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N-duo and SRB rings

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Abstract

We study in this paper the strong regularity of right SSF-rings and obtain the following result: Let R be a right SSF-ring. If R satisfies one of the following conditions, then R is a strongly regular ring:

1) R is a IFP ring;
2) R is a left N-duo ring;
3) R is a right N-duo ring;
4) R is a SRB ring;
5) R is a SLB ring.

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1. Introduction

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary. A ring R is called strongly regular if for each $a \in R$ there exists $b \in R$ such that $a = a^2b$. It is well-known that a ring R is strongly regular if and only if R is reduced and regular if and only if R is regular and right (left) duo. Following Ramamurthy [10], a ring R is called left (right) SF-ring if each simple left (right) R-modules are flat. It is well-known that regular rings are left and right SF-rings. And ring R is strongly regularI if and only if R is reduced right SF-rings. Ramamurthy [10] initiaed the von

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Neumann regularity of SF-rings, asking whether a right and left SF-rings is necessary von Neumann regular. This question has drawn the atention of many authors [3], [6], [10], [11], etc. For example, Rege [11] proved that R is a reduced left (right) SF rings, then R is a strongly regular. Subedi [12] proved that R is a N-duo left (right) SF rings, then R is a strongly regular. Zhang and Du [16] proved that a ring R is strongly regular if and only if R is SRB right SF-ring.

Recall the following definitions and facts: A ring is R is called reduced if R has no nonzero nilpotent elements. A ring R is called right (left) duo if every right (left) ideal of R is two-sided. Following Wei [15], a ring R is called left N-duo if for any $a \in N(R)$, Ra is an ideal. Right N-duo rings is defined similarly. It is well-known that reduced rings are right (left) N-duo rings, the converse is not true in general.

A ring R is called right (left) duo if every right (left) ideal of R is two-sided. The notation of bounding a one-sided ideal by a two-sided ideal goes back at least to Jacobson [5]. He said that a right ideal of R is bounded if it contains a nonzero ideal of R. This concept has been extended in several ways. From Faith [4], a ring R is called SRB (SLB) if every nonzero right (left) ideal of R contains a nonzero two-sided ideal of R. A ring is called strongly bounded if it both strongly right bounded and strongly left bounded. The class of strongly one-sided bounded rings has been observed by many authors [2], [9], [11], etc. It is well-known that right duo rings are SRB rings, the converse is not true in general.

According to Mahmood [9], a ring R is called left (right) SSF-ring if simple singular left (right) R-modules are flat. In this note ,we study conditions under which right SSF-rings are strongly regular. We show that SSF-ring R is strongly regular if R is right (left) N-duo. We also show that SSF-ring R is strongly regular if R is SRB (SLB).

We denote the set of all nilpotent elements, the Jacobson radical, the left (right) singular ideal R by $N(R)$, $J(R)$, $Z_l(R)$ ($Z_r(R)$, resp.), respectively. For any nonempty subset S of R, $r(S)$ and $l(S)$ denote the right annihilator and the left annihilator of S in R, respectively. Especially, if $X = \{a\}$, we write $l(X) = l(a)$ and $r(X) = r(a)$.

Lemma 1.1. ([11]) Let R be a ring, and let I be a right ideal of R. Then R/I is a flat right ideal of R if and only if for each $a \in I$ there exists $b \in I$ such that $a = ba$.

Lemma 1.2. ([9]) Let R be a right (left) SSF-ring. If I is an ideal of R, then R/I is a right (left) SSF-ring.

Lemma 1.3. ([1]) If R is a reduced left (right) SSF-ring, then R is a strongly regular ring.

2. N-duo and SRB rings

A ring is R is called reduced if R has no nonzero nilpotent elements. A ring R is called right (left) duo if every right (left) ideal of R is two-sided. Following Wei [15], a ring R is called left N-duo if for any $a \in N(R)$, Ra is an ideal. Right N-duo rings is defined similarly. It is well-known that reduced rings are right (left) N-duo rings, the converse is not true in general [15].

The following lemmas gives properties of N-duo rings.

Lemma 2.1. For any $a \in N(R)$, Then;

(1) If R is a right N-duo ring, then $l(a)$ is an ideal.

(2) If R is a left N-duo ring, then $r(a)$ is an ideal.

Proof.

(1) Let $x \in l(a)$ and $r \in R$. Since R is right N-duo ring and $a \in N(R)$, aR is an ideal. Thus $ra \in aR$, and there exists $t \in R$, $ra = at$. By left multiplication x, $xra = xat$. Since $x \in l(a)$, $xra = 0$. Therefore $xR \subseteq l(a)$, and so $l(a)$ is an ideal.

(2) Similary proof of (1). ■

Recall that a ring R is semiprime if $aRa = 0$ implies $a = 0$ for any $a \in R$. Obviously, every reduced rings are left (right) N-duo rings. The converse holds if R is semiprime.

Lemma 2.2. [12, Proposition 3.1] The following conditions are equivalent for a ring R.

(1) R is a reduced ring.

(2) R is a semi-prime left (right) N-duo ring.

Theorem 2.3. [12, Theorem 3.7] The followings are equivalents

(1) R is a strongly regular ring.

(2) R is a right N-duo right SF-ring.

(3) R is a left N-duo right SF-ring.

The following theorem gives extending the result of Subedi [12].

Theorem 2.4. The followings are equivalents.

(1) R is a strongly regular ring.

(2) R is a right N-duo right SSF-ring.

(3) R is a left N-duo right SSF-ring.

Proof. Clearly (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). (2) $\Rightarrow$ (1): First we will show that R is a reduced ring. Let $a^2 = 0$. If $a = 0$, then we have done. If $a \neq 0$, then $l(a) \neq R$. Since $a \in N(R)$ and R is a right N-duo ring, $l(a)$ is an ideal by Lemma 2.1. There exists a maximal right ideal K of R such that $l(a) \subseteq K$. First observe that K is an essential right ideal of R. If not, then K is a direct summand of R. So we can write $K = r(e)$ for some $0 \neq e = e^2 \in R$. Thus $a \in K = r(e)$, and $ea = 0$. Hence $e \subseteq l(a) \subseteq K = r(e)$; whence $e = 0$. It is a contradiction. Therefore K is an essential right ideal of R. Thus R/K is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in K$ such that $a = ca$ by Lemma 1.1. Thus $(1 - c)a = 0$, and $1 - c \subseteq l(a) \subseteq K$; whence $1 \in K$. It is a contradiction. Thus $a = 0$, and so R is reduced. Therefore R is a strongly regular ring by Lemma 1.3.

(3) $\Rightarrow$ (1). First we will show that $J(R)$ is reduced. Let $0 \neq b \in J(R)$ such that $b^2 = 0$. We claim that $J(R) + l(Rb) = R$. If not, there exists a maximal right ideal L such that $J(R) + l(Ra) \subseteq L$. First observe that L is an essential right ideal of R. If not, then L is a direct summand of R. So we can write $L = r(e)$ for some $0 \neq e = e^2 \in R$. Since $b \in L = r(e)$, $eb = 0$ and $b \in r(e)$. By Lemma 2.1, $r(e)$ is an ideal. Thus $Rb \in r(e)$, and $e \in l(Rb) \subseteq L = r(e)$; whence $e = 0$. It is a contradiction. Therefore L is an essential right ideal of R. Thus R/L is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in L$ such that $b = cb$ by Lemma 1.1. Thus $(1 - c)b = 0$. If $1 - c = 0$, then $1 \in L$; It is a contradiction. If $1 - c \neq 0$, then $b \in r(1 - c)$. By Lemma 2.1, $r(1 - c)$ is an ideal.

Thus $Rb \in r(1 - c)$, and $1 - c \in l(Ra) \subseteq L$; whence $1 \in L$. It is a contradiction. Hence $J(R) + l(Rb) = R$. Thus there exists $x \in J(R)$ and $y \in l(Rb)$ such that $x + y = 1$. So $xb + yb = b$, and $(x - 1)b = 0$. Since $x \in J(R)$ and $x - 1$ is invertible, hence $b = 0$. Also it is a contradiction. Thus $J(R)$ is reduced, and so R is a semiprime ring. Hence R is reduced by Lemma 2.2. Therefore R is a strongly regular ring by Lemma 1.3. ■

A ring R is called right (left) duo if every right (left) ideal of R is two-sided. From Faith [4], a ring R is called SRB (SLB) if every nonzero right(left) ideal of R contains a nonzero two-sided ideal of R. A ring is called strongly bounded if it both strongly right bounded and strongly left bounded. It is well-known that right duo rings are SRB rings, the converse is not true in general [13].

Proposition 2.5. [2, Lemma 1] If R is a semiprime SRB (SLB) rings, then R is reduced.

Proposition 2.6. [16, Theorem 3] If R is SRB (SLB) right SF-ring, then R is strongly regular.

The following proposition shows properties of right SSF-ring.

Lemma 2.7. If R is a right SSF-ring, then $Z_l(R) \subseteq J(R)$.

Proof. For any $x \in Z_l(R)$, then $l(x)$ is an essential left ideal of R. Note that $l(x) \cap l((1 - x) = 0$. Thus $l((1 - x)) = 0$. We claim that $(1 - x)R = R$. Suppose not. Then there exists a maximal right ideal M of R containing $(1-x)R$. We claim that M is right essential in R. Assume that M is not right essential in R. Then there exists a nonzero right ideal I of R such that $M \cap I = 0$ and so $M \oplus I = R$. This implies $a + b = 1$ for some $a \in M$ and $b \in I$ and so $a(1-x) + b(1-x) = 1 - x$. Since $1 - x \in M$, we have $(1-x) - a(1-x) = b(1-x) \in M \cap I = 0$. Thus $(1-a)(1-x) = 0$. Since $\ell_R(1-x) = 0$, we have $a = 1$, which is a contradiction. Hence M is an essential maximal right ideal of R. Thus R/M is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in M$ such that $(1-x) = c(1-x)$ by Lemma 1.1. Thus $1 - c \in l(1-x) = 0$ and so $1 \in M$, which is also a contradiction. Therefore $(1-x)R = R$, and so $Z_l(R) \subseteq J(R)$. ■

In the following theorems we give partial answer to Ramamurthy question, extending the result of Zhang and Du [16].

Theorem 2.8. The following conditions are all equivalent:

(1) R is a strongly regular ring.

(2) R is a SRB right SSF-ring.

(3) R is a SLB right SSF-ring.

Proof. (1) ⇒ (2) and (1) ⇒ (3) are obvious. (2) ⇒ (1): We claim that R is reduced. Suppose that $a^2 = 0$ with $a \neq 0$. Then $rl(a)$ is nonzero right ideal of R, and so there exists a nonzero two-sided ideal I of R such that $I \subseteq rl(a)$. So $l(a) \subseteq l(I)$. We claim that $l(I) = R$. If not, $l(I) \subseteq M$ for some maximal right ideal M of R because $l(I)$ is two-sided. Note that M is right essential in R. Thus R/M is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in M$ such that $a = ca$ by Lemma 1.1. Thus $1 - c \in l(a) \subseteq l(I) \subseteq M$, and so $1 \in M$, which is also a contradiction. Thus R is reduced. Therefore R is strongly regular by Lemma 1.3.

(3) ⇒ (1): We first claim that $R/Z_l(R)$ is reduced. Assume that there exists $a \notin Z_l(R)$ such that $a^2 \in Z_l(R)$. Then there exists a nonzero left ideal I of R such that $l(a) \oplus I \subseteq l(a^2)$. Since R is SLB, there exists a nonzero two-sided ideal L such that $L \subseteq I$. Then $La^2 = 0$ and so $La \subseteq l(a) \cap L \subseteq l(a) \cap I = 0$. Thus $La = 0$ and hence $L \subseteq l(a) \cap I = 0$, which is a contradiction. Therefore $R/Z_l(R)$ is reduced. By applying Lemma 1.2, $R/Z_l(R)$ is strongly regular. Thus $J(R) \subseteq Z_l(R)$, and hence $J(R) = Z_l(R)$ by Lemma 2.7. This implies that $R/J(R)$ is strongly regular. Suppose $J(R) \neq 0$. Then there exists $0 \neq b \in J(R)$ such that $b^2 = 0$. We claim that $J(R) + l(b) = R$ for any $b \in J(R)$. If not, then there exist $b \in J(R)$ such that $J(R) + l(b) \neq R$. There exists a maximal right ideal K such that $J(R) + l(a) \subseteq K$. First observe that L is an essential right ideal of R. If not, then K is a direct summand of R. So we can write $K = r(e)$ for some $0 \neq e = e^2 \in R$. Since $b \in K$, $ea = 0$, and $e \in l(a) \subseteq K = r(e)$; whence $e = 0$. It is a contradiction. Therefore K is an essential right ideal of R. Thus R/K is a simple singular right R-module and flat by hypothesis. Now since R is right SSF-ring, there exist $c \in K$ such that $a = ca$ by Lemma 1.1. Thus $(1-c)a = 0$, and $1 - c \in l(a) \subseteq K$;

whence $1 \in K$. It is a contradiction. Hence $J(R) + l(b) = R$ for any $b \in J(R)$. Thus there exists $x \in J(R)$ and $y \in l(b)$ such that $x + y = 1$. So $xb + yb = b$, and $(x - 1)b = 0$. Since $b \in J(R)$ and $x - 1$ is invertible, hence $b = 0$. Also it is a contradiction. Therefore $J(R) = 0$, and so R is a strongly regular ring. ■

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