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An Application of Dynamic Programming Principle for Pricing Multi-assets Barrier Options

Komang Dharmawan
Department of Mathematics,
Udayana University,
Kampus Bukit Jimbaran, Bali, Indonesia.

Abstract

This paper is concerned with the stochastic control problem arising in pricing barrier option consisting of two or more assets. We refer to a barrier option where the volatilities of the underlying assets are stochastically moving within specified interval. The interval can be the maximum or the minimum value of the volatilities during the life of the contract. These values of the volatilities may correspond to the best and the worst case scenarios of the future positions in the portfolio of the options. The concept of suprehedging strategies in pricing an option is applied. Furthermore, in this case, the strategy is considered as a certain exit time control problem. First, we prove that the control u is lower semicontinuous. Then, under certain assumptions, we show the value function is bounded and nonnegative. Next, by applying probability methods, we prove that the value function of the exit control problem is continuous on the boundary. Finally, we prove that superhedging prices of multi-asset barrier options can be represented in the dynamic programming principle (DPP) for an exit control problem.

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1. Introduction

It has been claimed in many publications that the volatility of underlying assets of an option is not constant as in Black-Scholes framework. This has been identified by Rubinstein and Reiner [13] who claimed that constant volatilities as in the Black-Scholes model can not explain the observed market price for options. The use of constant volatilities may result in an over or under price of an option, hence hedging strategy using this option may cause a problem. To overcome this problem, some researchers, El-Karoui, Jeanblanc-Picque, and Shreve [4] provided conditions under which the Black-Scholes formula is robust with respect to a misspecification of volatility. Gozzi and Vargiolu [6], proposed a new method for the study of robustness of the Black-Scholes formulae for several assets. The volatilities are considered to follow stochastic process but lying on an interval band $[\sigma_{\min}, \sigma_{\max}]$. Then bounds on the option prices are obtained by setting the volatility equal to $\sigma_{\min}$ and $\sigma_{\max}$ depending on the convexity or concavity of the option price functions.

In the situation of pricing barrier options, the method proposed in [4] may not applicable. This is because the barrier option prices may not increase monotonically as volatilities stochastically change. Moreover, the value function of the option is neither convex nor concave. Hence, we apply a superhedge strategy to price barrier options. This strategy may be considered as a certain exit time control problem.

Research on the application of the exit time control on pricing of the exotic options is still a great challenge. The main difficulties come from the treatment of the boundary conditions. Researchers such as Lions [10] and Barles and Rouy [1] relax the value function to be continuous in the open set $\mathcal{O}$ and can be extended continuously to $\bar{\mathcal{O}}$. Sofiene and Abergel [14] applies the optimal stochastic control for analytical expressions to find the optimal bid and ask quotes of the market maker. Another version of stochastic control problem is dynamic programming principle (DPP) which has also attracted a great interest among researchers in financial mathematics, see for example paper by Huang and Wu [7]. In this paper, the application of DPP on the maximizing investor's optimal portfolio proportion is discussed. For the existence of the solution for dynamic programming principle, one may consult to Liu and Schikorra [11].

There are two main results of the present article. The first result is Theorem 4.6, under a certain assumption, we show that the value function problem satisfies the properties of the dynamic programming relation. Second result is Theorem 4.7. We show, by a probability approach, that the value function of the exit control problem v is continuous with respect to time t and space of price x, and is regular enough to apply the Ito formula.

The article is organized as follows. In Section 2 we describe the tools and assumption that we use to prove the theorems. In Section 3 we set up the financial model to which actually, we apply the exit control problems. In section 4, we establish the dynamic programming principle for the exit control problem and the main results are presented here. In section 5, we give an illustration example for the case of two assets. In section 6, we summarize our results and make some remarks.

2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}, t \geq 0\}$ satisfying the usual conditions. Given a bounded Borel subset $U \subset \mathbb{R}^n$, we denote by $\mathcal{U}$ a set of progressively measurable processes $u = (u_t,\ t \geq 0)$ defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that $\mathbb{P}(u_t \in U) = 1$ for all $t \geq 0$. The elements of $\mathcal{U}$ are called *admissible control processes*. For each control process $(u_t) \in \mathcal{U}$, we consider a stochastic differential equation, for $t \geq s$,

$$\begin{cases} dX_t^{s,x_0,u} = \mu(t, X_t^{s,x_0,u}, u_t)dt + \sigma(t, X_t^{s,x_0,u}, u_t)dW_t, \\ X_s = x_0 \end{cases} \tag{2.1}$$

where $X_t^{s,x_0,u} \in \mathbb{R}^d$, and $\mu : \mathbb{R}_+ \times \mathbb{R}^d \times U \to \mathbb{R}^d$, and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times n}$ are assigned Lipschitz continuous functions for each $u \in \mathbb{R}^n$. The process $X_t = X_t(\omega)$ is interpreted as the state of the system at time t. By a pathwise solution of this equation, we mean an $(\mathcal{F}_t)$-adapted continuous stochastic process $X_t^{s,x,u}$ satisfying

$$X_t^{s,x_0,u} = x_0 + \int_s^t \mu(r, X_r^{s,x_0,u}, u_r)dr + \int_s^t \sigma(r, X_r^{s,x_0,u}, u_r)dW_r,\ 0 \leq s \leq t. \tag{2.2}$$

If the above equation has a unique solution $X_t^{s,x_0,u}$, the process (X_t) is called a controlled process.

Definition 2.1. Let x_0 be an $\mathcal{F}_s$-measurable random variable and for $p \geq 2$ such that $\mathbb{E}|x_0|^p < \infty$, the stopping time τ_n is defined as

$$\tau_n = \begin{cases} \inf\{t \in [s, T]; |X_t^{s,x_0,u}| \geq n\}, \quad n \geq 1, \\ \\ T, \quad \text{if } \{t\ [s, T]; |X_t^{s,x_0,u}| \geq n\} = \emptyset \end{cases} \tag{2.3}$$

The stopping times τ_n are well defined since the process $X_t^{s,x_0,u}$ is continuous in $t \in [s, T]$. Then following (2.2) we have

$$X_{t\wedge\tau_n}^{s,x_0,u} = x_0 + \int_s^{t\wedge\tau_n} b(r, X_r^{s,x_0,u}, u_r)dr + \int_s^{t\wedge\tau_n} \sigma(r, X_r^{s,x_0,u}, u_r)dW_r. \tag{2.4}$$

Assumption 2.2. For each $T > 0$ there exists a constant $K > 0$ such that for all $u \in U, s \leq T$ and $x, y \in \mathbb{R}^d$

$$|b(s, x, u) - b(s, y, u)| + |\sigma(s, x, u) - \sigma(s, y, u)| \leq K|x - y| \tag{2.5}$$

$$|b(s, x, u)| + |\sigma(s, x, u)| \leq K(1 + |x|) \tag{2.6}$$

This assumption is very standard in control theories where functions μ and σ appearing in the control system. The existence of constant $K > 0$ such that for all $u \in U, s \leq T$ and $x, y \in \mathbb{R}^d$ is granted. Assumption 2.2 yields the existence of a unique strong solution of $(X_t^{s,x_0,u})$ of (2.1), for each $s > 0$, each initial condition x_0, and each $u \in U$. Moreover, $(X_t^{s,x_0,u})$ is continuous on $[s, T]$.

Assumption 2.3. The function $\phi > 0$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz, that is, there exists $K > 0$ such that

$$|\phi(x) - \phi(y)| \leq K|x - y|, \quad x, y \in \mathbb{R}^d.$$

Proposition 2.4. Let x be an $\mathcal{F}_s$-measurable random variable and for $p \geq 2$ such that $\mathbb{E}|x_0|^p < \infty$. Then there exists a constant $K(T, p) > 0$ which is independent of u such that for all $0 \leq s \leq t \leq T$,

$$\mathbb{E}|X_t^{s,x_0,u}|^p \leq K\mathbb{E}(1 + |\xi|^p). \tag{2.7}$$

Proposition 2.5. Let X_t^n be the solution of the stochastic differential equation

$$X_t^{s,x_0,u} = x_0^n + \int_s^t \mu(r, X_r^{s,x_0^n,u}, u_r)dr + \int_s^t \sigma(r, X_r^{s,x_0^n,u}, u_r)dW_r,\ 0 \leq s \leq t \tag{2.8}$$

Let (X_t) be the solution of (2.4) and assume that for a certain $p \geq 2$

$$\mathbb{E}\left(\left|\xi^n\right|^p + |\xi|^p\right) < \infty, \quad n \geq 1.$$

Then there exists a constant $C(p, K, T)$ which is independent of u such that

$$\mathbb{E}\sup_{t\leq T}\left|X_t^n - X_t\right|^p \leq C(p, K, T)\mathbb{E}\left|\xi^n - \xi\right|^p.$$

Proposition 2.6. Let $X_t^{s_n,x_0,u}$, where $0 \leq s_n \leq t \leq T$ be a solution of the stochastic differential equation

$$X_{t\wedge T}^{s_n,x_0,u} = x_0 + \int_{s_n}^{t\wedge T} b(r, X_r^{s_n,x_0,u}, u_r)dr + \int_{s_n}^{t\wedge T} \sigma(r, X_r^{s_n,x_0,u}, u_r)dW_r. \tag{2.9}$$

Then for all $p \geq 2$ there exists a constant $C(T, p)$ which is independent of u and such that

$$\mathbb{E}\sup_{\bar{s}\leq t\leq T}\left|X_t^{s_n,x_0,u} - X_t^{s,x_0,u}\right|^p \leq C(T, p)|s_n - s|^{p/2}$$

where $\bar{s} = \max(s, s_n)$.

Proposition 2.7. Let Assumption 2.2 hold. For each $p \geq 1$, $T > 0$, $t \geq s_2 > s_1 > 0$,

$$\mathbb{E}\sup_{s_2\leq t\leq T}\left|X_t^{s_2,x_2,u} - X_t^{s_1,x_2,u}\right|^p \leq C_1(|x_2 - x_1|^p + |s_2 - s_1|^{p/2}),$$

where C_1, is independent of u, s, s_n, ξ.

Reader who may be interested in the proof of Preposition 2.4–2.7 may refer to [3].

3. The financial Market Model

Given a finite time horizon $T > 0$, consider a complete probability $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, equipped with a Wiener process $W_t = \{(W_t^1, \ldots, W_t^n),\ 0 \leq t < T\}$ valued in $\mathbb{R}^n$ with respect to filtration $\mathbb{F} = \{\mathcal{F}_t\}$. The filtration $\mathbb{F} = \{\mathcal{F}_t; 0 \le t \le T\}$ is $\mathbb{P}$-augmentation of the natural filtration

$$\mathcal{F}_t^W = \sigma(W_s, 0 \le s \le t), \quad 0 \le t \le T.$$

Throughout this paper, we shall consider the market for d risky assets, $\{S_t^i,\ i = 1, \ldots, d; 0 \le t \le T\}$. The assets are traded continuously in a frictionless market (no transaction cost, no tax). The risky asset prices $\{S_t^i, i = 1, \ldots, d; 0 \le t \le T\}$ are modelled by the linear stochastic differential equation

$$\begin{aligned} dS_t^i &= S_t^i \left[b_t^i dt + \sum_{j=1}^n \sigma_t^{ij} dW_t^j \right] \\ S_0^i &= x^i, \ \ i = 1, \ldots, d. \end{aligned} \tag{3.10}$$

In this model, the sources of randomness are given by the independent components of the Wiener processes $W_t = (W_t^1, \cdots, W_t^n)', 0 \le t \le T$. The vector process $\{b_t = (b_t^1, \ldots, b_t^d)', 0 \le t \le T\}$ is the vector of appreciation rates which is assumed to be bounded and adapted to $\mathbb{F}$. The matrix-process of volatilities

$$\{\sigma_t = \left(\sigma_t^{ij}\right)_{1 \le i \le d, 1 \le j \le n}, 0 \le t \le T\},$$

whose rows are $\sigma_t^1, \ldots, \sigma_t^d$, is assumed to be bounded.

Consider a payoff function for a barrier option $h(S_T)1_{\{\tau > T\}}$, where τ is the first moment of time where the stock price hits a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ pre-specified level of barrier. The payoff function h in this case is discontinues and the value function is

$$v(t, x) = \mathbb{E}_{\mathbb{Q}} \left[h(S_{T-t}^{t,x,\sigma}) 1_{\{\tau > (T-t)\}} | \mathcal{F}_t \right]$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation operator with respect to measure $\mathbb{Q}$ and τ is the first moment of time when the stock price S_t hits the the boundary barrier boundary defined by

$$\tau = \inf\{t > s;\ S_t^{s,x,\sigma} \in \partial\mathcal{O}\}. \tag{3.11}$$

In this case, h is not a convex function. Based on the position of the barrier, we categorize the multi-asset barrier options into three different types. The first one is the external barrier option. The value of the option at maturity and the hitting time τ are determined by different (both tradeable) assets. If a certain stock hits a predetermined level then the value of such an option is zero. The second one is the basket barrier option. The value of this option depends on whether the underlying assets in the basket hit a certain level of barrier or not. The third one termed the max/min barrier option is a barrier option where the value of the option depends on whether the maximum of the underlying assets hits a certain level of barrier or not. To begin with let us define a price process for the multi-asset barrier option.

Definition 3.1. A price process for a barrier option is any adapted process $\{v_t;\ 0 \le t \le T\}$ satisfying

$$v_T = h(S_T^{t,x,\sigma})\mathbf{1}_{\{\tau > T\}}, \quad a.s.$$

where $h : \mathbb{R}_+^d \to [0, \infty)$ is a given function and τ is the first moment of time when S_t hits the barrier, defined as

$$\tau = \inf\{t > 0;\ \ S_t \in \partial\mathcal{O}\}. \tag{3.12}$$

Here S_t is the solution of (3.10), $\mathcal{O} \subset \mathbb{R}^{d+1}$ and $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$.

We propose here a payoff function for a multi-asset barrier option with an external barrier in which the terminal payoff is characterized by

$$h(S_T) = (\max(S_T^2, \dots, S_T^d) - K)^+\mathbf{1}_{\{\tau > T\}}.$$

We adopt the usual Black-Scholes assumptions on the capital market and we assume that the volatilities are fixed. In the risk-neutral assumption, the stock price S_t^i, $i = 1, \dots, d$ follow the lognormal diffusion processes. Let ρ_{ij} denote the correlation coefficients between dW^i and dW^j which are constant. Let H denote the upper barrier. The call option will be knocked out when $S_t^1 \ge H$ at any time before expiry time T. The value of the multi-asset barrier option with barrier level H is given by

$$v(t, x) = \mathbb{E}_{\mathbb{Q}}\left[h(S_{T-t}^{x,\sigma})\mathbf{1}_{\{\tau > (T-t)\}}\right] \tag{3.13}$$

This formulation gives rise to the analytical evaluation of the expectation integral in many dimensions. This is beyond our discussion. Instead we convert the problem into the partial differential equation given by the following theorem.

Theorem 3.2. Suppose that v is a solution of the partial differential equation

$$\frac{\partial}{\partial t}v(t, x) + \frac{1}{2}\mathrm{tr}(D_x^2 v(t, x)(\sigma\bar{x})(\sigma\bar{x})') = 0, \ \ 0 \le t < T,\ x_1 < H, \tag{3.14}$$

with terminal and boundary conditions, where $x = (x_1, x_2, \dots x_d)$.

$$v(T, x) \ = \ h(x^1), \quad x_1 < H, \tag{3.15}$$

$$v(t, x) \ = \ 0, \quad x_1 = H,\ \ 0 \le t \le T. \tag{3.16}$$

Then v is given by (3.13).

4. Main Results

Before we discuss the results, we assume that $\mathcal{O}$ is an open bounded domain in $\mathbb{R}^d$ with C^2 boundary and $Q = [0, T) \times \mathcal{O}$. In addition to (2.2), we also assume that

$$\sigma(s, x, u) = s\sigma_0(s, x, u), \quad \text{and} \quad \left(\sigma_0^*\sigma_0(s, x, u)h, h\right) \ge c_n|h|^2, \quad h \in \mathbb{R}^d, \tag{4.17}$$

uniformly in $s \leq T$, $u \in U$ and $|x| \leq n$. Most of the results that follow can be proved under more general assumptions but the above condition is usually satisfied in problems arising in option pricing.

The first moment of time when X_t hits the boundary is given by

$$\tau = \inf\{t > s;\ X_t^{s,x,u} \in \partial\mathcal{O}\}. \tag{4.18}$$

Lemma 4.1. Let

$$\tau^{s,x,u} = \inf\{t > s;\quad X_t^{s,x,u} \in \partial\mathcal{O}\}.$$

Then for each u the function $(s, x) \to \tau^{s,x,u}$ is lower semicontinuous.

Proof. All quantities defined here are standard. For $s_n \to s$ and $x_n \to x$, $x_n, x \in \mathcal{O}$, we need to show that

$$\tau^{s,x,u} \leq \liminf_{n\to\infty} \tau^{s_n,x_n,u}. \tag{4.19}$$

Assume that (4.19) is not true. Then

$$\mathbb{P}\left(\tau^{s,x,u} > \liminf_{n\to\infty} \tau^{s_n,x_n,u}\right) > 0. \tag{4.20}$$

For any rational $r_1 < r_2$, let

$$A_{r_1,r_2} = \left\{\omega\ ;\ \tau^{s,x,u} > r_2 > r_1 > \liminf_{n\to\infty} \tau^{s_n,x_n,u}\right\}.$$

Then

$$\left\{\tau^{s,x,u} > \liminf_{n\to\infty} \tau^{s_n,x_n,u}\right\} = \bigcup_{r_1<r_2} A_{r_1,r_2}$$

and therefore

$$\mathbb{P}(A_{r_1,r_2}) > 0$$

for at least one pair $r_1 < r_2$. Let

$$B_m = \left\{\inf_{t\leq r_2}\ \mathrm{dist}(X_t^{s,x,u}, \partial\mathcal{O}) \geq \frac{1}{m}\right\}.$$

Then $A_{r_1,r_2} \subset \bigcup_{m=1}^{\infty} B_m$. Hence, for a certain m,

$$\mathbb{P}(A_{r_1,r_2} \cap B_m) > 0.$$

Let $X_t^k = X_t^{s_{n_k},x_{n_k},u}$ for any subsequence $n_k \to \infty$. By Proposition (2.7),

$$\mathbb{P}\left(\sup_{t\leq r_2} |X_t^k - X_t| \geq \varepsilon\right) < \frac{1}{\varepsilon^2}\left(|x_{n_k} - x|^2 + |s_{n_k} - s|\right). \tag{4.21}$$

Let n_k be such that

$$\tau^{s_{n_k},x_{n_k},u} \to \liminf_{k\to\infty} \tau^{s_{n_k},x_{n_k},u}$$

and let $\varepsilon = \dfrac{1}{2m}$. We also have

$$A_{r_1,r_2} = \left\{\tau^{s,x,u} > r_2\right\} \cap \bigcup_j \bigcap_i \left\{\tau^{s_{i+j},x_{i+j},u} < r_1\right\}$$

and therefore, for a certain j

$$\mathbb{P}\left(\left\{\tau^{s,x,u} > r_2\right\} \cap \bigcap_i B_{i+j}\right) > 0.$$

Hence

$$\begin{aligned} 0 \;\; < \;\; & \mathbb{P}\left(\left\{\tau^{s,x,u} > r_2\right\} \cap \bigcap_i B_{i+j}\right) \leq \\ & \mathbb{P}\left(\tau^{s_{i+j},x_{i+j},u} > r_2 > r_1 > \tau^{i+j}\right), \end{aligned}$$

where by (4.21)

$$\begin{aligned} \mathbb{P}\left(\tau^{s_{i+j},x_{i+j},u} > r_2 > r_1 > \tau^{i+j}\right) \;\; & \leq \;\; \mathbb{P}\left(\sup_{t\leq r_2} |X_t^{s_{i+j},x_{i+j},u} - X_t^{i+j}| > \frac{1}{2m}\right) \\ & \leq \;\; (|x_{i+j} - x|^2 + |s_{i+j} - s|)4m^2. \end{aligned}$$

Since

$$\lim_{i,j\to\infty} |x_{i+j} - x|^2 + |s_{i+j} - s| = 0,$$

we obtain

$$\mathbb{P}\left(\tau^{s_{i+j},x_{i+j},u} > r_2 > r_1 > \tau^{i+j}\right) = 0.$$

This is a contradiction with (4.20). So,

$$\tau^{s,x,u} \leq \liminf_{n\to\infty} \tau^{s_n,x_n,u}.$$

■

Now, we shell establish the dynamic programming principle for the exit time control problem given a stochastic differential equation as described in (2.1)–(2.2). Let τ be the exit time of X_t from the open domain $\mathcal{O}$. This choice is related to the multi-asset barrier options. The value function v is automatically 0 when (X_t) exits from the open domain $\mathcal{O}$. This condition may result in non-smoothness of the value function v. Moreover, this may create degeneracy in the partial differential equations, see [9] for non-degeneracy case.

For any admissible progressively measurable control process $u(\cdot)$, the payoff function is given by

$$J(s, x; u) = \mathbb{E}\left[\phi(X^{s,x,u}_{T-s})\mathbf{1}_{\tau>T-s}\right] \quad s \in [0, T), \tag{4.22}$$

where $\phi \in C_b(\mathcal{O})$ and $(s, x) \in Q$ is given. $C_b(\mathcal{O}$ is a set of bounded functions in $C(\mathcal{O})$. The problem now is to choose $u(\cdot)$ to maximize J. The terminal payoff function for this problem is given by equation (4.22). The general form of the value function contains an integral term which is not relevant in this work. Formally, the optimal control problem is then formulated as follows. For any given $(s, x) \in [0, T) \times \mathcal{O}$, find $\bar{u} \in U$, such that

$$J(s, x; \bar{u}) = \sup_{u \in U} J(s, x; u) = v(s, x). \tag{4.23}$$

Any process $u \in U$ which is adapted to the natural filtration $\mathbb{F}$ of the associated state process is called *feedback control*. A process $u \in \mathcal{U}$ which can be written in the form $u_s = \tilde{u}(s, X_s)$ for some measurable $\tilde{u} : [0, T) \times \mathcal{O} \to U$, is called *Markovian control*.

We will make use of the following standing assumption on the function ϕ appearing in the value function. We start by examining the value function $v(s, \cdot)$ for fixed $s \in [0, \tau \wedge T)$.

Theorem 4.2. Assume that Assumption 2.2 and 2.3 hold. There exists $K > 0$, such that for any $s \in [0, T)$ and $x \in \mathbb{R}^d$, it holds that

$$0 \leq v(s, x) \leq K(1 + |x|). \tag{4.24}$$

Proof. The value function is obviously nonnegative. The second inequality is proved as follows. Let $u \in \mathcal{U}$ be a control process. Then by the Lipschitz property of ϕ we obtain

$$\begin{aligned} \mathbb{E}[\phi(X^{s,x,u}_{T-s})\mathbf{1}_{\tau>T-s}] &\leq \mathbb{E}[K_1(1 + |X^{s,x,u}_{T-s}|)]. \\ &\leq K_1 + K_1\mathbb{E}|X^{s,x,u}_{T-s}| \end{aligned}$$

By Lemma 2.4, and noting that $X_s = x$, we have

$$\begin{aligned} \mathbb{E}[\phi(X^{s,x,u}_{T-s})\mathbf{1}_{\tau>T-s}] &\leq K_1 + K_1K_2(1 + |x|) \\ &\leq K(1 + |x|), \end{aligned}$$

where $K = \max(K_1(1 + K_2), K_1K_2))$. Since this holds for every $u \in \mathcal{U}$, it also holds for value function v. ∎

Now, we can claim the continuity of the value function by proposing the following theorem.

Theorem 4.3. Let assumption 2.2 and 2.3 hold. Then for any fixed $s \in [0, T)$ the function $x \to v(s, x)$ is continuous on $\mathcal{O}$.

$$|v(s, x) - v(s, y)| \leq K|x - y|. \tag{4.25}$$

Proof. Let us first prove the inequality above without the exit time in the value function, so our first estimate

$$\begin{aligned} |v(s,x) - v(s,y)| &= |\sup_{u\in U} \mathbb{E}\phi(X^{s,x,u}_{T-s}) - \sup_{u\in U} \mathbb{E}\phi(X^{s,y,u}_{T-s})| \\ &\leq \sup_{u\in U} |\mathbb{E}\phi(X^{s,x,u}_{T-s}) - \mathbb{E}\phi(X^{s,y,u}_{T-s})|. \end{aligned}$$

Then Assumption 2.2 and Lemma 2.5 give

$$\begin{aligned} |v(s,x) - v(s,y)| &\leq K\mathbb{E}|X^{s,x,u}_{T-s} - X^{s,y,u}_{T-s}| \\ &\leq K|x-y| \end{aligned}$$

Now we include the exit time in the value function. For $x, y \in \mathcal{O}$ and $u \in U$, we first estimate that:

$$\begin{aligned} |v(s,x) - v(s,y)| \leq \sup_{u\in U} \mathbb{E}|\phi(X^{s,x,u}_{T-s})\mathbf{1}_{\tau>T-s} \\ - \phi(X^{s,y,u}_{T-s})\mathbf{1}_{\tau>T-s}|. \end{aligned}$$

Let us take $B_r(x) \subset \mathcal{O}$, a small ball with radius r in $\mathcal{O}$. For a small $T > 0$, we define

$$\mathbb{P}(\tau^{s,z,u} < T) = \mathbb{P}\left[\sup_{0\leq t\leq T} |X^{s,z,u}_t| > r\right] \leq C_T e^{-r\beta} \quad z \in B_r(x), \tag{4.26}$$

where $\beta > 0$ and $C_T \to 0$ for $T \to 0$. For some $s \in [0, T]$, we have

$$\begin{aligned} |v(s,x) - v(s,y)| \leq \sup_{u\in U} \mathbb{E}|(\phi(X^{s,x,u}_{T-s}) - \phi(X^{s,y,u}_{T-s})) - \phi(X^{s,x,u}_{T-s})\mathbf{1}_{(\tau^{s,x,u}<T-s)} \\ + \phi(X^{s,y,u}_{T-s})\mathbf{1}_{(\tau^{s,y,u}<T-s)}| \end{aligned}$$

or

$$\begin{aligned} |v(s,x) - v(s,y)| \leq \sup_{u\in U} \mathbb{E}|\phi(X^{s,x,u}_{T-s}) - \phi(X^{s,y,u}_{T-s})| + \sup_{u\in U} \mathbb{E}|\phi(X^{s,y,u}_{T-s})|\mathbf{1}_{(\tau^{s,y,u}<T-s)} \\ + \sup_{u\in U} \mathbb{E}|\phi(X^{s,y,u}_{T-s})|\mathbf{1}_{(\tau^{s,y,u}<T-s)}. \end{aligned}$$

The first term above is Lipschitz continuous by the first part of the proof. For the last two terms we may assume that $T - s$ is arbitrarily small because $X^{s,x,u}_t$ is a unique strong solution of SDE. Therefore, (4.26) implies that that

$$\sup_{u\in U} \mathbb{E}\left|\phi(X^{s,y,u}_{T-s})\right| \mathbf{1}_{(\tau^{s,y,u}<T-s)} \leq C\mathbb{P}\left(\tau^{s,y,u} < T-s\right)$$

can be made arbitrarily small uniformly in $y \in B_r(x)$ hence the continuity follows. ■

Now using approximations as proposed by Fleming and Soner [5], we will show that the exit control problem satisfies the Dynamic Programming Principle. Let us replace the payoff function ϕ by a function $\bar{\phi} \in C_b^2(\mathbb{R}^d)$ such that $\bar{\phi}(x) = \phi(x)$ for $x \in \mathcal{O}$. Let us now introduce the approximation of the control problem J. The payoff function is approximated by a sequence of problems defined on the entire space $\bar{Q} = [0, T) \times \mathbb{R}^d$. For $\varepsilon > 0$, $(s, x) \in \bar{Q}$,

$$J_\varepsilon(s, x; u) = \mathbb{E}[e^{-g_\varepsilon(X_t^{s,x,u})} \bar{\phi}(X_T^{s,x,u})] \quad s \in [0, T), \tag{4.27}$$

where

$$g_\varepsilon(X_t^{s,x,u}; s \le t) = \frac{1}{\varepsilon} \int_s^t c(X_r^{s,x,u}) dr$$

and $c \in C_b^2(\mathbb{R}^d)$. Furthermore, $c = 0$ on $\mathcal{O}$ and $c > 0$ on $\bar{\mathcal{O}}^c$. Now the value function is given by

$$v_\varepsilon(s, x) = \sup_{u \in U} J_\varepsilon(s, x; u). \tag{4.28}$$

The following lemma provides properties of the approximations.

Lemma 4.4. The following holds:

(i) $g_\varepsilon(X_t^{s,x,u}) \ge g_{\varepsilon'}(X_t^{s,x,u})$ for $\varepsilon' > \varepsilon$, $g_\varepsilon(X_t^{s,x,u}) = 0$ for $\tau > t$, and $g_\varepsilon(X_t^{s,x,u}) \to \infty$ as $\varepsilon \to 0$ for $\tau < t$.

(ii) $J_\varepsilon(t, x; u) \ge J(t, x; u)$ $\forall (t, x) \in Q$ and $u \in U$.

(iii) $\lim_{\varepsilon \to 0} J_\varepsilon(t, x; u) = J(t, x; u)$.

Proof.

(i) It is obvious that as $\varepsilon \to 0$, $g_\varepsilon(X_t^{s,x,u}) \to \infty$ for $\tau < t$.

(ii) The positivity of $\bar{\phi}$ implies that $J_\varepsilon(t, x; u) \ge J(t, x; u)$ $\forall (t, x) \in \bar{Q}$.

(iii) Using the fact that $g_\varepsilon(X_t^{s,x,u}) \to 0$ as $\varepsilon \to \infty$ for $\tau < t$, so $e^{g_\varepsilon(X_t^{s,x,u})} \to 1$ as $\varepsilon \to \infty$ for $\tau < t$. This implies that $\lim_{\varepsilon \to 0} J_\varepsilon(t, x; u) = J(t, x; u)$. ■

Theorem 4.5. Then the sequence of the functions $v^\varepsilon \ge 0$ is non-increasing in ε and

$$\lim_{\varepsilon \to 0} v^\varepsilon(s, x) = v(s, x).$$

Proof. Note first that by Lemma 4.4 part (i) v^ε is non-increasing. Let

$$\bar{v}(s, x) = \lim_{\varepsilon \to 0} v^\varepsilon(s, x).$$

From Lemma 4.4 part (ii), we have $J_\varepsilon(t, x; u) \ge J(t, x; u)$ $\forall (t, x) \in Q$. Hence $v_\varepsilon(s, x) \ge v(s, x)$ $\forall (s, x) \in Q$ and $u \in U$. This results in

$$v_\varepsilon(t, x) \ge \bar{v}(s, x) \ge v(s, x)$$

for all $\varepsilon > 0$ and $(s, x) \in Q$. Assume that for a certain (s, x)

$$v_\varepsilon(s, x) \geq \bar{v}(s, x) > v(s, x).$$

Hence, there exists $c > 0$ such that

$$v_\varepsilon(s, x) > c > v(s, x) \geq J(s, x; u).$$

Therefore, there exists a sequence (u_ε^n) such that

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} J_\varepsilon(s, x; u_\varepsilon^n) > c > J(s, x; u) = \lim_{\varepsilon \to 0} J_\varepsilon(s, x; u_\varepsilon),$$

where u is arbitrary but fixed. Then for any ε_1 and n,

$$J_\varepsilon(s, x; u_\varepsilon^n) > c > J_{\varepsilon_1}(s, x; u).$$

Finally, we have

$$J_\varepsilon(s, x; u_\varepsilon^n) > c \geq \lim_{\varepsilon_1 \to 0} \sup_{u \in U} J_{\varepsilon_1}(s, x; u) = \bar{v}(s, x).$$

This yields a contradiction: that $\bar{v}(s, x) > c \geq \bar{v}(s, x)$. So we can conclude that $v(s, x) = \bar{v}(s, x)$. Hence,

$$\lim_{\varepsilon \to 0} v^\varepsilon(s, x) = v(s, x).$$

■

Theorem 4.6. Assume that U is compact, and that Assumptions 2.2 and 2.3 hold. Let $\theta \in [0, T - s]$ be a stopping time. Then the function v satisfies the following property of the dynamic programming relation.

$$v(s, x) = \sup_{u \in U} \mathbb{E}\left[v(s + \theta, X_\theta^{s,x,u})\mathbf{1}_{\{\tau > \theta\}}\right], \quad s \in [0, T), x \in \bar{\mathcal{O}}. \tag{4.29}$$

In more general terms,

(a) For any given $(s, x) \in Q$ and control process u

$$v(s, x) \geq \mathbb{E}\left[v(s + \theta, X_\theta^{s,x,u})\mathbf{1}_{\{\tau > \theta\}}\right]. \tag{4.30}$$

(b) For every $\delta > 0$, there exists a control process $u \in U$ such that

$$v(s, x) - \delta \leq \mathbb{E}\left[v(s + \theta, X_\theta^{s,x,u})\mathbf{1}_{\{\tau > \theta\}}\right]. \tag{4.31}$$

Proof. Let $K_t^{s,x,u,\varepsilon} = v^\varepsilon(s+t, X_t^{s,x,u})$. By Theorem 3.5 in Krylov [8], p.149, the process $(K_t^{s,x,u,\varepsilon})$ is a supermartingale with respect to $(\mathcal{F}_t)$. Then, we have

$$\begin{aligned} v^\varepsilon(s,x) &\geq \mathbb{E}|K_0^{s,x,u,\varepsilon}| \geq \mathbb{E}\left[v^\varepsilon(T-s, X_\theta^{s,x,u})\mathbf{1}_{\{\tau>\theta\}}\right] \\ &\geq \sup_{u\in U}\mathbb{E}|K_\theta^{s,x,u,\varepsilon}| \\ &\geq \sup_{u\in U}\mathbb{E}[v^\varepsilon(s+\theta, X_\theta^{s,x,u})\mathbf{1}_{\{\tau>\theta\}}]. \end{aligned}$$

Taking limits of both sides and applying Theorem 4.5, we have

$$\begin{aligned} v(s,x) &= \lim_{\varepsilon\to 0}\sup_{u\in U}\mathbb{E}[v^\varepsilon(s+\theta, X_\theta^{s,x,u})\mathbf{1}_{\{\tau>\theta\}}] \\ &= \sup_{u\in U}\mathbb{E}[v(s+\theta, X_\theta^{s,x,u})\mathbf{1}_{\{\tau>\theta\}}]. \end{aligned}$$

Then v satisfies the dynamic programming principle. ■

Our approach here is to approximate the exit time control problem by a sequence of problem with state space $\bar{Q}$. Then, we can use the verification theorem to show the uniqueness of the problems.

Theorem 4.7. Assume that U is compact and Assumption 2.2, 2.3 hold. Then the function $v : [0,T]\times\bar{\mathcal{O}} \to \mathbb{R}$ is continuous.

Proof. We provide here a simple proof that v is lower-semicontinuous. Let $s_n \to s$, $x_n \to x$, where $(x_n) \subset \bar{\mathcal{O}}$. We will show first that v is a lower semi-continuous function. It is enough to show that the function

$$(s,x) \to J(s,x,u)$$

is lower semi-continuous for each $u \in \mathcal{U}$. Let $X_t^n = X_t^{s_n,x_n,u}$ and $\tau^n = \tau^{s_n,x_n,u}$. By Lemma 4.1,

$$\tau = \tau^{s,x,u} \leq \liminf_{n\to\infty}\tau^n.$$

Then, invoking the Fatou Lemma (see [12], page 82) we obtain

$$\liminf_{n\to\infty} J(s_n,x_n,u) \geq \mathbb{E}\liminf_{n\to\infty}\phi(X_{T-s_n}^n)\mathbf{1}_{\tau^n>(T-s_n)}.$$

By Proposition 2.7 and since ϕ is continuous we obtain

$$\liminf_{n\to\infty} J(s_n,x_n,u) \geq \mathbb{E}\phi(X_{T-s})\liminf_{n\to\infty}\mathbf{1}_{\tau^n>(T-s_n)}.$$

Note that

$$\liminf_{n\to\infty}\mathbf{1}_{\tau^n>(T-s_n)} \geq \mathbf{1}_{\tau>(T-s)}. \tag{4.32}$$

Indeed, if $\mathbf{1}_{\tau>(T-s)} = 0$ then (4.32) holds. Let

$$\mathbf{1}_{\tau>(T-s)} = 1.$$

Then

$$\liminf_{n\to\infty} \tau^n \geq \tau > (T-s)$$

and therefore

$$\liminf_{n\to\infty} \left(\tau^n - (T-s_n)\right) > 0.$$

It follows that

$$\liminf_{n\to\infty} \mathbf{1}_{\tau^n > (T-s_n)} = 1,$$

hence (4.32) follows. Finally

$$\begin{aligned}\liminf_{n\to\infty} J(s_n, x_n, u) &\geq \mathbb{E}\phi(X_{T-s})\mathbf{1}_{\tau>(T-s)} \\ &= J(s, x)\end{aligned}$$

■

By using the result of Theorem 4.7, we can state the dynamic programming equation in the following theorem.

Theorem 4.8. Let $f \in C(\mathcal{O})$. We assume here that $v \in C^{1,2}((0,T)\times\mathcal{O})$. Then the dynamic programming equation is given by

$$\frac{\partial}{\partial s}v(s,x) + \sup_{u\in U}\frac{1}{2}\mathrm{tr}(D_x^2 v(s,x)(\sigma\sigma')(s,x,u)) = 0, \quad (t,x) \in Q \tag{4.33}$$

with boundary condition

$$v(T,x) = f(x) \tag{4.34}$$

$$v(s,x) = 0, \; x \in \partial O. \tag{4.35}$$

Proof. Let us introduce the linear second order operator $\mathcal{L}^u$ associated with the process (X_t) controlled by the control process u:

$$\mathcal{L}^u\varphi(s,x) = \frac{1}{2}\mathrm{tr}\left(D_x^2\varphi(s,x)\sigma(s,x,u)\sigma'(s,x,u)\right). \tag{4.36}$$

With this notation, by Ito's formula we have

$$\begin{aligned}\varphi(t, X_t^{s,x,u}) - \varphi(s,x) &= \int_s^t \left(\frac{\partial}{\partial r} + \mathcal{L}^u\right)\varphi(r, X_r^{s,x,u})dr + \\ &\int_s^t D_x\varphi(s, X_r^{s,x,u})\sigma(s,x,u)dW_r \qquad t > s\end{aligned} \tag{4.37}$$

for all smooth functions $\varphi \in C^{1,2}([0,T)\times\mathcal{O})$ and all admissible control processes $u \in U$. In this case, D_x and D_x^2 denote the gradient and the Hessian operator with respect to the x variable, respectively.

Fix $(s, x) \in Q$ and a control process u. Let X_t be the solution of (2.1) with control u and the initial data $X_s = x$. Ito's formula (4.37) yields

$$\mathbb{E}w(T \wedge \tau, X^{s,x,u}_{T\wedge\tau}) \;=\; w(s,x) + \mathbb{E}\int_s^{T\wedge\tau}\left[\frac{\partial}{\partial r}w(r, X^{s,x,u}_r) + \mathcal{L}^u w(r, X^{s,x,u}_r)\right]dr,$$

since w solves (4.33), and

$$\frac{\partial}{\partial r}w(r, X^{s,x,u}_r) + \mathcal{L}^u w(r, X^{s,x,u}_r) \geq 0$$

Combining (4.34),(4.35) and the above inequalities, we obtain $w(s,x) \geq J(s,x,u)$. (b) Suppose that $\bar{X}_s$ is the solution of (2.1) with control process $\bar{u}$. Then by definition of $\bar{u}$ we have the dynamic programming equation, that is

$$\frac{\partial}{\partial r}w(s,x) + \mathcal{L}^{\bar{u}}(s,x) = 0, \quad s < T, \quad x \in \mathcal{O}.$$

Therefore

$$\frac{\partial}{\partial r}w(r, \bar{X}^{s,x,\bar{u}}_r) + \mathcal{L}^{\bar{u}_r} w(r, \bar{X}^{s,x,\bar{u}}_r) = 0$$

for almost every (s, ω). Using (4.34) and (4.35), we obtain $w(s,x) = J(s,x,\bar{u})$. ■

5. Financial Applications

Consider a market consist of two risky asset ($d = 2$) as shown by equation (3.10). Without loss of generality, assume that =0. The risk neutral price processes for the two assets S^1_t and S^2_t follow the stochastic differential equations

$$dS^1_t = \sigma^1_t S^1_t dW^1_t \tag{5.38}$$

$$dS^2_t = \sigma^2_t S^2_t dW^2_t \tag{5.39}$$

$$S^1_0 = x_1, \; S^2_0 = x_2. \tag{5.40}$$

Let ρ denote the correlation coefficient of the Brownian motion dW^1_t and dW^2_t. Assume that the interest rate is zero. We can write the dynamics of the two assets in a more compact vectorial notation:

$$d\begin{pmatrix} S^1_t \\ S^2_t \end{pmatrix} = \begin{pmatrix} S^1_t & 0 \\ 0 & S^2_t \end{pmatrix}\begin{pmatrix} (\sigma^1_t)^2 & \rho\sigma^1_t\sigma^2_t \\ \rho\sigma^1_t\sigma^2_t & (\sigma^2_t)^2 \end{pmatrix} d\begin{pmatrix} W^1_t \\ W^2_t \end{pmatrix} \tag{5.41}$$

or

$$dS_t = \bar{S}_t\sigma_t dW_t. \tag{5.42}$$

Here

$$\bar{S}_t = \operatorname{diag}(S_t) = \begin{pmatrix} S^1_t & 0 \\ 0 & S^2_t \end{pmatrix}$$

and (σ_t) is a two-dimensional process such that $\sigma_t \in \mathcal{A}(\Sigma)$, where $\mathcal{A}(\Sigma)$ is a set of admissible volatilities which is progressively measurable with respect to $(\mathcal{F})$.

Consider a price process $\{v_t; 0 \le t \le T\}$ for a two-asset barrier option satisfying

$$v_T = (\max(S_T^1, S_T^2) - K)^+ \mathbf{1}_{\{\tau > T\}}, \quad 0 \le t \le T.$$

Let $S_t^* = \max(S_t^1, S_t^2),\ 0 \le t \le T$. Then the first moment of time when the process S_t^* hits the barrier H is given by

$$\tau = \inf\{t \ge 0; \quad S_t^* \ge H\}, \tag{5.43}$$

where $0 < K < H$. The value of the option at time t is given by

$$v(0, x_1, x_2) = \mathbb{E}\left[\min((S_T^1, S_T^2) - K)^+ \mathbf{1}_{\{\tau > T\}} | S_t^1 = x_1, S_t^2 = x_2\right]. \tag{5.44}$$

For constant volatilities $\sigma_t = \sigma$, the option price v can be computed by solving the partial differential equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}(\sigma^1)^2 x_1^2 \frac{\partial^2 v}{\partial x_1^2} + \frac{1}{2}(\sigma^2)^2 x_2^2 \frac{\partial^2 v}{\partial x_2^2} + \rho\sigma^1\sigma^2 x_1 x_2 \frac{\partial^2 v}{\partial x_1 \partial x_2} = 0 \tag{5.45}$$

with terminal and boundary conditions

$$v(T, x_1, x_2) = (\min(x_1, x_2) - K)^+, \quad 0 \le \max(x_1, x_2) < H \tag{5.46}$$

$$v(t, x_1, x_2) = 0, \qquad \max(x_1, x_2) \ge H, \ 0 \le t \le T. \tag{5.47}$$

Equation (5.45) can be written as

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}(D_x^2 v(\bar{x}\sigma)(\bar{x}\sigma)' = 0 \tag{5.48}$$

where the vector $x = (x_1, x_2)$ and the matrix

$$\bar{x} D_x^2 v \bar{x}' = \begin{pmatrix} x_1^2 \dfrac{\partial^2 v}{\partial x_1^2} & x_1 x_2 \dfrac{\partial^2 v}{\partial x_1 \partial x_2} \\ x_1 x_2 \dfrac{\partial^2 v}{\partial x_1 \partial x_2} & x_2^2 \dfrac{\partial^2 v}{\partial x_2^2} \end{pmatrix}.$$

Now we assume that the true volatilities are not known and limited to move in a certain interval. We write the set of admissible volatilities as follows:

$$\Sigma = \left\{ \gamma \in \mathbb{R}^{2\times 2} \;\middle|\; \gamma = \begin{pmatrix} \sigma_t^1 & \rho\sigma_t^1 \\ \rho\sigma_t^2 & \sigma_t^2 \end{pmatrix}, \sigma_1^- \le \sigma_t^1 \right.$$

$$\left. \le \sigma_1^+, \sigma_2^- \le \sigma_t^2 \le \sigma_2^+, \rho^- \le \rho \le \rho^+ \right\}.$$

Then we have

$$\gamma\gamma^* = \begin{pmatrix} (\sigma_t^1)^2 & \rho\sigma_t^1\sigma_t^2 \\ \rho\sigma_t^1\sigma_t^2 & (\sigma_t^2)^2 \end{pmatrix}.$$

Let $A_{t,x} = A$ be

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then the dynamic programming programming equation can be written as:

$$\frac{\partial v}{\partial t} + \frac{1}{2} \max_{\gamma \in \mathcal{A}(\Sigma)} \operatorname{tr}(A\gamma\gamma^*) = 0, \tag{5.49}$$

with terminal and boundary condition

$$v(T, x_1, x_2) = (\max(x_1, x_2) - K)^+, \quad 0 \leq \max(x_1, x_2) < H \tag{5.50}$$

$$v(t, S_T^1, S_T^2) = 0, \qquad , \max(x_1, x_2) \geq H, \quad 0 \leq t \leq T. \tag{5.51}$$

the optimization problem appearing in the DPP equation (5.49). First, we write the function to be maximized as follows:

$$f(\sigma_1, \sigma_2, \rho) = \operatorname{tr}(A\gamma\gamma^*) = a\sigma_1^2 + 2b\rho\sigma_1\sigma_2 + c\sigma_2^2, \tag{5.52}$$

where $\sigma_1^- \leq \sigma_t^1 \leq \sigma_1^+,\ \sigma_2^- \leq \sigma_t^2 \leq \sigma_2^+,\ \rho^- \leq \rho \leq \rho^+$.

The numerical solution of problem (5.49)-(5.51) is beyond our discussion. One may refer to [3] for the outline solution of problem.

6. Concluding remark

Our discussion above shows that the exit time control problem can be solved by approximating the problem by a sequence of problems with state space $\bar{Q}$. We have also shown that the approximation satisfies the dynamic programming principle, Theorem 4.6. If $v \in C^{2,1}(\bar{Q})$ then the nonlinear Black-Scholes equation is easily derived from the dynamic programming principles via Ito's formula (see Theorem 4.8).

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