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Characterizations on the Mazur-Ulam theorem in non-Archimedean fuzzy n-normed spaces

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Abstract

We investigate a notion of non-Archimedean fuzzy anti-n-normed spaces and prove the Mazur-Ulam theorem in the spaces.

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1. Introduction

A mapping $f : X \longrightarrow Y$ is called an *isometry* if f satisfies $d_y(f(x),\ f(y)) = d_x(x,\ y)$, for all $x, y \in X$, where $d_X(\cdot,\ \cdot)$ and $d_Y(\cdot,\ \cdot)$ denote the metrices in the spaces X and Y, respectively.

The theory of isometric mappings had been originated in the classical paper [5] by S. Mazur and S. Ulam in 1932:

Mazur-Ulam Theorem. Every isometry of a real normed linear space onto a real normed linear space is a linear mapping up to translation.

J.A. Baker [2] proved that an isometry from one real normed linear space into a strictly convex real normed linear space is affine. Recently, M.S. Moslehian and G. Sadeghi [7] intorduced the Mazur-Ulam theorem in the non-Archimedean strictly convex normed

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spaces. In particular, M. Amyari and G. Sadeghi [1] proved that Mazur-Ulam theorem under some conditions in non-Archimedean 2-normed spaces.

A *non-Archimedean field* is a field $\mathcal{K}$ equipped with a (valuation) function from $\mathcal{K}$ into $[0, \infty)$ satisfying the following properties: (1) $|a| \geq 0$ and equality holds if and only if $a = 0$, (2) $|ab| = |a|\,|b|$, (3) $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathcal{K}$, Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. An example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$; see [4]. Also, the most important examples of non-Archimedean spaces are p-adic numbers; see [6]. Moreover, A.K. Mirmostafaee and M.S. Moslehian [6] introduced a non-Archimedean fuzzy norm on a linear space over a non-Archimedean field. In addition, fuzzy n-normed linear spaces were studied by Al. Narayanan and S. Vijayabalaji;see [3].

Now we give our definitions of a non-Archimedean n-normed linear space and non-Archimedean fuzzy n-normed linear space over a non-Archimedean field.

Definition 1.1. Let X be a linear space over a non-Archimedean field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $||\cdot, \cdots, \cdot|| : X^n \longrightarrow [0, \infty)$ is said to be a *non-Archimedean n-norm* if it satisfies the following properties:

1. $||x_1, \ldots, x_n|| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent
2. $||x_1, \ldots, x_n|| = ||x_{\sigma(1)}, \ldots, x_{\sigma(n)}||$, where σ is a permutation
3. $||cx_1, \ldots, x_n|| = |c|\,||x_1, \ldots, x_n||$
4. $||x_1, \ldots, x_{n-1}, y + z|| \leq \max\{||x_1, \ldots, x_{n-1}, y||, ||x_1, \ldots, x_{n-1}, z||\}$,

for all $x_1, \ldots, x_n, y, z \in X$ and $c \in \mathcal{K}$, Then $(X, ||\cdot, \ldots, \cdot||)$ is called a *non-Archimedean n-normed space*.

Definition 1.2. Let X be a linear space over a non-Archimedean field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$. A function $N : X^n \times \mathbb{R} \longrightarrow [0, 1]$ is said to be a *non-Archimedean fuzzy anti-n-norm* on X if for all $x_1, \ldots, x_n, x, y \in X$ and all $s, t \in \mathbb{R}$,

(aN1) $N(x_1, \ldots, x_n, t) = 1$ for $t \leq 0$,

(aN2) for $t > 0$, $N(x_1, \ldots, x_n, t) = 0$ if and only if $x_1, \ldots, x_{n-1}$ and x_n are linearly dependent,

(aN3) $N(x_1, \ldots, x_n, t) = N(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, t)$ where σ is a permutation,

(aN4) $N(cx_1, \ldots, x_n, t) = N(x_1, \ldots, x_n, \frac{t}{|c|})$ for $c \in \mathcal{K}$ $(c \neq 0)$,

(aN5) $N(x+y, x_2, \ldots, x_n, \max\{s, t\}) \leq \max\{N(x, x_2, \ldots, x_n, s), N(y, x_2, \ldots, x_n, t)\}$,

(aN6) $N(x_1, \ldots, x_n, *)$ is a non-increasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x_1, \ldots, x_n, t) = 0$.

The pair (X, N) is called a *non-Archimedean fuzzy anti-n-normed space.*

The property (aN4) implies that $N(-x_1, x_2, \ldots, x_n, t) = N(x_1, x_2, \ldots, x_n, t)$ for all $x_1, \ldots, x_n \in X$ and $t > 0$. It is easy to show that (aN5) is equivalent to the following condition:

$$N(x + y, x_2, \ldots, x_n, t) \leq \max\{N(x, x_2, \ldots, x_n, t), N(y, x_2, \ldots, x_n, t)\},$$

for all $x, y, x_2, \ldots, x_n \in X$ and $t \in \mathbb{R}$.

Example 1.3. Let $(X, ||\cdot, \ldots, \cdot||)$ be a non-Archimedean n-normed space. Define

$$N(x_1, x_2, \ldots, x_n, t) = \begin{cases} \dfrac{||x_1, x_2, \ldots, x_n||}{t + ||x_1, x_2, \ldots, x_n||} & \text{when } t > 0,\ t \in \mathbb{R} \\ 1 & \text{when } t \leq 0, \end{cases}$$

where $x_1, x_2, \ldots, x_n \in X$. Then (X, N) is a non-Archimedean fuzzy anti-n-normed space. Indeed,

(aN1) The definition of N implies that $N(x_1, \ldots, x_n, t) = 1$ for $t \leq 0$,

(aN2) Let $t > 0$, $N(x_1, \ldots, x_n, t) = 0 \Leftrightarrow ||x_1, \ldots, x_n|| = 0 \Leftrightarrow x_1, \ldots, x_n$ are linearly dependent.

(aN3) $N(x_1, \ldots, x_n, t) = \dfrac{||x_1, \ldots, x_n||}{t + ||x_1, \ldots, x_n||} = \dfrac{||x_{\sigma(1)}, \ldots, x_{\sigma(n)}||}{t + ||x_{\sigma(1)}, \ldots, x_{\sigma(n)}||}$ $= N(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, t)$, for any permutation σ,

(aN4)

$$\begin{aligned} N(cx_1, \ldots, x_n, t) &= \frac{||cx_1, \ldots, x_n||}{t + ||cx_1, \ldots, x_n||} \\ &= \frac{||x_1, \ldots, x_n||}{\frac{t}{|c|} + ||x_1, \ldots, x_n||} = N(x_1, \ldots, x_n, \frac{t}{|c|}) \end{aligned}$$

for all $c \neq 0$,

(aN5) Let $x_1, \ldots, x_{n-1},\ y, z \in X$ and $s, t \in \mathbb{R}$, If $s, t \leq 0$, then $N(x_1, \ldots, x_{n-1}, y + z, \max\{s, t\}) = 1$ and

$$\max\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, t)\} = 1,$$

If $s < 0 < t$, then

$$\begin{aligned} &N(x_1, \ldots, x_{n-1}, y + z, \max\{s, t\}) \leq 1 \\ &\quad = \max\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, t)\}, \end{aligned}$$

Now, let $s, t > 0$. We may assume that $\max\{s, t\} = s \geq t$. Since $||x_1, \ldots, x_{n-1}, y+z|| \leq \max\{||x_1, \ldots, x_{n-1}, y||, ||x_1, \ldots, x_{n-1}, z||\}$. We may consider two cases where (a) $\max\{||x_1, \ldots, x_{n-1}, y||, ||x_1, \ldots, x_{n-1}, z||\} = ||x_1, \ldots, x_{n-1}, y||$ and (b) $\max\{||x_1, \ldots, x_{n-1}, y||, ||x_1, \ldots, x_{n-1}, z||\} = ||x_1, \ldots, x_{n-1}, z||$. The case (a):

$$\begin{aligned} N(x_1, \ldots, x_{n-1}, y+z, \max\{s,t\}) &= \frac{||x_1, \ldots, x_{n-1}, y+z||}{s + ||x_1, \ldots, x_{n-1}, y+z||} \\ &\leq \frac{||x_1, \ldots, x_{n-1}, y||}{s + ||x_1, \ldots, x_{n-1}, y||} \\ &\leq \max\{N(x_1, \ldots, x_{n-1}, y, s), \\ &\quad N(x_1, \ldots, x_{n-1}, z, t)\}, \end{aligned}$$

The case (b):

$$\begin{aligned} N(x_1, \ldots, x_{n-1}, y+z, \max\{s,t\}) &= \frac{||x_1, \ldots, x_{n-1}, y+z||}{s + ||x_1, \ldots, x_{n-1}, y+z||} \\ &\leq \frac{||x_1, \ldots, x_{n-1}, z||}{s + ||x_1, \ldots, x_{n-1}, z||} \\ &\leq \frac{||x_1, \ldots, x_{n-1}, z||}{t + ||x_1, \ldots, x_{n-1}, z||} \\ &\leq \max\{N(x_1, \ldots, x_{n-1}, y, s), \\ &\quad N(x_1, \ldots, x_{n-1}, z, t)\}, \end{aligned}$$

These cases imply that

$$\begin{aligned} &N(x_1, \ldots, x_{n-1}, y+z, \max\{s,t\}) \\ &\quad \leq \max\{N(x_1, \ldots, x_{n-1}, y, s), N(x_1, \ldots, x_{n-1}, z, t)\}, \end{aligned}$$

(aN6) Let $s < t \leq 0$ and let $s \leq 0 < t$. These cases imply that $N(x_1, \ldots, x_n, s) = 1 \geq N(x_1, \ldots, x_n, t)$. If $0 < s < t$, then

$$N(x_1, \ldots, x_n, s) - N(x_1, \ldots, x_n, t) = \frac{(s-t)||x_1, \ldots, x_n||}{(s + ||x_1, \ldots, x_n||)(t + ||x_1, \ldots, x_n||)} \leq 0,$$

Hence $N(x_1, \ldots, x_n, *)$ is a non-increasing function of $\mathbb{R}$. Also,

$$\lim_{t\to\infty} N(x_1, \ldots, x_n, t) = \lim_{t\to\infty} \frac{||x_1, \ldots, x_n||}{t + ||x_1, \ldots, x_n||} = 0,$$

for all $x_1, \ldots, x_n \in X$.

Definition 1.4. A non-Archimedean fuzzy anti-n-normed space is called *strictly convex* if for every $x_2, \ldots, x_n, y, z \in X$ and $s,\ t \in \mathbb{R}$,

$$N(y+z, x_2, x_3, \ldots, x_n,\ \max\{s,\ t\}) \\ = \max\{N(y, x_2, \ldots, x_n, s), N(z, x_2, \ldots, x_n, t)\}$$

and for any $z_1, \ldots, z_n \in X$,

$$N(x_1, x_2, \ldots, x_n, s) = N(z_1, z_2, \ldots, z_n, t)$$

imply $x_1 = z_1, x_2 = z_2, \ldots,\ x_n = z_n$ and $s = t$.

Definition 1.5. Let (X, N) and (Y, N) be two non-Archimedean fuzzy anti-n-normed spaces. We call $f : (X, N) \longrightarrow (Y, N)$ a *fuzzy n-isometry* if

$$N(x_1 - z, x_2 - z, \ldots, x_n - z, t) = N(f(x_1) - f(z), \ldots, f(x_n) - f(z), t),$$

for all $x_1, \ldots, x_n, z \in X$.

Definition 1.6. Let X be a non-Archimedean fuzzy anti-n-normed space and a, b, c mutually disjoint elements of X. Then a, b and c are said to be *2-collinear* if $b - c = r(a - c)$ for some $r \in \mathcal{K}$.

In this paper, we will investigate that Mazur-Ulam Theorem holds under some conditions in non-Archimedean fuzzy anti-n-normed space over a non-Archimedean field $\mathcal{K}$ with a non-Archimedean valuation $|\cdot|$.

We denote the set of all elements of $\mathcal{K}$ whose norms are 1 by $\mathcal{C}$, that is,

$$\mathcal{C} = \{r \in \mathcal{K} \mid |r| = 1\}.$$

2. Main Results

Lemma 2.1. Let (X, N) is a non-Archimedean fuzzy anti-n-normed space over a non-Archimedean field $\mathcal{K}$. Then

$$N(x_1, \ldots, x_n, t) = N(x_1, x_2 + rx_1, x_3, \ldots, x_n, t),\ \text{for all}\ r \in \mathcal{K}.$$

Proof. Let $x_1, \ldots, x_n \in X$ and let $r \in \mathcal{K}$. Without a loss generality, we may assume $t > 0$. Then

$$\begin{aligned} N(x_1, x_2 + rx_1, x_3, \ldots, x_n, t) &\leq \max\{N(x_1, \ldots, x_n, t), N(x_1, rx_1, x_3, \ldots, x_n, t)\} \\ &= N(x_1, \ldots, x_n, t), \end{aligned}$$

Conversely,

$$\begin{aligned} N(x_1, \ldots, x_n, t) &= N(x_1, x_2 + rx_1 - rx_1, x_3, \ldots, x_n, t) \\ &\leq \max\{N(x_1, x_2 + rx_1, x_3, \ldots, x_n, t), N(x_1, rx_1, x_3, \ldots, x_n, t)\} \\ &= N(x_1, x_2 + rx_1, x_3, \ldots, x_n, t), \end{aligned}$$

Thus $N(x_1, \dots, x_n, t) = N(x_1, x_2 + rx_1, x_3, \dots, x_n, t)$, as desired. ■

Lemma 2.2. Let (X, N) is a non-Archimedean fuzzy anti-n-normed space over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{2^n \mid n \in \mathbb{Z}\}$. Let $x, y, z \in X$ and $t > 0$, Suppose X is strictly convex. Then $\alpha = \dfrac{x+y}{2}$ is the unique element of X such that

$$\begin{aligned} N(x-z, x-\alpha, x_3-z, \dots, x_n-z, t) &= N(y-\alpha, y-z, x_3-z, \dots, x_n-z, t) \\ &= N(x-z, y-z, x_3-z, \dots, x_n-z, t) \end{aligned}$$

where $x_3, \dots, x_n \in X$ and α, x, y are 2-collinear.

Proof. Let $x,\ y,\ z,\ x_3, \dots, x_n \in X$ and $t > 0$. By Lemma 2.1, we have

$$\begin{aligned} &N(x-z, x-\alpha, x_3-z, \dots, x_n-z, t) = N(x-z, x-\frac{x+y}{2}, x_3-z, \dots, x_n-z, t) \\ &= N(y-z, x-y, x_3-z, \dots, x_n-z, t) = N(y-z, \frac{y-x}{2}, x_3-z, \dots, x_n-z, t) \\ &= N(y-z, y-\alpha, x_3-z, \dots, x_n-z, t) = N(y-\alpha, y-z, x_3-z, \dots, x_n-z, t), \end{aligned}$$

Also, we get

$$N(x-z, x-y, x_3-z, \dots, x_n-z, t) = N(x-z, y-z, x_3-z, \dots, x_n-z, t),$$

Hence we may say that

$$\begin{aligned} N(x-z, x-\alpha, x_3-z, \dots, x_n-z, t) &= N(y-\alpha, y-z, x_3-z, \dots, x_n-z, t) \\ &= N(x-z, y-z, x_3-z, \dots, x_n-z, t), \end{aligned}$$

This shows the existence part. To show the uniqueness part, suppose that β is such an element of X such that

$$\begin{aligned} N(x-z, x-\beta, x_3-z, \dots, x_n-z, t) &= N(y-\beta, y-z, x_3-z, \dots, x_n-z, t) \\ &= N(x-z, y-z, x_3-z, \dots, x_n-z, t) \end{aligned}$$

where $x_3, \dots, x_n \in X$ and β, x, y are 2-collinear. Since β, x, y are 2-collinear, there exists a $s \in \mathcal{K}$ such that

$$\beta = sx + (1-s)y,$$

We may assume $s \neq 0$ and $s \neq 1$,

$$\begin{aligned} &N(x-z, y-z, x_3-z, \dots, x_n-z, t) = N(x-z, x-\beta, x_3-z, \dots, x_n-z, t) \\ &= N(x-z, x-(sx+(1-s)y), x_3-z, \dots, x_n-z, t) \\ &= N\left(x-z, y-z, x_3-z, \dots, x_n-z, \frac{t}{|1-s|}\right), \end{aligned}$$

Similarly, we have

$$\begin{aligned} N(x-z, y-z, x_3-z, \ldots, x_n-z, t) &= N(y-\beta, y-z, x_3-z, \ldots, x_n-z, t) \\ &= N\left(x-z, y-z, x_3-z, \ldots, x_n-z, \frac{t}{|s|}\right), \end{aligned}$$

Hence we have

$$\begin{aligned} N(x-z, y-z, x_3-z, \ldots, x_n-z, t) &= N\left(x-z, y-z, x_3-z, \ldots, x_n-z, \frac{t}{|s|}\right) \\ &= N\left(x-z, y-z, x_3-z, \ldots, x_n-z, \frac{t}{|1-s|}\right), \end{aligned}$$

By the strict convexity of X, we have

$$1 = \frac{1}{|1-s|} = \frac{1}{|s|},$$

Then there exist elements t_1 and t_2 in $\mathbb{Z}$ such that $1-s = 2^{t_1}$ and $s = 2^{t_2}$. Since $2^{t_1} + 2^{t_2} = 1$, we know that $t_1, t_2 < 0$. Without a loss of generality, we let $1-s = 2^{-n_1}$ and $s = 2^{-n_2}$ with $n_1 \geq n_2$. If $n_1 \gneqq n_2$ then

$$1 = 2^{-n_1} + 2^{-n_2} = 2^{-n_1}(1 + 2^{n_1-n_2}),$$

Hence $2^{n_1} = 1 + 2^{n_1-n_2}$. This is a contradiction. Thus $n_1 = n_2$, that is, $s = \frac{1}{2}$. This implies that $\beta = \frac{a+b}{2} = \alpha$. Therefore the proof is completed. ■

Theorem 2.3. Let X and Y are non-Archimedean fuzzy anti-n-normed spaces and strict convexities. Suppose $f : (X, N) \longrightarrow (Y, N)$ is a fuzzy n-isometry.

(1) For every a, b and $c \in X$, if a, b and c are 2-collinear, then $f(a)$, $f(b)$ and $f(c)$ are 2-collinear

(2) If $f(0) = 0$, then

$$N(x_1, \ldots, x_n, t) = N(f(x_1), \ldots, f(x_n), t),$$

for $x_1, \ldots, x_n \in X$ and $t > 0$.

Proof.

(1) Since a, b and c are 2-collinear, then there exists a $s \in \mathcal{K}$ ($s \neq 0$) such that $a - c = s(b-c)$. For each $x_2, \ldots, x_n \in X$ and $t > 0$, we have

$$\begin{aligned} &N(f(a)-f(c), f(x_2)-f(c), \ldots, f(x_n)-f(c), t) \\ &\quad = N(a-c, x_2-c, \ldots, x_n-c, t) \\ &\quad = N(s(b-c), x_2-c, \ldots, x_n-c, t) = N\left(b-c, x_2-c, \ldots, x_n-c, \frac{t}{|s|}\right) \\ &\quad = N(s(f(b)-f(c)), f(x_2)-f(c), \ldots, f(x_n)-f(c), t), \end{aligned}$$

By the strict convexity of X, we get $f(a) - f(c) = s(f(b) - f(c))$, that is, $f(a)$, $f(b)$ and $f(c)$ are 2-collinear.

(2) For $x_1, \ldots, x_n \in X$ and $t > 0$,

$$\begin{aligned} &N(x_1, \ldots, x_n, t) = N(x_1 - 0, \ldots, x_n - 0, t) \\ &= N(f(x_1) - f(0), \ldots, f(x_n) - f(0), t) = N(f(x_1), \ldots, f(x_n), t). \end{aligned}$$

■

Theorem 2.4. Let X and Y are non-Archimedean fuzzy n-normed spaces and strict convexities. Suppose $f : X \longrightarrow Y$ is a fuzzy n-isometry. Then f is additive.

Proof. Let $g(x) = f(x) - f(0)$. Since f is a fuzzy n-isometry, so is g. Also, we have $g(0) = 0$, Since $g : X \longrightarrow Y$ is a fuzzy n-isometry, Lemma 2.1 implies that

$$\begin{aligned} &N\left(g(a) - g(c),\ g(a) - g\left(\frac{a+b}{2}\right), g(x_3) - g(c), \ldots, g(x_n) - g(c), t\right) \\ &N\left(g(a) - g(c),\ g(a) - g\left(\frac{a+b}{2}\right), g(x_3) - g(a), \ldots, g(x_n) - g(a), t\right) \\ &= N\left(g(c) - g(a),\ g\left(\frac{a+b}{2}\right) - g(a), g(x_3) - g(a), \ldots, g(x_n) - g(a), t\right) \\ &= N\left(c - a,\ \frac{a+b}{2} - a, x_3 - a, \ldots, x_n - a, t\right) \\ &= N(c - a,\ b - a, x_3 - a, \ldots, x_n - a, t) \\ &= N(g(c) - g(a),\ g(b) - g(a), g(x_3) - g(a), \ldots, g(x_n) - g(a), t) \\ &= N(g(a) - g(c),\ g(b) - g(c), g(x_3) - g(c), \ldots, g(x_n) - g(c), t) \end{aligned}$$

for $x_3, \ldots, x_n \in X$, Similarly, we get

$$\begin{aligned} &N\left(g(b) - g\left(\frac{a+b}{2}\right),\ g(b) - g(c), g(x_3) - g(c), \ldots, g(x_n) - g(c), t\right) \\ &= N(g(a) - g(c),\ g(b) - g(c), g(x_3) - g(c), \ldots, g(x_n) - g(c), t), \end{aligned}$$

Hence

$$\begin{aligned} &N\left(g(a) - g(c),\ g(a) - g\left(\frac{a+b}{2}\right), g(x_3) - g(c), \ldots, g(x_n) - g(c), t\right) \\ &= N\left(g(a) - g(c),\ g(b) - g(c), g(x_3) - g(c), \ldots, g(x_n) - g(c), t\right) \\ &= N\left(g(b) - g\left(\frac{a+b}{2}\right),\ g(b) - g(c), g(x_3) - g(c), \ldots, g(x_n) - g(c), t\right), \end{aligned}$$

By the uniqueness of Lemma 2.2, we have $g\left(\frac{a+b}{2}\right) = \frac{g(a) + g(b)}{2}$, for all $a,\ b \in X$, Thus $f(x) - f(0)$ is additive. ■

In the following, we will investigate that the interior preserving mapping carries the barycenter of a triangle to the barycenter point of the corresponding triangle. By using this result, we will prove a Mazur-Ulam Theorem on non-Archimedean fuzzy anti-n-normed space X over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{3^n \mid n \in \mathbb{Z}\}$.

Lemma 2.5. Let (X, N) be a non-Archimedean fuzzy anti-n-normed space over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{3^n \mid n \in \mathbb{Z}\}$ and let $a, b, c \in X$ and $t > 0$, Suppose X is strictly convex. Then $\alpha = \dfrac{a+b+c}{3}$ is the unique element of X such that

$$\begin{aligned} &N(a-\alpha, b-\alpha, x_3, \ldots, x_n, t) = N(b-\alpha, c-\alpha, x_3, \ldots, x_n, t) \\ &= N(a-\alpha, c-\alpha, x_3, \ldots, x_n, t) = N(a-b, a-c, x_3, \ldots, x_n, t) \end{aligned}$$

where $x_3, \ldots, x_n \in X$ and

$$\alpha \in \{t_1 a + t_2 b + t_3 c \mid t_1 + t_2 + t_3 = 1, t_i \in \mathcal{K}, t_i > 0, i = 1, 2, 3\}.$$

Proof. Let $\alpha = \dfrac{a+b+c}{3} \in X$ and $t > 0$. By Lemma 2.1, we have

$$\begin{aligned} N(a-\alpha, b-\alpha, x_3, \ldots, x_n, t) &= N(a-\alpha, b-a, x_3, \ldots, x_n, t) \\ &= N(2a-b-c, b-a, x_3, \ldots, x_n, |3|t) \\ &= N(a-c, b-a, x_3, \ldots, x_n, t) \\ &= N(a-c, b-c, x_3, \ldots, x_n, t) \\ &= N(a-b, a-c, x_3, \ldots, x_n, t). \end{aligned}$$

Similarly, we get

$$\begin{aligned} N(b-\alpha, c-\alpha, x_3, \ldots, x_n, t) &= N(a-b, a-c, x_3, \ldots, x_n, t) \\ N(a-\alpha, c-\alpha, x_3, \ldots, x_n, t) &= N(a-b, a-c, x_3, \ldots, x_n, t). \end{aligned}$$

Hence we have

$$\begin{aligned} &N(a-\alpha, b-\alpha, x_3, \ldots, x_n, t) = N(b-\alpha, c-\alpha, x_3, \ldots, x_n, t) \\ &= N(a-\alpha, c-\alpha, x_3, \ldots, x_n, t) = N(a-b, a-c, x_3, \ldots, x_n, t), \end{aligned}$$

that is, the existence part holds. To show the uniqueness part, assume that β is such an element of X such that

$$\begin{aligned} &N(a-\beta, b-\beta, x_3, \ldots, x_n, t) = N(b-\beta, c-\beta, x_3, \ldots, x_n, t) \\ &= N(a-\beta, c-\beta, x_3, \ldots, x_n, t) = N(a-b, a-c, x_3, \ldots, x_n, t) \end{aligned}$$

where $\beta \in \{t_1 a + t_2 b + t_3 c \mid t_1 + t_2 + t_3 = 1, t_i \in \mathcal{K}, t_i > 0, i = 1, 2, 3\}$. Hence we may let $\beta = s_1 a + s_2 b + s_3 c$ where $s_1 + s_2 + s_3 = 1$. Then we have

$$\begin{aligned}
& N(a-b, a-c, x_3, \ldots, x_n, t) \\
= \; & N(a-\beta, b-\beta, x_3, \ldots, x_n, t) \\
= \; & N((1-s_1)a - s_2 b - (1-s_1-s_2)c,\ b-a, x_3, \ldots, x_n, t) \\
= \; & N((1-s_1-s_2)a - (1-s_1-s_2)c,\ b-a, x_3, \ldots, x_n,\ t) \\
= \; & N(a-c,\ b-a, x_3, \ldots, x_n,\ \frac{t}{|1-s_1-s_2|}) \\
= \; & N(a-b,\ a-c, x_3, \ldots, x_n,\ \frac{t}{|1-s_1-s_2|}),
\end{aligned}$$

Also, we have

$$\begin{aligned}
& N(a-b, a-c, x_3, \ldots, x_n, t) \\
= \; & N(a-\beta, c-\beta, x_3, \ldots, x_n, t) \\
= \; & N((1-s_1)a - s_2 b - (1-s_1-s_2)c,\ c-a, x_3, \ldots, x_n,\ t) \\
= \; & N(s_2 a - s_2 b,\ c-a, x_3, \ldots, x_n,\ t) \\
= \; & N\left(a-b,\ a-c, x_3, \ldots, x_n,\ \frac{t}{|s_2|}\right),
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
N(a-b, a-c, x_3, \ldots, x_n, t) &= N(b-\beta,\ c-\beta, x_3, \ldots, x_n,\ t) \\
&= N\left(a-b,\ a-c, x_3, \ldots, x_n,\ \frac{t}{|s_1|}\right),
\end{aligned}$$

that is,

$$\begin{aligned}
N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|1-s_1-s_2|}\right) &= N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|s_2|}\right) \\
&= N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|s_1|}\right),
\end{aligned}$$

We note that

$$\begin{aligned}
& N\left(a-b+a-b+a-b, a-c, x_3, \ldots, x_n, \max\left\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\right\}\right) \\
& \leq \max\left\{N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|s_1|}\right), N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|s_2|}\right),\right. \\
& \quad \left. N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|1-s_1-s_2|}\right)\right\} \\
& = N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|s_1|}\right) = N\left(a-b, a-c, x_3, \ldots, x_n \frac{t}{|s_2|}\right) \\
= \; & N\left(a-b, a-c, x_3, \ldots, x_n, \frac{t}{|1-s_1-s_2|}\right),
\end{aligned}$$

and

$$
\begin{aligned}
&N\left(a-b+a-b+a-b, a-c, x_3, \ldots, x_n, \max\left\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\right\}\right) \\
&= N\left(3(a-b), a-c, x_3, \ldots, x_n, \max\left\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\right\}\right) \\
&= N\left(a-b, a-c, x_3, \ldots, x_n, \max\left\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\right\}\right),
\end{aligned}
$$

The strict convexity of X implies that $|s_1| = |s_2| = |1 - s_1 - s_2| = 1$. Then there exist elements k_1, k_2 and k_3 in $\mathbb{Z}$ such that $s_1 = 3^{k_1}, s_2 = 3^{k_2}$ and $1 - s_1 - s_2 = 3^{k_3}$. Since $3^{k_1} + 3^{k_2} + 3^{k_3} = 1$, we know that $k_1, k_2, k_3 < 0$. Without a loss of generality, we let $s_1 = 3^{-n_1}, s_2 = 3^{-n_2}$ and $1 - s_1 - s_2 = 3^{-n_3}$ with $n_1 \geq n_2 \geq n_3$. Then

$$
1 = 3^{-n_1} + 3^{-n_2} + 3^{-n_3} = 3^{-n_1}(1 + 3^{n_1-n_2} + 3^{n_1-n_3}),
$$

Hence $3^{n_1} = 1 + 3^{n_1-n_2} + 3^{n_1-n_3}$. This is a contradiction. Thus $s_1 = s_2 = s_3 = \frac{1}{3}$. This implies that $\beta = \frac{a+b+c}{3} = \alpha$. Therefore the proof is completed. ∎

Theorem 2.6. Let X and Y be non-Archimedean fuzzy anti-n-normed spaces over a linear ordered non-Archimedean field $\mathcal{K}$ with $\mathcal{C} = \{3^n \mid n \in \mathbb{Z}\}$. Let X and Y be strict convexities. If $f : X \longrightarrow Y$ is an interior preserving fuzzy n-isometry, then $f(x) - f(0)$ is additive.

Proof. Let $g(x) = f(x) - f(0)$. Since f is a fuzzy n-isometry, so is g. For a, b and $c \in X$, let $\triangle abc$ be a triangle determined by the points a, b and c, and let x be an interior point of $\triangle abc$. Since f is an interior preserving mapping, we may write

$$
f(x) = s_1 f(a) + s_2 f(b) + s_3 f(c),
$$

where $s_i \in \mathcal{K}, s_i > 0\,(i = 1, 2, 3)$ with $s_1 + s_2 + s_3 = 1$, Then we have

$$
\begin{aligned}
g(x) &= s_1 f(a) + s_2 f(b) + s_3 f(c) - f(0) \\
&= s_1(f(a) - f(0)) + s_2(f(b) - f(0)) + s_3(f(c) - f(0)) \\
&= s_1 g(a) + s_2 g(b) + s_3 g(c),
\end{aligned}
$$

Hence $g(x)$ is an interior point of $\triangle g(a)g(b)g(c)$, that is, g is also an interior preserving mapping.

Since $g : X \longrightarrow Y$ is a fuzzy 2-isometry, we have

$$\begin{aligned}
&N\left(g(a) - g\left(\frac{a+b+c}{3}\right),\ g(b)\right.\\
&\quad\left.-g\left(\frac{a+b+c}{3}\right), g(x_3) - g(c), \ldots, g(x_n) - g(c),\ t\right)\\
&= N\left(a - \frac{a+b+c}{3},\ b - \frac{a+b+c}{3}, x_3 - c, \ldots, x_n - c,\ t\right)\\
&= N\left(a - b,\ b - \frac{a+b+c}{3}, x_3 - c, \ldots, x_n - c,\ t\right)\\
&= N(a - b,\ 2b - a - c, x_3 - c, \ldots, x_n - c,\ |3|t)\\
&= N(a - b,\ a - c, x_3 - c, \ldots, x_n - c,\ t)\\
&= N(g(a) - g(b),\ g(a) - g(c), g(x_3) - g(c), \ldots, g(x_n) - g(c),\ t),
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&N\left(g(b) - g\left(\frac{a+b+c}{3}\right),\ g(c) - g\left(\frac{a+b+c}{3}\right),\right.\\
&\quad\left. g(x_3) - g(c), \ldots, g(x_n) - g(c),\ t\right)\\
&= N(g(a) - g(b),\ g(a) - g(c), g(x_3) - g(c), \ldots, g(x_n) - g(c),\ t)\\
&= N\left(g(a) - g\left(\frac{a+b+c}{3}\right),\ g(c)\right.\\
&\quad\left.-g\left(\frac{a+b+c}{3}\right), g(x_3) - g(c), \ldots, g(x_n) - g(c),\ t\right),
\end{aligned}$$

Since $\frac{a+b+c}{3}$ is an interior point of the triangle $\triangle abc$ and g is an interior preserving mapping, $g\left(\frac{a+b+c}{3}\right)$ is an interior point of the triangle $\triangle g(a)g(b)g(c)$. By the uniqueness of Lemma 2.5, we have

$$g\left(\frac{a+b+c}{3}\right) = \frac{g(a) + g(b) + g(c)}{3}$$

for all $a,\ b,\ c \in X$. Thus $f(x) - f(0)$ is additive, as desired. ■

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