Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3021-3035

http://www.ripublication.com/gjpam.htm

Common fixed point for generalized-(ψ, α, β)-weakly contractive mappings in generalized metric spaces

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Abstract

In this paper, we establish some common fixed point theorems for generalized-(ψ, α, β)-weakly contractive mappings in generalized metric spaces which extends

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the results of Isik et al. [3]. We present an example in support of our theorem.

AMS subject classification: 47H10, 54H25.
Keywords: Generalized metric, generalized weakly contractive condition, weakly compatible maps, E.A. property, (CLR) property.

1. Introduction and preliminaries

In 2000, Branciari [2] introduced the concept of a generalized metric space as follows:

Definition 1.1. Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (the rectangular inequality).

Then (X, d) is called a generalized metric space (or for short g.m.s.).

Definition 1.2. Let (X, d) be a generalized metric space. A sequence $\{x_n\}$ in X is said to be

(i) g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $\{x_n\} \to x$ as $n \to \infty$ or $lim_{n\to\infty}$â'¡$x_n = x$

(ii) g.m.s. Cauchy sequence if and only if for each $\epsilon > 0$ there exists a natural number $n(\epsilon)$ such that for all $n > m > n(\epsilon)$, $d(x_n, x_m) < \epsilon$.

(iii) complete g.m.s. if every g.m.s. Cauchy sequence is g.m.s. convergent in X.

We denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(ψ1) ψ is continuous and monotone nondecreasing,

(ψ2) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ the set of functions ± : $[0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(α1) α is continuous,

(α2) $\alpha(t) = 0$ if and only if $t = 0$.

We denote by Γ the set of functions $\beta : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(β1) β is lower semi-continuous,

(β2) $\beta(t) = 0$ if and only if $t = 0$.

Definition 1.3. A mapping $T : X \to X$ is said to be (ψ, α, β) weak contraction if there exists three maps $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ such that $\psi(d(Tx, Ty)) \leq \alpha(d(x, y))$âŁ“$\beta(d(x, y))$, where

(i) ψ is continuous and monotone non-decreasing,

(ii) α is continuous,

(iii) β is lower semi-continuous,

(iv) $\psi(t) = 0 = \alpha(t) = \beta(t)$, if and only if, $t = 0$.

Now, we introduce the following notions:

Definition 1.4. A mapping $T : X \to X$ is said to be generalized (ψ, α, β) weak contraction if there exists three maps $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ such that $\psi(d(Tx, Ty)) \leq \alpha(M(x, y))\beta(M(x, y))$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

(i) ψ is continuous and monotone non-decreasing,

(ii) α is continuous,

(iii) β is lower semi-continuous,

(iv) $\psi(t) = 0 = \alpha(t) = \beta(t)$, if and only if, $t = 0$.

Definition 1.5. A mapping $g : X \to X$ is said to be generalized (ψ, α, β) weak contraction with respect to $f : X \to X$ if there exists three maps $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ such that

$$\psi(d(gx, gy)) \leq \alpha(N(x, y))\beta(N(x, y)),$$

where

$$N(fx, fy) = \max\left\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx, gx)d(fy, gy)}{1 + d(fx, fy)}, \frac{d(fx, gx)d(fy, gy)}{1 + d(gx, gy)}\right\},$$

(i) ψ is continuous and monotone non-decreasing,

(ii) α is continuous,

(iii) β is lower semi-continuous,

(iv) $\psi(t) = 0 = \alpha(t) = \beta(t)$, if and only if, $t = 0$.

In 1996, Jungck et al. [4] introduced the concept of weakly compatible maps as follows:

Definition 1.6. Two maps f and g defined on a self map X are said to be weakly compatible if they commute at their coincidence points.

In 2002, Aamri et al. [1] introduced the notion of E.A. property as follows:

Definition 1.7. Two self-mappings f and g of a metric space (X, d) are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty}$â'¡$gx_n = t$ for some $t \in X$

In 2011, Sintunavarat et al. [5] introduced the notion of (CLRg) property as follows:

Definition 1.8. Two self-mappings f and g of a metric space (X, d) are said to satisfy (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

2. Main Results

For proving our main results, we need the following Lemma:

Lemma 2.1. Let $\{a_n\}$ be a sequence of non-negative real numbers. If

$$\psi(a_{n+1}) \leq \alpha(a_n)\beta(a_n) \tag{2.1}$$

for all $n \in N$, where $\psi \in \Psi, \alpha \in \Phi, \beta \in \Gamma$ and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \tag{2.2}$$

for all $t > 0$, then the following hold:

(i) $a_{n+1} \leq a_n$ if $a_n > 0$,

(ii) $a_n \to 0$ as $n \to \infty$.

Theorem 2.2. Let f and g be self mappings of a Hausdorff g.m.s. (X, d) satisfying the followings:

$$gX \subseteq fX, \tag{2.3}$$

$$fX \text{ or } gX \text{ is a complete subspace of } X, \tag{2.4}$$

$$\psi(d(gx, gy)) \leq \alpha(N(fx, fy))\beta(N(fx, fy)), \text{ for all } x, y \in X, \tag{2.5}$$

where $\psi \in \Psi$, $\alpha \in \Phi$ and $\beta \in \Gamma$ and satisfy condition (2.2) with

$$N(fx, fy) = \max\left\{d(fx, fy), d(fx, gx), d(fy, gy), \frac{d(fx, gx)d(fy, gy)}{1 + d(fx, fy)}, \frac{d(fx, gx)d(fy, gy)}{1 + d(gx, gy)}\right\}.$$

Then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $gX \subseteq fX$, we can define the sequences x_n and y_n in X by

$$y_{2n} = fx_{2n+1} = gx_{2n} \text{ for all } n \geq 0. \tag{2.6}$$

Moreover, we assume that if $y_{2n} = y_{2n+1}$ for some $n \in \mathbb{N}$, then there is nothing to prove. Now, we assume that $y_{2n} \neq y_{2n+1}$ for all $n \in \mathbb{N}$. We assert that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.7}$$

Substituting $x = x_{2n}$ and $y = x_{2n+1}$ in (2.5), using (2.6), we have

$$\begin{aligned}\psi(d(y_2n, y_{2n+1})) &= \psi(d(gx_{2n}, gx_{2n+1}))\\ &\leq \alpha(N(fx_2n, fx_{2n+1}))\beta(N(fx_{2n}, fx_{2n+1}))\\ &= \alpha(N(y_{2n-1}, y_{2n}))\beta(N(y_{2n-1}, y_{2n})),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}N(y_{2n-1}, y_{2n}) &= \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})}, \frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})}\right\}\\ &= \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\},\end{aligned}$$

since

$$\frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})} \leq d(y_{2n}, y_{2n+1})$$

and

$$\frac{d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} \leq d(y_{2n-1}, y_{2n}).$$

If $d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1})$, then from (2.8), we get

$$\psi(d(y_{2n}, y_{2n+1})) \leq \alpha(d(y_{2n}, y_{2n+1}))\beta(d(y_{2n}, y_{2n+1})),$$

which implies that, $d(y_{2n}, y_{2n+1}) = 0$, that is, $y_{2n} = y_{2n+1}$, which is a contradiction. So

$$d(y_{2n}, y_{2n+1}) < d(y_{2n-1}, y_{2n}),$$

then from (2.8), we obtain

$$\psi(d(y_{2n}, y_{2n+1})) \leq \alpha(d(y_{2n-1}, y_{2n}))\beta(d(y_{2n-1}, y_{2n})). \tag{2.9}$$

Similarly, we also conclude that

$$\psi(d(y_{2n+1}, y_{2n+2})) \leq \alpha(d(y_{2n}, y_{2n+1}))\beta(d(y_{2n}, y_{2n+1})). \tag{2.10}$$

Generally, we have that for each $n \in \mathbb{N}$

$$\psi(d(y_n, y_{n+1})) \leq \alpha(d(y_{2n}, y_{2n+1}))\beta(d(y_{2n}, y_{2n+1})). \tag{2.11}$$

From (ii) of Lemma 2.1, we obtain that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Next, we prove that $\{y_n\}$ is a g.m.s. Cauchy sequence. Suppose that $\{y_n\}$ is not a g.m.s. Cauchy sequence. Then there exists $\epsilon > 0$ such that for $k \in \mathbb{N}$, there are $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) > k$ satisfying

(a) $m(k)$ is even and $n(k)$ is odd

(b) $d(y_{n(k)}, y_{m(k)}) \leq \epsilon$

(c) $m(k)$ is the smallest even number such that the condition (b) holds

Taking into account (b) and (c), we have that

$$\begin{aligned}\epsilon &\leq d(y_{n(k)}, y_{m(k)})\\ &\leq d(y_{n(k)}, y_{m(k)-2}) + d(y_{m(k)-2}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})\\ &\leq \epsilon + d(y_{n(k)}, y_{n(k)-2}) + d(y_{n(k)-2}, y_{n(k)-1}).\end{aligned} \tag{2.12}$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(y_{n(k)}, y_{m(k)}) = \epsilon,$$

$$\epsilon \leq d(y_{n(k)-1}, y_{m(k)-1}) \tag{2.13}$$

$$\begin{aligned}&\leq d(y_{n(k)-1}, y_{m(k)-3}) + d(y_{m(k)-3}, y_{m(k)-2}) + d(y_{m(k)-2}, y_{m(k)-1})\\ &\leq \epsilon + d(y_{m(k)-3}, y_{m(k)-2}) + d(y_{m(k)-2}, y_{m(k)-1}).\end{aligned} \tag{2.14}$$

Making $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(y_{n(k)-1}, y_{m(k)-1}) = \epsilon \tag{2.15}$$

Substituting $x = x_{n(k)} and y = x_{m(k)}$ in (2.5), we have

$$\psi(d(gx_{n(k)}, gx_{m(k)})) \leq \alpha(N(fx_{n(k)}, fx_{m(k)}))\beta(N(fx_{n(k)}, fx_{m(k)})), \text{ that is,}$$

$$\psi(d(y_{n(k)}, y_{m(k)})) \leq \alpha(N(y_{n(k)-1}, y_{m(k)-1}))\beta(N(y_{n(k)-1}, y_{m(k)-1})), \tag{2.16}$$

where

$$\begin{aligned} d(y_{n(k)-1}, y_{m(k)-1}) \leq N(y_{n(k)-1}, y_{m(k)-1}) \\ = \max\Big\{d(y_{n(k)-1}, y_{m(k)-1}), d(y_{n(k)-1}, y_{n(k)}), d(y_{m(k)-1}, y_{m(k)}), \\ \frac{d(y_{n(k)-1}, y_{n(k)})d(y_{m(k)-1}, y_{m(k)})}{1+d(y_{n(k)-1}, y_{m(k)-1})}, \frac{d(y_{n(k)-1}, y_{n(k)})d(y_{m(k)-1}, y_{m(k)})}{1+d(y_{n(k)}, y_{m(k)})}\Big\}. \\ = \max\{d(y_{n(k)-1}, y_{m(k)-1}), d(y_{n(k)-1}, y_{n(k)}), d(y_{m(k)-1}, y_{m(k)})\}. \end{aligned}$$

Letting $k \to \infty$ in (2.16) and using the lower semi-continuity of β and the continuities of ψ and α, we obtain $\psi(\epsilon) \leq \alpha(\epsilon)\beta(\epsilon)$, which implies that $\epsilon = 0$, by (2.2), a contradiction with $\epsilon > 0$. It follows that $\{y_n\}$ is a g.m.s. Cauchy sequence.

Since fX is complete, so there exists a point u in fX such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{n+1} = u \tag{2.17}$$

Since $u \in fX$, so we can find $p \in X$ such that $fp = u$. We claim that $fp = gp$. From (2.5), we have

$$\begin{aligned} \psi(d(fx_{n+1}, gp)) = \psi(d(gx_n, gp)) \\ \leq \alpha(N(gx_n, gp))\beta(N(gx_n, gp)). \end{aligned}$$

Letting limit as $n \to \infty$ and using the continuity of α and semi-continuity of β, we get

$$\psi(d(fp, gp)) \leq \alpha(lim_{n\to\infty}N(gx_n, gp)) - \beta(lim_{n\to\infty}N(gx_n, gp)), \tag{2.18}$$

where

$$\begin{aligned} N(gx_n, gp) = \max\Big\{d(fx_n, fp), d(fx_n, gx_n), d(fp, gp), \\ \frac{d(fx_n, , gx_n)d(fp, gp)}{1+d(fx_n, fp)}, \frac{d(fx_n, gx_n)d(fp, gp)}{1+d(gx_n, gp)}\Big\}. \end{aligned}$$

Making limit as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} N(gx_n, gp)) = \max\Big\{d(fp, fp), d(fp, fp), d(fp, gp), \\ \frac{d(fp, fp)d(fp, gp)}{1+d(fp, fp)}, \frac{d(fp, gp)d(fp, gp)}{1+d(fp, gp)}\Big\} \\ = d(fp, gp). \end{aligned} \tag{2.19}$$

So, from (2.18) and (2.19), we have

$$\psi(d(fp, gp)) \leq \alpha(d(fp, gp)) - \beta(d(fp, gp)),$$

which implies that, $d(fp, gp) = 0$, that is,

$$fp = gp = u. \tag{2.20}$$

Therefore, p is a point of coincidence of f and g. The uniqueness of the point of coincidence is a consequence of condition (2.5). Now, we show that there exists a common fixed point of f and g. Since f and g are weakly compatible, by (2.20), we have $gfp = fgp$, and

$$gu = gfp = fgp = fu. \tag{2.21}$$

If $p = u$, then p is a common fixed point of f and g. If $p \neq u$, then by (2.5), we have

$$\psi(d(gp, gu)) \leq \alpha(N(gp, gu)) - \beta(N(gp, gu)),$$

where,

$$\begin{aligned} N(gp, gu) = \max \Big\{ & d(fp, fu), d(fp, gp), d(fu, gu), \\ & \frac{d(fp, gp)d(fu, gu)}{1 + d(fp, fu)}, \frac{d(fp, gp)d(fu, gu)}{1 + d(gp, gu)} \Big\} \\ = & max\{d(u, gu), d(u, u), 0, 0, 0\} \\ = & d(u, gu). \end{aligned}$$

Therefore, we have

$$\psi(d(u, gu)) \leq \alpha(d(u, gu)) - \beta(d(u, gu)),$$

which by (2.2) implies that, $d(u, gu) = 0$, that is, $u = gu = fu$. Consequently, u is the unique common fixed point of f and g. ■

Denote by $\wedge$ the set of functions $\gamma : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(h1) γ is a Lebesgue-integrable mapping on each compact subset of $[0, \infty)$,

(h2) for every $\epsilon > 0$, we have

$$\int_0^{\epsilon} \gamma(s)ds < \epsilon.$$

We have the following result.

Theorem 2.3. Let (X, d) be a Hausdorff g.m.s. and $f, g : X \to X$ be self mappings satisfying (2.3), (2.4) and the following:

$$\begin{aligned} \int_0^{d(gx,gy)} \gamma_1(s)ds \leq & \int_0^{N(fx,fy)} \gamma_2(s)ds \\ & - \int_0^{N(fx,fy)} \gamma_3(s)ds, \end{aligned}$$

for all $x, y \in X$, where $\gamma_1, \gamma_2, \gamma_3 \in \wedge$ and satisfy condition (2.2). If f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. On taking $\psi(t) = \int_0^t \gamma_1(s)ds$, $\alpha(t) = \int_0^t \gamma_2(s)ds$ and $\beta(t) = \int_0^t \gamma_3(s)ds$ in Theorem 2.2, we get Theorem 2.3. ■

Taking $\gamma_3(s) = (1-k)\gamma_2(s)$ for $k \in [0, 1)$ in Theorem 2.3, we obtain the following result:

Corollary 2.4. Let (X, d) be a Hausdorff g.m.s. and $f, g : X \to X$ be self mappings satisfying (2.3), (2.4) and the following:

$$\int_0^{d(gx,gy)} \gamma_1(s)ds \leq k \int_0^{N(fx,fy)} \gamma_2(s)ds,$$

for all $x, y \in X$, where $\gamma_1, \gamma_2 \in \wedge$ and satisfy condition (2.2). If f and g are weakly compatible, then f and g have a unique common fixed point.

Remark 2.5. If $N(fx, fy) = d(fx, fy)$, then (2.5) reduces to

$$\psi(d(gx, gy)) \leq \alpha(d(fx, fy)) - \beta(d(fx, fy)), \tag{2.22}$$

which is condition (2.3) of Theorem 1 [3].

Remark 2.6. If f is the identity mapping, then (2.22) reduces to

$$\psi(d(gx, gy)) \leq \alpha(d(x, y)) - \beta(d(x, y)). \tag{2.23}$$

Example 2.7. Let $X = [0, 10] \cup 11, 12, 13, \ldots$ and

$$d(x, y) = \begin{cases} |x-y|, & \text{if } x, y \in [0, 10], x \neq y; \\ x+y, & \text{if atleast one of } x \text{ or } y \notin [0, 10] \text{ and } x \neq y; \\ 0, & \text{if } x = y. \end{cases} \tag{2.24}$$

Then (X, d) is a Hausdorff and g.m.s.

Let $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ be defined as

$$\psi(t) = \alpha(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 10; \\ t^2, & \text{if } t > 10 \quad \text{and} \end{cases}$$

$$\beta(t) = \begin{cases} \frac{1}{5}t^2, & \text{if } 0 \leq t \leq 10; \\ \frac{1}{5}, & \text{if } t > 10. \end{cases}$$

Let $g : X \to X$ be defined as

$$g(x) = \begin{cases} x - \frac{1}{5}x^2, & \text{if } 0 \le x \le 10; \\ x - 10, & \text{if } x \in \{11, 12, 13, \ldots\}. \end{cases}$$

Without loss of generality, we assume that $x > y$ and discuss the following cases:

Case 1. $(x \in [0, 10])$. Then

$$\begin{aligned} \psi(d(gx, gy)) &= \left(x - \frac{1}{5}x^2\right) - \left(y - \frac{1}{5}y^2\right) \\ &= (x - y) - \frac{1}{5}(x - y)(x + y) \le (x - y) - \frac{1}{5}(x - y)^2 \\ &= d(x, y) - \frac{1}{5}(d(x, y))^2 \\ &= \alpha(d(x, y)) - \beta(d(x, y)). \end{aligned}$$

Case 2. $(x \in \{12, 13, \ldots\})$. Then

$$d(gx, gy) = d\left(x - 10, y - \frac{1}{5}y^2\right), \text{ if } y \in [0, 10],$$
$$\text{or, } d(gx, gy) = x - 10 + y - \frac{1}{5}y^2 \le x + y - 10.$$

and

$$d(gx, gy) = d(x - 10, y - 10), \text{ if } y \in \{11, 12, 13, \ldots\},$$
$$\text{or, } d(gx, gy) = x - 10 + y - 10 < x + y - 10.$$

Consequently, we have

$$\begin{aligned} \psi(d(gx, gy)) &= (d(gx, gy))^2 \le (x + y - 10)^2 < (x + y - 10)(x + y + 10) \\ &= (x + y)^2 - 100 < (x + y)^2 - \frac{1}{5} \\ &= \alpha(d(x, y)) - \beta(d(x, y)). \end{aligned}$$

Case 3. $(x = 11)$. Then $y \in [0, 10]$, $gx = 1$ and $d(gx, gy) = 1 - \left(y - \frac{1}{5}y^2\right) \le 1$.

So, we have $\psi(d(gx, gy)) \le \psi(1) = 1$. Again $d(x, y) = 11 + y$. So,

$$\begin{aligned} \alpha(d(x, y)) - \beta(d(x, y)) &= (11 + y)^2 - \frac{1}{5} \\ &= 121 + y^2 + 22y - \frac{1}{5} \\ &= \frac{604}{5} + 22y + y^2 > 1 = \psi(d(gx, gy)). \end{aligned}$$

Considering all the above cases, we conclude that the inequality (2.23) remains valid for ψ, α, and β constructed as above and consequently, g has a unique fixed point.

Clearly, it is seen that 0 is the unique fixed point of g. ■

3. Weakly compatible and E.A. property

Theorem 3.1. Let f and g be self mappings of a Hausdorff g.m.s (X, d) satisfying (2.3), (2.5) and the following:

$$f \text{ and } g \text{ are weakly compatible,} \tag{3.25}$$

$$f \text{ and } g \text{ satisfy the E.A. property.} \tag{3.26}$$

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the E.A. property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = z, \text{ for some } z \text{ in } X. \tag{3.27}$$

Since $gX \subseteq fX$, there exists a sequence $\{y_n\}$ in X such that $gx_n = fy_n$. Hence $lim_{n\to\infty} fx_n = z$. Now, we shall show that $lim_{n\to\infty} gy_n = z$. Let us suppose that $lim_{n\to\infty} gy_n = t$. From (2.5), we have

$$\psi(d(gx_n, gy_n)) \leq \alpha(N(fx_n, fy_n))\beta(N(fx_n, fy_n)).$$

Letting $n \to \infty$, we have

$$\psi(d(z,t)) \leq \alpha(lim_{n\to\infty} N(fx_n, fy_n))\beta(lim_{n\to\infty} N(fx_n, fy_n)), \tag{3.28}$$

where,

$$\begin{aligned} N(fx_n, fy_n) = \max \{&d(fx_n, fy_n), d(fx_n, gx_n), d(fy_n, gy_n), \\ &\frac{d(fx_n, gx_n)d(fy_n, gy_n)}{1 + d(fx_n, fy_n)}, \frac{d(fx_n, gx_n)d(fy_n, gy_n)}{1 + d(gx_n, gy_n)}\Big\}. \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} N(fx_n, fy_n) &= \max \left\{d(z,z), d(z,z), d(z,t), \frac{d(z,z)d(z,t)}{1 + d(z,z)}, \frac{d(z,z)d(z,t)}{1 + d(z,t)}\right\} \\ &= d(z,t). \end{aligned} \tag{3.29}$$

Thus, from (3.4) and (3.5), we get

$$\psi(d(z,t)) \leq \alpha(d(z,t))\beta(d(z,t)),$$

which implies that $d(z, t) = 0$, that is, $z = t$. Hence, $\lim_{n\to\text{â'¡}\infty} gy_n = z$. Now, suppose that fX is complete subspace of X. Then, there exists u in X such that $z = fu$. Subsequently, we have

$$\begin{aligned}\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} \\ fy_n = \lim_{n\to\infty} gy_n = z = fu.\end{aligned}$$

Now, we show that $fu = gu$. From (2.5), we have

$$\psi(d(gx_n, gu)) \leq \alpha(N(fx_n, fu))\beta(N(fx_n, fu)).$$

Letting $n \to \infty$, we have

$$\psi(d(z, gu)) \leq (\lim_{n\to\infty} N(fx_n, fu))(\lim_{n\to\infty} N(fx_n, fu)), \tag{3.30}$$

where,

$$\begin{aligned}N(fx_n, fu) = \max \Bigg\{ & d(fx_n, fu), d(fx_n, gx_n), \\ & \frac{d(fu, gu), d(fx_n, gx_n)d(fu, gu)}{1 + d(fx_n, fu)}, \frac{d(fx_n, gx_n)d(fu, gu)}{1 + d(gx_n, gu)} \Bigg\}.\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} N(fx_n, fu) = \max \Bigg\{ & d(z, z), d(z, z), d(z, gu), \\ & \frac{d(z, z)d(z, gu)}{1 + d(z, fu)}, \frac{d(z, z)d(z, gu)}{1 + d(z, gu)} \Bigg\} = d(z, gu).\end{aligned} \tag{3.31}$$

Thus, from (3.6) and (3.7), we get

$$\psi(d(z, gu)) \leq \alpha(d(z, gu))\beta(d(z, gu)),$$

which implies that, $d(z, gu) = 0$, that is, $z = gu = fu$. Since f and g are weakly compatible, therefore, $gfu = fgu$, implies that, $ffu = fgu = gfu = ggu$. Now, we claim that gu is the common fixed point of f and g. From (2.5), we have

$$\begin{aligned}\psi(d(gu, ggu)) &\leq \alpha(N(fu, ffu))\beta(N(fu, ffu)) \\ &= \alpha(d(fu, ffu))\beta(d(fu, ffu)) \\ &= \alpha(d(gu, ggu))\beta(d(gu, ggu)),\end{aligned}$$

which implies that, $gu = ggu = ffu$.

Therefore, gu is the common fixed point of f and g. For the uniqueness, let z and w be two common fixed points of f and g. From (2.5), we have

$$\begin{aligned}\psi(d(z,w)) &= \psi(d(gz,gw)) \\ &\leq \alpha(N(fz,fw))\beta(N(fz,fw)),\end{aligned} \tag{3.32}$$

where,

$$\begin{aligned}N(fz,fw) = \max\Bigg\{&d(fz,fw), d(fz,gz), d(fw,gw), \\ &\frac{d(fz,gz)d(fw,gw)}{1+d(fz,fw)}, \frac{d(fz,gz)d(fw,gw)}{1+d(gz,gw)}\Bigg\} \\ = \max&\{d(z,w),0,0,0,0\} = d(z,w).\end{aligned} \tag{3.33}$$

From (3.8) and (3.9), we get

$$\psi(d(z,w)) \leq \alpha(d(z,w))\beta(d(z,w)),$$

which implies that, $d(z,w) = 0$, that is, $z = w$. Therefore, f and g have a unique common fixed point in X. ■

4. Weakly compatible and (CLR_f) property

Theorem 4.1. Let f and g be self mappings of a Hausdorff g.m.s (X,d) satisfying (2.3), (2.5), (3.1) and the following:

$$f \text{ and } g \text{ satisfy } (CLR_f) \text{ property.} \tag{4.34}$$

Then f and g have a unique common fixed point in X.

Proof. Since f and g satisfy the (CLR_f) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx, \text{ for some } x \in X.$$

From (2.5), we have

$$\psi(d(gx_n,gx)) \leq \alpha(N(fx_n,fx))\beta(N(fx_n,fx)).$$

Letting $n\to\infty$, we have

$$\psi(d(fx,gx)) \leq \alpha(\lim_{n\to\infty} N(fx_n,fx))\beta(\lim_{n\to\infty} N(fx_n,fx)), \tag{4.35}$$

where,

$$\begin{aligned}N(fx_n,fx) = \max\Bigg\{&d(fx_n,fx), d(fx_n,gx_n), d(fx,gx), \\ &\frac{d(fx_n,gx_n)d(fx,gx)}{1+d(fx_n,fx)}, \frac{d(fx_n,gx_n)d(fx,gx)}{1+d(gx_n,gx)}\Bigg\}.\end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} N(fx_n, fx) = \max\Big\{d(fx, fx), d(fx, fx), d(fx, gx), \frac{d(fx, fx)d(fx, gx)}{1 + d(fx, fx)}, \frac{d(fx, fx)d(fx, gx)}{1 + d(fx, gx)}\Big\} = d(fx, gx). \tag{4.36}$$

Thus, from (4.3) and (4.4), we get

$$\psi(d(fx, gx)) \leq \alpha(d(fx, gx))\beta(d(fx, gx)),$$

which implies that $d(fx, gx) = 0$, that is, $gx = fx$. Now, let $z = fx = gx$. Since f and g are weakly compatible, therefore, $fgx = gfx$, implies that, $fz = fgx = gfx = gz$. Now, we claim that $gz = z$. From (2.5), we have

$$\psi(d(gz, z)) = \psi(d(gz, gx)) \leq \alpha(N(fz, fx))\beta(N(fz, fx)). \tag{4.37}$$

where,

$$N(fz, fx) = \max\Big\{d(fz, fx), d(fz, gz), d(fx, gx), \frac{d(fz, gz)d(fx, gx)}{1 + d(fz, fx)}, \frac{d(fz, gz)d(fx, gx)}{1 + d(gz, gx)}\Big\} = \max\{d(gz, z), 0, 0, 0, 0\} = d(gz, z). \tag{4.38}$$

From (4.5) and (4.6), we get

$$\psi(d(gz, z)) \leq \alpha(d(gz, z))\beta(d(gz, z)),$$

which implies that, $d(gz, z) = 0$, that is, $gz = z$. Hence, $gz = z = fz$. So, z is the common fixed point of f and g. For the uniqueness, let w be another common fixed point of f and g. From (2.5), we have

$$\psi(d(z, w)) = \psi(d(gz, gw)) \leq \alpha(N(fz, fw))\beta(N(fz, fw)), \tag{4.39}$$

where,

$$N(fz, fw) = \max\Big\{d(fz, fw), d(fz, gz), d(fw, gw), \frac{d(fz, gz)d(fw, gw)}{1 + d(fz, fw)}, \frac{d(fz, gz)d(fw, gw)}{1 + d(gz, gw)}\Big\} = \max\{d(z, w), 0, 0, 0, 0, 0\} = d(z, w). \tag{4.40}$$

From (4.7) and (4.8), we get

$$\psi(d(z, w)) \leq \alpha(d(z, w))\beta(d(z, w)),$$

which implies that, $d(z, w) = 0$, that is, $z = w$. Therefore, f and g have a unique common fixed point in X. ■

References

[1] Aamri, M., Moutawakil, D. El., 2002, "Some new common fixed point theorems under strict contractive conditions", J. Math. Anal. Appl. 27(1), 181–188.

[2] Branciari A., 2000, "A fixed point theorem of Banach–Cacciopoli type on a class of generalized metric spaces", Publ. Math. (Debr.) 57, 31–37.

[3] Isik H. and Turkoglu D., 2013, "Common fixed points for (ψ, α, β)-weakly contractive mappings in generalized metric spaces", Fixed point theory and applications 2013:131.

[4] Jungck G., 1996, "Common fixed points for non-continuous non-self mappings on non-metric spaces", Far East J. Math. Sci. 4(2), 199–212.

[5] Sintunavarat W. and Kumam P., 2011, "Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces", Journal of Applied mathematics, Article ID 637958, 14 pages.