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Fuzzy BP-ideal

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Abstract

In this paper, we define the notion of Fuzzy BP-ideal. We discuss the properties of Fuzzy BP-ideals and prove some results.

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1. Introduction

In 1966 Y. Imai and K. Iseki introduced new classes of abstract algebra, BCK algebras and BCI algebras [3, 4]. While studying the algebraic structures of BCK and BCI logic, many algebraic structures were studied by several authors, which are generalization of BCI and BCI algebras. Q.P. Hu and X. Li introduced the notion of BCH algebras [2]. J. Neggers and H.S. Kim introduced the notion of d-algebras which is another generalization of BCK algebras [6]. In 1975 Iseki introduced the concept of implicative ideals [5]. In 1971 A.Rosenfeld initiated the study of fuzzy algebraic structures [7]. In 1965 L.A. Zadeh introduced the notion of fuzzy sets which give a complete picture of uncertainty in real physical world [9]. In 2012 Sun Shin Ahn and Jeong Soon Han introduced the notion of BP-Algebras [8]. In our earlier paper we introduced the notion of fuzzy structures in BP-algebras [1]. In this paper we introduce the notion of fuzzy BP-ideal and discuss some results.

2. Preliminares

In this section we recall some basic definitions that are needed for our work.

Definition 2.1. A BP algebra (X,$*$, 0) is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions: for all $x, y, z \in X$

(i) $x * x = 0$

(ii) $x * (x * y) = y$

(iii) $(x * z) * (y * z) = x * y$.

Definition 2.2. A non-empty subset A of a BP-Algebra X is said to be a BP sub-Algebra if x$*$y $\in$ A, $\forall\, x, y \in A$.

Definition 2.3. Let S be a non-empty set. A mapping $\mu : S \to [0, 1]$ is called a fuzzy subset of S.

Definition 2.4. A fuzzy subset μ of a BP-algebra $(X, *, 0)$ is called a fuzzy BP sub algebra if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.5. A non-empty subset I of BP-algebra (X, $*$, 0) is said to be a BP-Ideal of X if it satisfies the following conditions:

(i) $0 \in$ I

(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I, \forall x, y \in I$.

Definition 2.6. Let λ and μ be the fuzzy set in a set X. The Cartesian product $\lambda \times \mu : X \times X \to [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(x)\} \forall x \in X$.

Definition 2.7. Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be BP-algebras. A mapping $f : X_1 \to X_2$ is called a homomorphism if, $f(x *_1 y) = f(x) *_2 f(y) \forall x, y \in X$.

Definition 2.8. Let f be any function from the BP-algebra X_1 to the BP-algebra X_2. Let μ be any fuzzy BP-sub algebra of X_1 and σ be any fuzzy BP-sub algebra of X_2. The image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of X_2 defined by

$$f(\mu(x)) = \begin{cases} Sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

where $y \in X_2$. The pre image of σ under f, symbolized by $f^{-1}(\sigma)$, is a fuzzy subset of X_1 defined by $(f^{-1}(\sigma))(x) = \sigma(f(x)) \forall x \in X_1$.

3. Fuzzy BP-Ideals

In this section we introduce the notion of Fuzzy BP ideal and prove some simple results.

Definition 3.1. Let X be a BP-algebra. A fuzzy set μ of X is said to be a fuzzy BP-ideal of X if it satisfies the following conditions: $\forall x, y \in X$

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

Example 3.2. Let $(X = \{0, 1, 2, 3\}, *, 0)$ be a BP-algebra with the following Cayley table.

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define $\mu : X \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.6 & \text{if } x = 2 \\ 0.3 & \text{if } x = 1, 3 \end{cases}$$

Therefore μ is a fuzzy BP-ideal of the BP-algebra X.

Proposition 3.3. Intersection of two fuzzy BP-ideals of X is again a fuzzy BP-ideal of X.

Proof. Let μ and ψ be any two fuzzy BP-ideals of X.

$$\begin{aligned}
(\mu \cap \psi)(0) &= (\mu \cap \psi)(x * x) \\
&\geq \min\{\mu(x * x), \psi(x * x)\} \\
&\geq \min\{\min\{\mu(x), \mu(x)\}, \min\{\psi(x), \psi(x)\}\} \\
&= \min\{\min\{\mu(x), \psi(x)\}, \min\{\mu(x), \psi(x)\}\} \\
&= \min\{(\mu \cap \psi)(x), (\mu \cap \psi)(x)\} \\
&= \min\{(\mu \cap \psi)(x)\} \\
\therefore (\mu \cap \psi)(0) &\geq (\mu \cap \psi)(x) \\
(\mu \cap \psi)(x) &= \min\{\mu(x), \psi(x)\} \\
&\geq \min\{\min(\mu(x * y), \mu(y)), \min(\psi(x * y), \psi(y))\} \\
&= \min\{\min(\mu(x * y), \psi(x * y))), \min(\mu(y), \psi(y))\} \\
&= \min\{(\mu \cap \psi)(x * y), (\mu \cap \psi)(y))\}, \text{ for all } x, y \in X.
\end{aligned}$$

Hence $\mu \cap \psi$ is a fuzzy BP-ideal of X. ■

Proposition 3.4. If μ is a fuzzy BP-ideal of a BP-algebra (X, $*$, 0), then $\forall x, y \in X$.

(i) μ is order reversing; that is, $x \leq y$ implies $\mu(x) \geq \mu(y)$

(ii) $\mu(x * (x * y)) \geq \mu(y)$.

Proof. Since μ is a fuzzy BP-ideal of X. Let

$$\begin{aligned} x \leq y \Rightarrow x * y &= 0 \\ \Rightarrow \mu(x * y) &= \mu(0) \\ \therefore \mu(x * y) &= \mu(0) \geq \mu(x). \\ \mu(x) &\geq \min\{\mu(x * y), \mu(y)\} \\ &\geq \min\{\mu(0), \mu(y)\} \\ &= \mu(y) \\ \therefore \mu(x) &\geq \mu(y) \\ \text{By definition 2.1(ii)} x * (x * y) &= y \\ \therefore (x * (x * y)) * y &= y * y \\ \Rightarrow (x * (x * y)) * y &= 0 \\ \Rightarrow x * (x * y) \leq y & \end{aligned}$$

By (1) μ is order reversing, $\mu(x * (x * y)) \geq \mu(y) \forall x, y \in X$. ∎

Proposition 3.5. If μ is a fuzzy ideal of a BP-algebra $(X, *, 0)$ and

$$\mu_\alpha(x) = \min\{\alpha, \mu(x)\}$$

$\forall$ x$\in$X and α $\in$[0, 1], then $\mu_\alpha(x)$ is fuzzy BP-ideal of X.

Proof. Let μ be a fuzzy ideal of the BP-algebra (X, $*$, 0) and $\alpha \in [0, 1]$. Therefore $\mu(0) \geq \mu(x) \forall x \in X$. Now,

$$\begin{aligned} \mu_\alpha(x)(0) &= \min\{\alpha, \mu(0)\} \\ &\geq \min\{\alpha, \mu(x)\} \\ &= \mu_\alpha(x) \quad \forall x \in X. \end{aligned}$$

Also, μ is a fuzzy ideal of X shows that

$$\begin{aligned} \mu(x) &\geq \min\{\mu(x * y), \mu(y)\} \forall x, y \in X \\ \mu_\alpha(x) &= \min\{\alpha, \mu(x)\} \\ &\geq \min\{\alpha, \min(\mu(x * y), \mu(y))\} \\ &= \min\{\min(\alpha, \mu(x * y)), \min(\alpha, \mu(y))\} \\ &= \min\{\mu_\alpha(x)(x * y), \mu_\alpha(x)(y)\} \end{aligned}$$

$\Rightarrow \mu_\alpha(x)$ (x) is a fuzzy ideal of X. Since this is true for all $\alpha \in [0, 1]$, $\mu_\alpha(x)$ is fuzzy BP-ideal of X for all $\alpha \in [0, 1]$. ■

Theorem 3.6. A fuzzy subset of a BP-algebra (X, $*$, 0) is a fuzzy BP-ideal if and only if for any $\lambda \in [0, 1]$, $U(\mu, \lambda) = \{x : x \in X, \mu(x) \geq \lambda\}$ is an ideal of X where $U(\mu, \lambda) \neq \phi$.

Proof. Suppose μ is a fuzzy ideal of X and $U(\mu, \lambda) \neq \phi$ for $\lambda \in [0, 1]$.

Let $x \in U(\mu, \lambda)$, then $\mu(x) \geq \lambda$. By definition of fuzzy BP-ideal, We have $\mu(0) \geq \mu(x) \geq \lambda$. Thus $0 \in U(\mu, \lambda)$. Suppose $x * y \in U(\mu, \lambda)$ and $y \in U(\mu, \lambda)$. Therefore, $\mu(x * y) \geq \lambda$ and $\mu(y) \geq \lambda$. By definition, we have

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq \lambda$$

$\therefore x \in U(\mu, \lambda)$. Hence (μ, λ) is an BP-ideal of X.

Conversely, suppose that for each $\lambda \in [0, 1]$, $U(\mu, \lambda)$ is either empty or an ideal of X. For any $x \in X$, let $\mu(x) = \lambda$. Then $x \in U(\mu, \lambda)$. Since $U(\mu, \lambda) \neq \phi$ is an ideal of X, we have $0 \in U(\mu, \lambda)$ and hence $\mu(0) \geq \lambda = \mu(x)$. Thus $\mu(0) \geq \mu(x) \forall x \in X$. Assume $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \forall x, y \in X$ is not true. Then there exists $x_o, y_o \in$ X such that

$$\begin{aligned} \mu(x_o) &\leq \min\{\mu(x_o * y_o), \mu(y_o)\} \\ \Rightarrow \mu(x_o) &< \lambda_o < \min\{\mu(x_o * y_o), \mu(y_o)\} \end{aligned}$$

We have $x_o * y_o, y_o \in U(\mu, \lambda_o)$ and $U(\mu, \lambda_o) \neq \phi$. But U$(\mu, \lambda_o)$ is an ideal of X. So $x_o \in U(\mu, \lambda_o)$ by the definition of BP-ideal, $\mu(x_o) \geq \lambda_o$, contradicting $(\mu(0) \geq \mu(x) \forall x \in X)$. Therefore $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$. ■

Theorem 3.7. A fuzzy subset μ of a BP-algebra $(X, *, 0)$ is a fuzzy BP-ideal if and only if every nonempty level subset of $U(\mu, s)$, $s \in Im(\mu)$ is a BP-ideal.

Proof. Let μ be a fuzzy BP-ideal.
Claim: $U(\mu, s)$, $s \in Im(\mu)$ is a BP-ideal.
Since $U(\mu, s) \neq \phi$ there exist $x \in U(\mu, s)$ such that $\mu(x) \geq s$. Since μ is a fuzzy BP-ideal, $\mu(0) \geq \mu(x) \forall x \in X$. Hence for this $x \in U(\mu, s)$, $\mu(0) \geq s$ which shows that $0 \in U(\mu, s)$. Now, for any $x, y \in X$, assume that $x * y \in U(\mu, s)$, and $y \in U(\mu, s)$.

$$x * y \in U(\mu, s) \quad \Rightarrow \quad \mu(x * y) \geq s$$

Also

$$\begin{aligned} y \in U(\mu, s) &\Rightarrow \mu(y) \geq s \\ \therefore \mu(x * y) &\geq s \text{ and } \mu(y) \geq s. \\ \Rightarrow \min\{\mu(x * y), \mu(y)\} &\geq s \end{aligned}$$

Since μ is a fuzzy BP-ideal, $\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq s$. Thus proving $x \in U(\mu, s)$. This proves that $U(\mu, s)$ is a BP-ideal of X.

Conversely, let $U(\mu, s), s \in Im(\mu)$ is a BP-ideal of X.
Claim: μ is a fuzzy BP-ideal.
Let $x, y \in X$. For any $s \in Im(\mu)$, let $s = \min\{\mu(x * y), \mu(y)\}$. Therefore, $\mu(x * y) \geq s$ and $\mu(y) \geq s$. This shows that $x * y, y \in U(\mu, s)$. Since $U(\mu, s)$ is a BP-ideal we have $x \in U(\mu, s)$. This proves that $\mu(x) \geq s = \min\{\mu(x * y), \mu(y)\}$. This shows that μ is a fuzzy BP-ideal of X. ■

Theorem 3.8. Let μ be a fuzzy BP-ideal of BP-algebra X and let $x \in X$. Then $\mu(x) = t$ if and only if $x \in U(\mu, t)$ but $x \notin U(\mu, s) \forall s > t$.

Proof. Let μ be a fuzzy BP-ideal of X and let $x \in X$. Assume $\mu(x) = t$, so that $x \in U(\mu, t)$. If possible, let $x \in U(\mu, s)$ for $s > t$. Then $\mu(x) \geq s > t$. This contradicts the fact that $\mu(x) = t$, concludes that $x \notin U(\mu, s) \forall s > t$.

Conversely, let $x \in U(\mu, t)$ but $x \notin U(\mu, s) \forall s > t$. $x \in U(\mu, t) \Rightarrow \mu(x) \geq t$. Since $x \notin U(\mu, s) \forall s > t$, $\mu(x) = t$.

Theorem 3.9. Let X be a BP-algebra. Let λ and μ be the fuzzy BP-ideals of X. Then $\lambda \times \mu$ is a fuzzy BP-ideal of $X \times X$.

Proof. Let X be a BP-algebra and let λ and μ be fuzzy BP-ideals of X. For any $(x, y) \in X \times X$.

$$\begin{aligned}
(\lambda \times \mu)(0, 0) &= \min\{\lambda(0), \mu(0)\} \\
&\geq \min\{\lambda(x), \mu(x)\} \\
&= (\lambda \times \mu)(x).
\end{aligned}$$

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$$\begin{aligned}
(\lambda \times \mu)(x) &= (\lambda \times \mu)(x_1, x_2) \\
&= \min\{\lambda(x_1), \mu(x_2)\} \\
&\geq \min\{\min(\lambda(x_1 * y_1), \lambda(y_1)), \min(\mu(x_2 * y_2), \mu(y_2))\} \\
&= \min\{\min(\lambda(x_1 * y_1), \mu(x_2 * y_2)), \min(\lambda(y_1), \mu(y_2))\} \\
&= \min\{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2), (\lambda \times \mu)(y_1, y_2)\} \\
&= \min\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda x \mu)(y_1, y_2)\} \\
&= \min\{(\lambda \times \mu)(x, y), (\lambda \times \mu)(y)\}.
\end{aligned}$$

Thus $\lambda \times \mu$ is a fuzzy BP-ideal of $X \times X$. ■

Theorem 3.10. For any two fuzzy subsets λ and μ of X, if $\lambda \times \mu$ is a fuzzy BP-ideal of X, then either λ or μ is a fuzzy BP-ideal of X.

Proof. Let λ and μ be fuzzy subsets of X such that $\lambda x \mu$ is a fuzzy BP-ideal of X.

$$\begin{aligned} \therefore (\lambda \times \mu)(0,0) &\geq (\lambda \times \mu)(x,y) \;\; \forall\, (x,y) \in X \times X. \\ \text{Assume } \lambda(x) &> \lambda(0) \text{ and } \mu(y) > \mu(0) \text{ for some } x, y \in X. \\ \text{Then } (\lambda \times \mu)(x,y) &= \min\{\lambda(x), \mu(y)\} \\ &> \min\{\lambda(0), \mu(0)\} \\ &= (\lambda \times \mu)(0,0) \text{ which is a contradiction.} \end{aligned}$$

Thus $\lambda(x) \geq \lambda(0)$ or $\mu(0) > \mu(y) \forall x, y \in X$.
Let $x = (x_1, x_2)$ and $y = (y_1, y_2)) \in X \times X$

$$\begin{aligned} (\lambda \times \mu)(x) &\geq \min\{(\lambda \times \mu)(x * y), (\lambda x \mu)(y)\} \\ &= \min\{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2), (\lambda \times \mu)(y_1, y_2)\}\} \\ &= \min\{\min(\lambda(x_1 * y_1), \mu(x_1, x_2)), \min(\lambda(y_1), \mu(y_2))\} \\ \Rightarrow \min\{(\lambda(x_1), \mu(x_2)\} &\geq \min\{\min(\lambda(x_1 * y_1), \lambda(y_1), \min(\mu(x_1, y_2), \mu(y_2))\} \\ \Rightarrow either\;\; \lambda(x_1) &\geq \min(\lambda(x_1 * y_1), (\lambda(y_1))\} \\ \text{or}\;\; \mu(x_2) &\geq \min(\mu(x_1, y_2)\mu(y_2)) \end{aligned}$$

$\Rightarrow$ λ or μ is fuzzy ideal of X. ■

Theorem 3.11. Let λ and μ be fuzzy BP-ideals of X_1 and X_2 respectively. Then $\lambda \times \mu$ is a fuzzy BP-ideal of $X_1 \times X_2$.

Proof. Let λ be a fuzzy BP-ideal of X_1. Let μ be a fuzzy BP-ideal of X_2.
Claim: $\lambda \times \mu$ is fuzzy BP-ideals of $X_1 \times X_2$. For any $(x, y) \in X_1 \times X_2$.

$$\begin{aligned} (\lambda \times \mu)(0,0) &= \min\{\lambda(0), \mu(0)\} \\ &\geq \min\{\lambda(x), \mu(y)\} \\ &= (\lambda \times \mu)(x,y) \end{aligned}$$

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$

$$\begin{aligned} (\lambda \times \mu)(x_1, x_2) &= \min\{\lambda(x_1), \mu(x_2)\} \\ &\geq \min\{\min(\lambda(x_1 * y_1), \lambda(y_1)), \min(\mu(x_2 * y_2), \mu(y_2))\} \\ &= \min\{\min(\lambda(x_1 * y_1), \mu(x_2 * y_2)), \min(\lambda(y_1), \mu(y_2))\} \\ &= \min\{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2)), (\lambda \times \mu(y_1, y_2)\} \\ &= \min\{(\lambda \times \mu)(x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\} \end{aligned}$$

Thus $\lambda \times \mu$ is a fuzzy BP-ideal of $X_1 \times X_2$. ■

Theorem 3.12. Let f: $X_1 \to X_2$ be an epimorphism of BP-algebras. Let μ be a fuzzy subset of X_2. If $f^{-1}(\mu)$ is a fuzzy BP-ideal of X_1, then μ is a fuzzy BP-ideal of X_2.

Proof. Let $f : X_1 \to X_2$ be an epimorphism of BP-algebras. Let μ be a fuzzy subset of X_2. Let $f^{-1}(\mu)$ is a fuzzy BP-ideal of X_1.
Claim: μ is a fuzzy BP-ideal of X_2.

$$\mu(0_{X_2})) = \mu(f(0_{X1}))) \geq f^{-1}(\mu(x_1)) = \mu(f(x_1)) = \mu(x_2)$$

Let $x_2, y_2 \in X_2$. Since f is an epimorphism, $x_1, y_1 \in X_1$ such that $f(x_1) = x_2$, and $f(y_1) = y_2$ that is, $x_1 = f^{-1}(x_2)$ and $y_1 = f^{-1}(y_2)$.

$$\begin{aligned} \mu(x_2) &= \mu(f(x_1) \\ &= f^{-1}(\mu(x_1)) \\ &\geq \min\{f^{-1}(\mu(X_1 * y_1)), f^{-1}(\mu(y_1))\} \\ &= \min\{\mu(f(x_1 * y_1)), \mu(f(y_1))\} \\ &= \min\{\mu(f(x_1) * f(y_1)), \mu(f(y_1))\} \\ &= \min\{\mu(x_2 * y_2)), \mu(y_2)\} \end{aligned}$$

$\therefore \mu$ is a fuzzy BP-ideal of X_2. ∎

Theorem 3.13. Inverse image of fuzzy BP-ideal is again a fuzzy BP-ideal.

Proof. Let $f : X_1 \to X_2$ be an epimorphism. Let σ be fuzzy BP-ideal of X_2.
To prove: $f^{-1}(\sigma)$ is a fuzzy BP-ideal of X_1.

$$\begin{aligned} (f^{-1}(\sigma))(x) &= \sigma(f(x)) \\ &\geq \min\{\sigma(f(x) * f(y)), \sigma(f(y))\} \\ &= \min\{\sigma(f(x * y), \sigma(f(y))\} \quad \text{(since } f \text{ is epimorphism)} \\ &= \min(f^{-1}(\sigma))(x * y), f^{-1}(\sigma))(y))) \end{aligned}$$

Thus $f^{-1}(\sigma)$ is a fuzzy BP-ideal of X_1. ∎

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