)	3 (6.0001)	2 (7.9999)

Based on the Table 1 one can see that order of convergence of the proposed method is eight.

The accuration of the proposed method and several other mehdods as a comparison are shown at Table 2.

Table 2: The absolute value of function $|f(x_{n+1})|$ under same total number of functional evaluation with TNFE = 12

f	x_0	N2	CH3 $\beta = 1/2$	VCH4 $\beta = 1/2$	VCH6 $\beta = 1$	M-8 $\theta = 1, \beta = 3/2$
f_1	2.9	4.2424e-09	9.6848e-37	1.7832e-77	1.7729e-62	1.6121e-123
	3.2	4.4796e-07	3.2665e-16	1.8752e-43	8.9121e-18	1.0792e-55
f_2	4.0	5.0253e-33	2.1103e-53	3.5672e-158	7.2789e-175	1.1120e-305
	4.5	3.1919e-52	5.2464e-76	1.4627e-232	5.5174e-196	1.1003e-451
f_3	-0.5	3.4884e-30	7.4037e-22	1.7116e-66	1.8429e-93	6.4582e-196
	1.5	3.7607e-64	1.1496e-51	4.9514e-202	5.1172e-160	3.3226e-442
f_4	1.5	1.8093e-11	6.3909e-24	9.7201e-60	3.8776e-62	1.2330e-115
	3.0	4.6449e-16	6.3909e-24	1.1038e-71	2.0932e-52	2.6533e-139
f_5	1.0	3.9823e-43	2.2349e-60	2.4510e-186	1.0457e-198	1.8792e-370
	2.0	1.2361e-37	4.6600e-52	3.6662e-162	1.5953e-148	4.1631e-322
f_6	-1.5	5.7389e-66	1.5261e-43	1.3689e-167	6.1185e-120	1.7304e-366
	0.0	1.9261e-65	6.3918e-26	1.1346e-153	8.9317e-108	5.5821e-275

5. Conclusion

We have developed a new eight-order convergence method for solving nonlinear equation that require evaluation of three function and one first derivative each iterative step with efficiency index equal to $8^{1/4} \approx 1, 8167$. The computation results show that the proposed method has better performance as compared with the other methods.

Acknowledgements

The author would like to thank to some colleagues for comments which improved the quality of this paper.

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