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On surfaces in $\mathbb{R}^3$

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Abstract

Let f be a C^∞ real valued function on $\mathbb{R}^3$, and c a real number. Let M be the subset of $\mathbb{R}^3$ such that $M = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = c\}$. On some teaching materials pertinent to Differential Geometry, a necessary and sufficient condition for M to be a surface in $\mathbb{R}^3$ is written as follows: M is a surface in $\mathbb{R}^3$ if and only if the differential df is not zero for each point of M. In this note, the authors find the following: the condition $df \neq 0$ for each point of M is a sufficient condition for M to be a surface in $\mathbb{R}^3$, but the condition $df \neq 0$ for each point of M is not a necessary condition for M to be a surface. And then, at an arbitrarily given point p which belongs to a surface M embedded isometrically in $\mathbb{R}^3$, we precisely make an explication the principal curvatures of M at p, considering M as a hypersurface of $\mathbb{R}^3$.

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1. Introduction

Let $\mathbb{E}^3$ be three dimensional Euclidean space $(\mathbb{R}^3, <\ ,\ >)$ with the usual Riemannian metric $<\ ,\ >$ on $\mathbb{R}^3$, f C^∞ real valued function, and c a real number. Let M $f = c$ the set of all points q such that $f(q) = c$. Then, on some teaching materials [3, 4, 7] pertinent to Differential Geometry, is written as follows: *M is a surface in $\mathbb{R}^3$ if and only if the differential df is not zero for each point of M.*

In this paper, we find the fact that *the condition $df \neq 0$ for every point of M is a sufficient condition for M to be a surface in $\mathbb{R}^3$, but the condition is not a necessary condition for M to be a surface* (cf. Proposition 2.1, Theorem 2.2).

Moreover, at an arbitrarily given point p which belongs to a surface M embedded isometrically in $\mathbb{E}^3(= (\mathbb{R}^3, <\ ,\ >)$, we precisely make an explication the principal curvatures of M at p, considering M as a hypersurface of $\mathbb{R}^3$ (cf. Propositions 3.1 and 3.4).

2. A surface in $\mathbb{R}^3$

Let f be a C^∞ real valued function defined on $\mathbb{R}^3$, and c a real number. Let M be the subset of $\mathbb{R}^3$ such that

$$M = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = c\}.$$

Now, assume the differential df of the function f is not zero at every point of M, and p is an arbitrarily given point of M. Then, from the hypothesis on df is equivalent to assuming that at least one of three partial derivatives is not zero at the point $p = (p_1, p_2, p_3) \in M$, say $(\partial f/\partial z)(p) \neq 0$. In this case, by the help of inverse function theorem we obtain the following:

there exist open neighborhoods $D(\subset \mathbb{R}^2)$ of (p_1, p_2), $I(\subset \mathbb{R})$ of p_3 and $J(\subset \mathbb{R})$ of the constant c such that

$$\Phi : D \times I \ni ((q_1, q_2), q_3) \longmapsto ((q_1, q_2), f(q_1, q_2, q_3)) \in D \times J$$

is a C^∞ diffeomorphism.

Putting $\Phi^{-1} =: \Psi = (\psi_1, \psi_2, \psi_3)$, $\psi_3(q_1, q_2, c) =: h(q_1, q_2)$ for each $(q_1, q_2) \in D$, we get

$$\begin{aligned}(q_1, q_2, c) &= (\Phi \circ \Psi)(q_1, q_2, c) = \Phi(q_1, q_2, h(q_1, q_2)) \\ &= (q_1, q_2, f(q_1, q_2, h(q_1, q_2)))\end{aligned}$$

for each $(q_1, q_2) \in D$. So, around the point $p(= (p_1, p_2, p_3))$ of M,

$$\{(q_1, q_2, h(q_1, q_2)) | (q_1, q_2) \in D\}$$

is a simple surface (2-dimensional C^∞ manifold embedded isometrically) in $\mathbb{E}^3(= (\mathbb{R}^3, <\ ,\ >))$. Since p is an arbitrarily given point of M, we can easily find the fact that M is a surface (2-dimensional C^∞ manifold embedded isometrically) in $\mathbb{E}^3$.

Proposition 2.1. Let f be a C^∞ real valued function defined on $\mathbb{R}^3$, and c a real number. Let M be the subset of $\mathbb{R}^3$ such that $M = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = c\}$. Then, if the differential df of f is not zero for each point of M, M is a surface (2-dimensional C^∞ manifold embedded isometrically) in $\mathbb{E}^3$.

On the other hand, the converse of Proposition 2.1 is not true. In fact, putting $f(x, y, z) := x^2 + y^2 - z^2$ and $M = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = c\}$, then $p = (0, 0, 0) \in M$ and $(df)_p = (2xdx + 2ydy - 2zdz)_p = 0$. But, M is a surface (2-dimensional C^∞ manifold embedded) in $\mathbb{R}^3$.

Thus, by virtue of Proposition 2.1 and the above fact, we obtain

Theorem 2.2. Let f be a C^∞ real valued function defined on $\mathbb{R}^3$, and c a real number. Let M be the subset of $\mathbb{R}^3$ such that $M = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = c\}$. Then, the condition $df \neq 0$ for every point of M is a sufficient condition for M to be a surface in $\mathbb{R}^3$, but the condition $df \neq 0$ for every point of M is not a necessary condition for M to be a surface in $\mathbb{R}^3$.

Remark 2.3. On [3, Theorem 1.4, p.127; 4, Theorem 4.1, p.127], a necessary and sufficient condition for M to be a surface (two dimensional C^∞ manifold embedded isometrically) in $\mathbb{R}^3$ is written as follows:

M is a surface in $\mathbb{R}^3$ if and only if the differential df is not zero for each point of M. But, by the help of Proposition 2.1 and Theorem 2.2, we get the fact that *the condition $df \neq 0$ for every point of M is a sufficient condition for M to be a surface in $\mathbb{R}^3$, but the condition is not a necessary condition for M to be a surface.*

3. Principal curvatures of a surface M at a point p belonging to M

In this section, we make a minute and detailed explication concerning principal directions of a surface M (in the 3-dimensional Euclidean space) at a point p which belongs to M.

Let $\mathbb{E}^3$ be the three dimensional Euclidean space $(\mathbb{R}^3, <\ ,\ >)$, M a surface (2-dimensional C^∞ manifold embedded isometrically) in $\mathbb{E}^3$, and ι the inclusion map of M into $\mathbb{R}^3$. And, let $g(:= \iota^\star(<\ ,\ >))$ be the Riemannian metric on M which is induced by ι and $<\ ,\ >$), and D the Levi-Civita connection on $\mathbb{E}^3$.

Let $E := \iota^{-1}T\mathbb{R}^3$ be the induced bundle over M of $T\mathbb{R}^3$ (the tangent bundle over $\mathbb{R}^3$) by the map ι, that is,

$$E := \{(x, u) | x \in M,\ u \in T_{\iota(x)}M (= T_x M)\}.$$

Then, the $C^\infty(M)$-module of all smooth sections of $E (= \iota^{-1}T\mathbb{R}^3)$, denoted by $\Gamma(M; E)$, is as follows:

$$\begin{aligned}\Gamma(M; E) = \{V \mid V : M \to T\mathbb{R}^3 \\ \text{is a } C^\infty - \text{mapping such that } V(x) \in T_{\iota(x)}\mathbb{R}^3 (= T_x\mathbb{R}^3) \quad (x \in M)\}.\end{aligned}$$

The *induced connection* $\tilde{D}$ on the induced bundle $E(=\iota^{-1}T\mathbb{R}^3)$ is defined as follows ([8, p. 126]):

For $X \in \mathfrak{X}(M)$ and $V \in \Gamma(M; E)$, we define $\tilde{D}_X V \in \Gamma(M; E)$ by

$$(3.1) \qquad (\tilde{D}_X V)(x) = (D_{\iota_\star X} V)(x) = \frac{d}{dt}\Big|_{t=0} {}^{D}P_{(\iota\circ\sigma)(t)}{}^{-1} V(\sigma(t)) \quad (x \in M),$$

where $t \mapsto \sigma(t) \in M$ is a C^∞-curve in M satisfying $\sigma(0) = x$, $\sigma'(0) = X_x \in T_x(M)$, and

$${}^{D}P_{(\iota\circ\sigma)(t)} : T_{\iota(x)}\mathbb{R}^3 \longrightarrow T_{(\iota\circ\sigma)(t)}\mathbb{R}^3$$

is the parallel transport along the curve $(\iota\circ\sigma)(t)$ with respect to the canonical Levi-Civita connection D on $(\mathbb{R}^3, <\ ,\ >)$.

Then, the formulas of Gauss and Weingarten [1, 5, 6] are

$$(3.2) \qquad \begin{aligned} &\tilde{D}_X \iota_\star Y = \iota_\star(\nabla_X Y) + h(X, Y)U \quad (X, Y \in \mathfrak{X}(M)), \\ &\tilde{D}_X U = \iota_\star(-S(X)) \quad (X \in \mathfrak{X}(M)), \end{aligned}$$

where h is the second fundamental form of M for the unit normal vector field U, and S is the shape operator of M (derived from U). And, the following lemma about the induced connection $\tilde{D}$ is well known ([8, Lemma 1.16, p.129], [6, Theorem 7.5, p.154]).

Lemma 3.1. For $X, Y \in \mathfrak{X}(M)$,

$$\tilde{D}_X \iota_\star Y - \tilde{D}_Y \iota_\star X - \iota_\star([X, Y]) = T^D(\iota_\star X, \iota_\star Y) = 0 \qquad (X, Y \in \mathfrak{X}(M)),$$

where T^D is the torsion tensor field on $(\mathbb{E}^3, D)$.

Since D is the Riemannian connection on $\mathbb{E}^3(= (\mathbb{R}^3, <\ ,\ >))$, by the help of (3.2) and Lemma 3.1 we see that ∇ is the Riemannian connection on $(M, g(:= \iota^\star <\ ,\ >))$ and for $X, Y \in \mathfrak{X}(M)$

$$(3.3) \qquad h(X, Y) = h(Y, X), \quad g(S(X), Y) = h(X, Y).$$

Let M be a surface (2-dimensional C^∞-manifold) which is isometrically embedded in $\mathbb{E}^3(= (\mathbb{R}^3, <\ ,\ >))$, p an arbitrarily given point of M, and (x^1, x^2) the standard coordinates of $\mathbb{R}^2$. For a local coordinate neighborhood (V, ϕ) around the point p, where $\phi : V \to \phi(V)(\subset \mathbb{R}^2)$, we define local coordinates $x^i (i = 1, 2)$ by

$$x^i := x^i \circ \phi : V \longrightarrow \mathbb{R} \quad (i = 1, 2).$$

Then each point of V can be uniquely expressed by the coordinate system (x^1, x^2). Putting

$$\phi^{-1} =: \mathbf{x} \qquad \text{and} \qquad \phi(V) =: D,$$

we get a C^∞-mapping $\mathbf{x}$ of D into $\mathbb{R}^3$ such that

$$\mathbf{x} : D \ni (u, v) \longrightarrow \mathbf{x}(u, v) \in V(\subset M \subset \mathbb{R}^3).$$

Here, we may also regard $\mathbf{x}(u, v),\ (u, v) \in D$, as $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ which belong to $\mathbb{R}^3$.

Let $\mathbf{u}$ be a unit vector tangent to M at the point p. Then, the number

$$\kappa_N(\mathbf{u}) := g(S(\mathbf{u}), \mathbf{u}) \tag{3.4}$$

is called the *normal curvature* of M in the $\mathbf{u}$ direction, where S is the shape operator of M (derived from the normal vector field U on M). The maximum and minimum values of the normal curvature $\kappa_N(\mathbf{u})$ of M at p are called the *principal curvatures* of M at p. Unit vectors in these directions are called *principal vectors* of M at the point p.

Now, we put

$$\iota_\star\left(\frac{\partial}{\partial u}\right) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) =: \mathbf{x}_u, \quad \iota_\star\left(\frac{\partial}{\partial v}\right) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) =: \mathbf{x}_v,$$

and consider the quantity

(3.5)

$$\begin{aligned}\mathbf{I} &=< d(\iota \circ \mathbf{x}), d(\iota \circ \mathbf{x}) > (= < d\mathbf{x}, d\mathbf{x} >) \\ &=< \mathbf{x}_u, \mathbf{x}_u > du \otimes du + < \mathbf{x}_u, \mathbf{x}_v > (du \otimes dv + dv \otimes du) + < \mathbf{x}_v, \mathbf{x}_v > dv \otimes dv.\end{aligned}$$

Since $\iota_\star\left(\frac{\partial}{\partial u}\right) = \mathbf{x}_u,\ \iota_\star\left(\frac{\partial}{\partial v}\right) = \mathbf{x}_v$ and $\iota^\star < , >= g$, we get from (3.5)

(3.6)

$$\begin{aligned}\mathbf{I} &= g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) du \otimes du + g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)(du \otimes dv + dv \otimes du) + g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) dv \otimes dv \\ &= \iota^\star(< , >) = g.\end{aligned}$$

Moreover, since the symmetric product $\alpha\beta$ of two 1-forms α and β is given by

$$\alpha\beta := 2^{-1}(\alpha \otimes \beta + \beta \otimes \alpha). \tag{3.7}$$

By the help of (3.6)and (3.7), we get

$$\begin{aligned}\mathbf{I} &= g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) du^2 + 2g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) dudv + g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) dv^2 \\ &= \iota^\star(< , >) = g.\end{aligned} \tag{3.8}$$

The tensor field $\mathbf{I}$ $(= \iota^\star(< , >) = g)$ of type (0,2) on the surface M is said to be the *first fundamental form* of $\mathbf{x} = \mathbf{x}(u, v)$.

And then, we consider the quantity

$$\mathbf{II} = - < d(\iota \circ \mathbf{x}), dU >= - < \mathbf{x}_u du + \mathbf{x}_v dv, U_u du + U_v dv > . \tag{3.9}$$

Since $<\mathbf{x}_u, U>=<\mathbf{x}_v, U>=0$ and $\mathbf{x}_{uv}=\mathbf{x}_{vu}$, we get

$$\tag{3.10} \begin{aligned} &<\mathbf{x}_u, U_u>=-<\mathbf{x}_{uu}, U>, \quad <\mathbf{x}_v, U_v>=-<\mathbf{x}_{vv}, U> \\ &<\mathbf{x}_u, U_v>=<\mathbf{x}_v, U_u>=-<\mathbf{x}_{uv}, U>. \end{aligned}$$

From (3.9) and (3.10), we have

(3.11)

$$\mathbf{II}=<\mathbf{x}_{uu}, U> du\otimes du+<\mathbf{x}_{uv}, U>(du\otimes dv+dv\otimes du)+<\mathbf{x}_{vv}, U> dv\otimes dv.$$

By virtue of (3.2), (3.3) and $\iota^\star(< \ , \ >=g$, we get

$$\tag{3.12} \begin{aligned} &<\mathbf{x}_u, U_u>=-g\left(\frac{\partial}{\partial u}, S\left(\frac{\partial}{\partial u}\right)\right)=-h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right), \\ &<\mathbf{x}_u, U_v>=<\mathbf{x}_v, U_u>=-g\left(\frac{\partial}{\partial u}, S\left(\frac{\partial}{\partial v}\right)\right)=-h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right), \\ &<\mathbf{x}_v, U_v>=-g\left(\frac{\partial}{\partial v}, S\left(\frac{\partial}{\partial v}\right)\right)=-h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right). \end{aligned}$$

By the help of (3.9), (3.10), (3.11) and (3.12), we obtain

(3.13)

$$\mathbf{II}=h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right)du\otimes du+h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)(du\otimes dv+dv\otimes du)+h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right)dv\otimes dv=h.$$

By the help of (3.7) and (3.13), (3.11) is written as

$$\tag{3.14} \mathbf{II}=h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right)du^2+2\,h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)du dv+h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right)dv^2=h.$$

The tensor field $\mathbf{II}$ $(=h)$ of type (0,2) on the surface M is said to be the *second fundamental form* of $\mathbf{x}=\mathbf{x}(u,v)$.

The tangent space of M at the point $p(\in M)$ is

$$\tag{3.15} T_pM=\left\{s\left(\frac{\partial}{\partial u}\right)_p+t\left(\frac{\partial}{\partial v}\right)_p \mid s,\ t\in\mathbb{R}\right\}.$$

By virtue of (3.4), we obtain the fact that the normal curvature of M in a direction $s\left(\frac{\partial}{\partial u}\right)_p+t\left(\frac{\partial}{\partial v}\right)_p=: \mathbf{v}(s,t)(\in T_pM)$ at the point $p(\in M)$ is

$$\tag{3.16} \kappa_N(s,t):=\kappa_N(\|\mathbf{v}(s,t)\|_g^{-1}\mathbf{v}(s,t))=\|\mathbf{v}(s,t)\|_g^{-2}g(S(\mathbf{v}(s,t)),\mathbf{v}(s,t)).$$

So, κ_N is a C^∞ function of $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$.

Putting

$$E_0 := g\left(\left(\frac{\partial}{\partial u}\right)_p, \left(\frac{\partial}{\partial u}\right)_p\right),$$
$$F_0 := g\left(\left(\frac{\partial}{\partial u}\right)_p, \left(\frac{\partial}{\partial v}\right)_p\right)$$

and

$$G_0 := g\left(\left(\frac{\partial}{\partial v}\right)_p, \left(\frac{\partial}{\partial v}\right)_p\right),$$

we get

$$\|\mathbf{v}(s,t)\|_g^2 = E_0 s^2 + 2F_0 st + G_0 t^2. \tag{3.17}$$

And, putting

$$L_0 := h\left(\left(\frac{\partial}{\partial u}\right)_p, \left(\frac{\partial}{\partial u}\right)_p\right),$$
$$M_0 := h\left(\left(\frac{\partial}{\partial u}\right)_p, \left(\frac{\partial}{\partial v}\right)_p\right)$$

and

$$N_0 := h\left(\left(\frac{\partial}{\partial v}\right)_p, \left(\frac{\partial}{\partial v}\right)_p\right),$$

we obtain from (3.3)

$$g(S(\mathbf{v}(s,t)), \mathbf{v}(s,t)) = h(\mathbf{v}(s,t), \mathbf{v}(s,t)) = L_0 s^2 + 2M_0 st + N_0 t^2. \tag{3.18}$$

By the help of (3.16), (3.17) and (3.18), we have

$$\kappa_N(s,t) = \frac{L_0 s^2 + 2M_0 st + N_0 t^2}{E_0 s^2 + 2F_0 st + G_0 t^2}. \tag{3.19}$$

Proposition 3.2. The normal curvature of M in a direction

$$\mathbf{v}(s,t) := s\left(\frac{\partial}{\partial u}\right)_p + t\left(\frac{\partial}{\partial v}\right)_p (\in T_pM)$$

at the point $p(\in M)$ is

$$\kappa_N(s,t) = \frac{L_0 s^2 + 2M_0 st + N_0 t^2}{E_0 s^2 + 2F_0 st + G_0 t^2}.$$

From the above proposition, we find the fact that $\kappa_N(s,t)$ *only depends upon the ratio* $s:t$.

Remark 3.3. There is the statement in [4, p. 143 ∼ p. 150] such that the normal curvature function κ_N defined on $T_p(M)$ depends only upon the ratio $du : dv$, where

$$\kappa_N := \frac{h(\frac{\partial}{\partial u},\frac{\partial}{\partial u})du^2 + 2h(\frac{\partial}{\partial u},\frac{\partial}{\partial v})dudv + h(\frac{\partial}{\partial v},\frac{\partial}{\partial v})dv^2}{g(\frac{\partial}{\partial u},\frac{\partial}{\partial u})du^2 + 2g(\frac{\partial}{\partial u},\frac{\partial}{\partial v})dudv + g(\frac{\partial}{\partial v},\frac{\partial}{\partial v})dv^2} = \frac{h}{g}.$$

But, the above statement is not appropriate, since $\{du, dv\}$ is the (locally defined) dual frame of $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$.

Moreover, we get the fact that a necessary and sufficient condition for $\|\mathbf{v}(s,t)\|_g^{-1}\mathbf{v}(s,t)$, $(s,t) \neq (0,0)$, to be a principal vector of M at the point $p(\in M)$ is

$$\frac{\partial \kappa_N(s,t)}{\partial s} = \frac{\partial \kappa_N(s,t)}{\partial t} = 0. \tag{3.20}$$

And, the condition (3.20) is equivalent to the following:

$$\begin{aligned} &(L_0 - \kappa_N(s,t)E_0)s + (M_0 - \kappa_N(s,t)F_0)t = 0, \text{ and} \\ &(M_0 - \kappa_N(s,t)F_0)s + (N_0 - \kappa_N(s,t)G_0)t = 0. \end{aligned} \tag{3.21}$$

Remark 3.4. In [2, Theorem 9.5, p. 183], a necessary and sufficient condition for κ_N to be a principal curvature of M is presented as follows: *A real number κ_N is a principal curvature at p in the direction $du : dv$ if and only if κ_N, du and dv satisfy*

$$\left(\begin{aligned} &(L_0 - \kappa_N(s,t)E_0)du + (M_0 - \kappa_N(s,t)F_0)dv = 0, \text{ and} \\ &(M_0 - \kappa_N(s,t)F_0)du + (N_0 - \kappa_N(s,t)G_0)dv = 0. \end{aligned}\right.$$

The ratio $du : dv$ at the phrase *'a principal curvature at p in the direction $du : dv$'* of the above theorem is not appropriate, since $\{du, dv\}$ is the (locally defined) dual frame of $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$.

Moreover, the homogeneous system (3.21) of equations has a nontrivial solution (s,t) if and only if

$$\begin{vmatrix} L_0 - \kappa_N(s,t)E_0 & M_0 - \kappa_N(s,t)F_0 \\ M_0 - \kappa_N(s,t)F_0 & N_0 - \kappa_N(s,t)G_0 \end{vmatrix} = 0. \tag{3.22}$$

So, the principal curvature $\kappa_N(s,t)$ of M at p is a solution of the equation

$$(E_0G_0 - F_0^2)\kappa_N(s,t)^2 - (E_0N_0 + G_0L_0 - 2F_0M_0)\kappa_N(s,t) + (L_0N_0 - M_0^2) = 0. \tag{3.23}$$

Thus we obtain

Proposition 3.5. A number κ is a principal curvature if and only if κ is a solution of the equation

$$(EG - F^2){\kappa_N}^2 - (EN + GL - 2FM)\kappa_N + (LN - M^2) = 0.$$

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