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Stability analysis of a coupled system of nonlinear fractional differential equations

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Abstract

Our aim in this work is to study the stability of the solution of a boundary value problem for a coupled system of nonlinear fractional differential equations. By using the Gronwall inequality of fractional type, we can derive sufficient conditions for the stability which depends on orders of fractional derivatives.

AMS subject classification:
Keywords: Fractional derivative, Boundary value problem, Green's function, Schauder fixed-point theorem, Gronwall-type inequality.

1. Introduction

The fractional calculus is a field of mathematics, which studies the integral and derivative operators with the possibility that a differential operator be of a fractional order.

In recent years, fractional differential equations have attracted the attention of many researchers due to their various applications in several fields of science and engineering, for example: electromagnetics, viscoelasticity and acoustics, (see [1, 3, 4, 8, 11]).

In [14] X. Su has studied the following system,

$$\begin{cases} D^{\alpha} u(t) = f\left(t, v(t), D^{\mu} v(t)\right), \quad 0 < t < 1 \\ D^{\beta} v(t) = g\left(t, u(t), D^{\nu} u(t)\right), \quad 0 < t < 1 \\ \quad u(0) = u(1) = v(1) = v(0) = 0, \end{cases} \tag{$(1)_{(\alpha,\beta)}$}$$

where $1 < \alpha, \beta < 2, \alpha - \nu \geq 1, \beta - \mu \geq 1, \nu, \mu > 0, f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and D^{α} is the standard Rieman–Liouville fractional derivative.

To show the existence of solutions by using the corresponding Green's function and the Schauder fixed point theorem, the author assume that f and g are bounded.

Motivated by the work of X. Su (see [14]), here we consider the system $(1)_{(\alpha,\beta)}$ with f and g unbounded.

First, in section 2, we deduct an integral equation of Fredholm type, equivalent to problem $(1)_{(\alpha,\beta)}$. Then, in section 3, by using the Schauder fixed point theorem we show the existence of solutions. Finally, in section 4, we consider the stability of solutions with respect to orders of fractional derivatives of the system $(1)_{(\alpha,\beta)}$.

2. Preliminaries

In this section, we present some necessary definitions and results of fractional calculus theory. Let $\alpha > 0$ and $n = [\alpha] + 1 = N + 1$, where N is the smallest integer less than or equal to α, Γ is the gamma function, $n \in \mathbb{N}$, and $x \in [a,b]$.

Definition 2.1. ([8]) The Riemann–Liouville integral fractional of a function $f \in L^1[a,b]$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^t (x-t)^{\alpha-1} f(t)\, dt. \tag{2}$$

Definition 2.2. ([8]) The Riemann–Liouville fractional derivative of order of $\alpha \in (n-1, n)$ of the function f is defined by

$$D_a^{\alpha} f(x) \ := D^n I_a^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\ dt\ . \tag{3}$$

Lemma 2.3. ([8]) Let $\alpha, \beta > 0$, $f \in C[a,b]$, and assume that the Riemann–Liouville fractional integral of f exists, then we have

$$I_a^{\alpha} I_a^{\beta} f(x) = I_a^{\alpha+\beta} f(x) = I_a^{\beta} I_a^{\alpha} f(x).$$

Lemma 2.4. ([8]) Let $\alpha > \beta > 0$ and $f \in L^1[a,b]$, then almost everywhere in $[a,b]$, we have

$$D_a^{\beta}\left(I_a^{\alpha} f\right)(x) = I_a^{\alpha-\beta} f(x). \tag{4}$$

Lemma 2.5. ([4]) (Gronwall inequality of fractional type) Let α, ϵ_1 and $\epsilon_2 \in \mathbb{R}_+ (0 < t < 1)$, suppose that $\delta : [0,T] \to \mathbb{R}$ is a continuous function such that

$$|\delta(x)| \leq \epsilon_1 + \frac{\epsilon_2}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |\delta(t)|\, dt; \forall x \in [0,T],$$

then

$$|\delta(x)| \leq \epsilon_1 E_{\alpha}\left(\epsilon_2 T^{\alpha}\right),$$

E_α is the Mittag-Leffler function:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \ \ (\alpha > 0) \,.$$

Next, we define the space

$$X \times Y = \left\{ (u, v) : (u, v) \in (C[0,1])^2 \ \text{et} \ \|(u, v)\|_{X\times Y} = \max\left\{ \|u\|_X, \|v\|_Y \right\} \right\}, \quad (5)$$

where,

$$\|u\|_X = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} \left| D^\nu u(t) \right|,$$

$$\|v\|_Y = \max_{t\in[0,1]} |v(t)| + \max_{t\in[0,1]} \left| D^\mu v(t) \right|.$$

Lemma 2.6. ([10]) $\left(X \times Y, \|(u, v)\|_{X\times Y}\right)$ is a Banach space.

Lemma 2.7. ([10]) If $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, then $(u, v) \in X \times Y$ is a solution of $(1)_{(\alpha,\beta)}$ if and only if $(u, v) \in X \times Y$ is a solution of the following system:

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t,s)\, f\left(s, v(s), D^\mu v(s)\right) ds, \\ v(t) = \displaystyle\int_0^1 G_2(t,s)\, g\left(s, u(s), D^\nu u(s)\right) ds, \end{cases} \quad (6)$$

where, for i=1, 2, $\sigma_1 = \alpha$ and $\sigma_2 = \beta$

$$G_i(t,s) = \begin{cases} \dfrac{(t-s)^{\sigma_i - 1} - [t(1-s)]^{\sigma_i - 1}}{\Gamma(\sigma_i)}, \ s \le t \\ \dfrac{-[t(1-s)]^{\sigma_i - 1}}{\Gamma(\sigma_i)}, \ t \le s \end{cases} \quad (7)$$

Here, (G_1, G_2) is the Green's function of the system $(1)_{(\alpha,\beta)}$.

3. Existence of the system

In the following theorem, we prove the exitence of solutions for the fractional boundary value problem $(1)_{(\alpha,\beta)}$.

Theorem 3.1. Let $f, g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two continuous functions, where,

$$\begin{aligned} p, q, r, \overline{r} > 1, \psi_1 (s, \alpha, \nu) &= \frac{(1-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}, \\ \psi_2 (s, \beta, \mu) &= \frac{(1-s)^{\beta-\mu-1}}{\Gamma(\beta-\mu)}, \\ c_a &= \max_{t \in I} \int_0^1 |G_1 (t,s)| \, a(t) \, ds, \\ c_b &= \max_{t \in I} \int_0^1 |G_2 (t,s)| \, b(t) \, ds, \\ c_{(\alpha,\nu)} &= \frac{1}{(\Gamma(\alpha-\nu+1))^r} \end{aligned}$$

and

$$c_{(\beta,\mu)} = \frac{1}{(\Gamma(\beta-\mu+1))^{\overline{r}}},$$

Such that, (H_1) There exists $a(t), b(t)$: $[a,b] \to \mathbb{R}$ such that,

$$\begin{aligned} |f(t,x,y)| &\leq a(t)(|x| + |y|), \\ |g(t,x,y)| &\leq b(t)(|x| + |y|), \end{aligned}$$

(H_2) There exists positive constants m, n such that, (H_3)

$$\max \left(c_a + 2mc_{(\alpha,\nu)}, c_b + 2nc_{(\beta,\mu)} \right) \leq 1,$$

Then, the system $(1)_{(\alpha,\beta)}$ has a solution.

Proof. We use the Schauder fixed-point theorem, to prove this result, let us consider the space

$$U = \left\{ (u,v) \setminus (u,v) \in X \times Y, \|(u,v)\|_{X \times Y} \leq R \right\},$$

where $X \times Y$ is the space defined by (5) and $R > 0$.

Let $T : X \times Y \to X \times Y$ be the operator defined as

$$\begin{aligned} T(u,v)(t) &= \left(\int_0^1 G_1(t,s) \, f\left(s, v(s), D^\mu v(s)\right) ds, \int_0^1 G_2(t,s) \, f\left(s, u(s), D^\nu u(s)\right) ds \right) \\ &=: (T_1 v(t), T_2 u(t)). \end{aligned}$$

We prove that $T : U \to U$ $(U \subset X \times Y)$ is bounded for any $(u,v) \in U$. Using (4) ,

(H_1) and $D^{\nu}t^{\alpha-1} = \dfrac{\Gamma(\alpha)\, t^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}$, we obtain,

$$\begin{aligned}
|T_1 v(t)| &= \left|\int_0^1 G_1(t,s)\, f\left(s, v(s), D^{\mu}v(s)\right) ds\right| \\
&\le \int_0^1 |G_1(t,s)| \left|f\left(s, v(s), D^{\mu}v(s)\right)\right| ds \\
&\le \int_0^1 |G_1(t,s)|\, a(s) \left(|v(s)| + \left|D^{\mu}v(s)\right|\right) ds \\
&\le \int_0^1 |G_1(t,s)|\, a(s) \left(\max_{s\in[0,1]} |v(s)| + \max_{s\in[0,1]} \left|D^{\mu}v(s)\right|\right) ds \\
&\le R\int_0^1 |G_1(t,s)|\, a(s)\, ds,
\end{aligned}$$

and

$$\begin{aligned}
\left|D^{\nu}T_1 v(t)\right| &= \left|D^{\nu}I^{\alpha} f\left(t, v(t), D^{\mu}v(t)\right) - D^{\nu}t^{\alpha-1} I^{\alpha}\left(f\left(t, v(t), D^{\mu}v(t)\right)\right)_{t=1}\right| \\
&= \Big|I^{\alpha-\nu} f\left(t, v(t), D^{\mu}v(t)\right) \\
&\quad -\frac{\Gamma(\alpha)}{\Gamma(\alpha-\nu)} t^{\alpha-\nu-1} I^{\alpha}\left(f\left(t, v(t), D^{\mu}v(t)\right)\right)_{t=1}\Big| \\
&= \Bigg|\int_0^t \frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} f\left(s, v(s), D^{\mu}v(s)\right) ds \\
&\quad -t^{\alpha-\nu-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\nu)} f\left(s, v(s), D^{\mu}v(s)\right) ds\Bigg| \\
&\le R\left\{\int_0^t \frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} a(s)\, ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha-\nu)} a(s)\, ds\right\} \\
&\le 2R\int_0^1 \frac{(1-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} a(s)\, ds \\
&= 2R\int_0^1 \psi_1(s,\alpha,\nu)\, a(s)\, ds,
\end{aligned}$$

we use the Minkowiski's inequality for $p, q > 1$ and (H_2), we have

$$\begin{aligned}
\int_0^1 \psi_1(s,\alpha,\nu)\, a(s)\, ds &= \int_0^1 (\psi_1(s,\alpha,\nu))^{r} (\psi_1(s,\alpha,\nu))^{1-r} a(s)\, ds \\
&\le \left(\int_0^1 (\psi_1(s,\alpha,\nu))^{rp} ds\right)^{\frac{1}{p}} \left(\int_0^1 (\psi_1(s,\alpha,\nu))^{(1-r)q} a^{q}(s)\, ds\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\le m\left(\int_0^1 \left(\psi_1\left(s,\alpha,\nu\right)\right)^{rp} ds\right)^{\frac{1}{p}} \\
&= m\left(\int_0^1 \left(\psi_1\left(s,\alpha,\nu\right)\right)^{rp} ds\right)^{\frac{1}{p}},
\end{aligned}
$$

if we set, $rp = 1$, we have

$$
\begin{aligned}
\int_0^1 \psi_1\left(s,\alpha,\nu\right) a\left(s\right) ds &\le m\left(\int_0^1 \left(\psi_1\left(s,\alpha,\nu\right)\right)^{rp} ds\right)^{\frac{1}{p}} \\
&= m\left(\int_0^1 \frac{(1-s)^{\alpha-\nu-1}}{\Gamma\left(\alpha-\nu\right)} ds\right)^{r} \\
&= m\left(\frac{1}{\Gamma\left(\alpha-\nu+1\right)}\right)^{r}.
\end{aligned}
$$

Then,

$$
\left|D^{\nu} T_1 v\left(t\right)\right| \le 2Rm\left(\frac{1}{\Gamma\left(\alpha-\nu+1\right)}\right)^{r}.
$$

Thus,

$$
\begin{aligned}
\left\|T_1 v\right\|_X &= \max_{t\in[0,1]} \left|T_1 v\left(t\right)\right| + \max_{t\in[0,1]} \left|D^{\mu} T_1 v\left(t\right)\right| \\
&\le R\left(\max_{t\in[0,1]} \int_0^1 \left|G_1\left(t,s\right)\right| a\left(t\right) ds + 2m\left(\frac{1}{\Gamma\left(\alpha-\nu+1\right)}\right)^{r}\right), \qquad (8)
\end{aligned}
$$

similary, we can prove that

$$
\left\|T_2 u\right\|_Y \le R\left(\max_{t\in[0,1]} \int_0^1 \left|G_2\left(t,s\right)\right| b\left(t\right) ds + 2n\left(\frac{1}{\Gamma\left(\beta-\mu+1\right)}\right)^{\overline{r}}\right).
$$

Finally, according to (8), (9) and (H_3), we obtain

$$
\left\|T\left(u,v\right)\right\|_{X\times Y} \le R \max\left(c_a + 2mc_{(\alpha,\nu)}, c_b + 2nc_{(\beta,\mu)}\right) \le R.
$$

Now, we show that T is a completely continuous operator. Let $t, \tau \in [0,1]$ $(t < \tau)$, then, by using (7) and (H_1) , we have

$$
\begin{aligned}
|T_1 v(\tau) - T_1 v(t)| &= \left| \int_0^1 (G_1(\tau, s) - G_1(t, s)) f\left(s, v(s), D^\mu v(s)\right) ds \right| \\
&\leq \int_0^1 |G_1(\tau, s) - G_1(t, s)| \left| f\left(s, v(s), D^\mu v(s)\right) \right| ds \\
&= C \left[\int_0^t |G_1(\tau, s) - G_1(t, s)| \, a(s) \, ds \right. \\
&\quad + \int_t^\tau |G_1(t, s) - G_1(\tau, s)| \, a(s) \, ds \\
&\quad \left. + \int_\tau^1 |G_1(\tau, s) - G_1(t, s)| \, a(s) \, ds \right] \\
&\leq \frac{C}{\Gamma(\alpha)} \left[\int_0^t \left((\tau - s)^{\alpha-1} - (t - s)^{\alpha-1} + (\tau(1 - s))^{\alpha-1} \right. \right. \\
&\quad \left. - (t(1 - s))^{\alpha-1} a(s) \, ds \right) \\
&\quad + \int_t^\tau \left((\tau - s)^{\alpha-1} - (t(1 - s))^{\alpha-1} + (\tau(1 - s))^{\alpha-1} \right) a(s) \, ds \\
&\quad \left. \int_\tau^1 \left((\tau(1 - s))^{\alpha-1} - (t(1 - s))^{\alpha-1} \right) a(s) \, ds \right] \\
&= \frac{C}{\Gamma(\alpha)} \left[\int_0^1 (1 - s)^{\alpha-1} \left(\tau^{\alpha-1} - t^{\alpha-1} \right) a(s) \, ds \right. \\
&\quad \left. + \int_0^\tau (\tau - s)^{\alpha-1} a(s) \, ds - \int_0^t (t - s)^{\alpha-1} a(s) \, ds \right] \\
&\leq C \left\{ \frac{m}{(\Gamma(\alpha + 1))^r} \left(\tau^{\alpha-1} - t^{\alpha-1} \right) + \frac{m}{(\Gamma(\alpha + 1))^r} \left(\tau^\alpha - t^\alpha \right) \right\} \\
&= \frac{Cm}{(\Gamma(\alpha + 1))^r} \left(\tau^{\alpha-1} - t^{\alpha-1} + \tau^\alpha - t^\alpha \right).
\end{aligned}
$$

Furthermore by (H_1) and (H_2), we have

$$
\begin{aligned}
\left| D^\nu T_1 v(\tau) - D^\nu T_1 v(t) \right| \quad = \quad & \left| I^{\alpha-\nu} \left(\tau, v(\tau), D^\mu v(\tau) \right) \right. \\
& - \frac{\Gamma(\alpha)}{\Gamma(\alpha - \nu)} \tau^{\alpha-\nu-1} I^\alpha f \left(\tau, v(\tau), D^\mu v(\tau) \right) \Big|_{\tau=1} \\
& \Big| - I^{\alpha-\nu} f \left(t, v(t), D^\mu v(t) \right) \\
& \left. - \frac{\Gamma(\alpha)}{\Gamma(\alpha - \nu)} t^{\alpha-\nu-1} I^\alpha \; f \left(t, v(t), D^\mu v(t) \right) \Big|_{t=1} \right|
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha-\nu)}\left[\int_0^\tau \left|(\tau-s)^{\alpha-\nu-1} f\left(s, v(s), D^\mu v(s)\right) ds\right.\right. \\
&\quad \left.\left. -\int_0^t (t-s)^{\alpha-\nu-1} f\left(s, v(s), D^\mu v(s)\right) ds\right|\right] \\
&\quad +\frac{\left(\tau^{\alpha-\nu-1}-t^{\alpha-\nu-1}\right)}{\Gamma(\alpha-\nu)} \int_0^1 (1-s)^{\alpha-1}\left|f\left(s, v(s), D^\mu v(s)\right)\right| ds \\
&\quad +\frac{\left(\tau^{\alpha-\nu-1}-t^{\alpha-\nu-1}\right)}{\Gamma(\alpha-\nu)} \int_0^1 (1-s)^{\alpha-1}\left|f\left(s, v(s), D^\mu v(s)\right)\right| ds \\
&= \frac{1}{\Gamma(\alpha-\nu)}\left[\left|\int_0^\tau (\tau-s)^{\alpha-\nu-1} f\left(s, v(s), D^\mu v(s)\right) ds\right.\right. \\
&\quad \left. -\int_0^\tau (t-s)^{\alpha-\nu-1} f\left(s, v(s), D^\mu v(s)\right) ds\right| \\
&\quad +\left|\int_0^\tau (t-s)^{\alpha-\nu-1} f\left(s, v(s), D^\mu v(s)\right) ds\right. \\
&\quad \left.\left. -\int_0^t (t-s)^{\alpha-\nu-1} f\left(s, v(s), D^\mu v(s)\right) ds\right|\right] \\
&\quad +\frac{\left(\tau^{\alpha-\nu-1}-t^{\alpha-\nu-1}\right)}{\Gamma(\alpha-\nu)} \int_0^1 (1-s)^{\alpha-1}\left|f\left(s, v(s), D^\mu v(s)\right)\right| ds \\
&\leq \frac{C}{\Gamma(\alpha-\nu)} \int_0^\tau \left((\tau-s)^{\alpha-\nu-1}-(t-s)^{\alpha-\nu-1}\right) a(s)\, ds \\
&\quad +\int_t^\tau (t-s)^{\alpha-\nu-1} a(s)\, ds+\left(\tau^{\alpha-\nu-1}-t^{\alpha-\nu-1}\right) \int_0^1 (1-s)^{\alpha-1} a(s)\, ds \\
&= \frac{R}{\Gamma(\alpha-\nu)}\left\{\int_0^\tau (\tau-s)^{\alpha-\nu-1} a(s)\, ds-\int_0^t (t-s)^{\alpha-\nu-1} a(s)\, ds\right. \\
&\quad \left. +\left(\tau^{\alpha-\nu-1}-t^{\alpha-\nu-1}\right) \int_0^1 (1-s)^{\alpha-1} a(s)\, ds\right\} \\
&\leq \frac{C}{\Gamma(\alpha-\nu)}\left\{\frac{m}{(\alpha-\nu)^r}\left(\tau^{\alpha-\nu}-t^{\alpha-\nu}\right)+\frac{m}{\alpha^r}\left(\tau^{\alpha-\nu-1}-t^{\alpha-\nu-1}\right)\right\} \\
&= \frac{Cm}{(\alpha-\nu)^r\, \Gamma(\alpha-\nu)}\left(\tau^{\alpha-\nu}-t^{\alpha-\nu}\right)+\frac{Cm}{\alpha^r \Gamma(\alpha-\nu)}\left(\tau^{\alpha-\nu-1}-t^{\alpha-\nu-1}\right),
\end{aligned}$$

similary, we can prove that

$$\left|T_2 u(\tau)-T_2 u(t)\right| \leq \frac{C}{\Gamma(\beta-\mu)}\left\{\frac{n}{(\beta-\mu)^{\bar{r}}}\left(\tau^{\beta-1}-t^{\beta-1}\right)+\frac{n}{\beta^r}\left(\tau^\beta-t^\beta\right)\right\},$$

$$\begin{aligned}
\left|D^\mu T_2 u(\tau)-D^\mu T_2 u(t)\right| &\leq \frac{Cn}{(\beta-\mu)^{\bar{r}}\, \Gamma(\beta-\mu)}\left(\tau^{\beta-\mu}-t^{\beta-\mu}\right) \\
&\quad +\frac{nC}{\beta^{\bar{r}} \Gamma(\beta-\mu)}\left(\tau^{\beta-\mu-1}-t^{\beta-\mu-1}\right),
\end{aligned}$$

since the functions t^{α}, $t^{\alpha-1}$, $t^{\alpha-\nu}$, $t^{\alpha-\nu-1}$, t^{β}, $t^{\beta-1}$, $t^{\beta-\mu}$, $t^{\beta-\mu-1}$ are uniformly continuous on $[0, 1]$, the family of functions TU is equicontinuous. Thus, we conclude that T is a completely continuous operator. Hence, according to the Schauder fixed point theorem, there exists a solution of $(1)_{(\alpha,\beta)}$. ■

4. Stability

In this section, we study the stability of the system of coupled equations given by $(1)_{(\alpha,\beta)}$.

Definition 4.1. The system $(1)_{(\alpha,\beta)}$ is stable (with respect to the orders of derivatives) if and only if

$$\forall \epsilon > 0,\ \exists K > 0,\ \ |\alpha - \overline{\alpha}|^{r} + \left|\beta - \overline{\beta}\right|^{\overline{r}} < K$$
$$\Rightarrow$$
$$|u - \overline{u}| + |v - \overline{v}| < \epsilon,$$

where (u, v) et $(\overline{u}, \overline{v})$ are respectively the solutions of $(1)_{(\alpha,\beta)}$ and $(1)_{(\overline{\alpha},\overline{\beta})}$.

Theorem 4.2. If f, g are continuous functions on $[0, 1] \times \mathbb{R} \times \mathbb{R}$, where, $p, q, r, \overline{r} > 1$, for $i = 1, 2$

$$(\rho_1 = \alpha, \rho_2 = \beta)\,,\ \Psi\left(t, s, \rho_i, \overline{\rho_i}\right) = \left| \frac{(t\,(1-s))^{\rho_i - 1}}{\Gamma(\rho_i)} - \frac{(t\,(1-s))^{\overline{\rho_i}-1}}{\Gamma\left(\overline{\rho_i}\right)} \right|,$$

$$\psi_{(\alpha,\nu)} = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\nu+1)}$$

and

$$\psi_{(\beta,\mu)} = \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta-\mu+1)}$$

satisfy:

(S_1) There exists K_f, K_g, L_f and L_g positive constants such that:

$$\begin{aligned} |f(t, x_1, y_1) - f(t, x_2, y_2)| &\leq K_f |x_1 - x_2| + L_f |y_1 - y_2| \\ \left|g\left(t, x_1', y_1'\right) - g\left(t, x_2', y_2'\right)\right| &\leq K_g \left|x_1' - x_2'\right| + L_g \left|y_1' - y_2'\right|, \end{aligned}$$

(S_2) There exists $h_1(t)\,,\ h_2(t) : [0, 1] \to \mathbb{R}^{+}$, satisfy

$$|f(t, x, y)| \leq h_1(t)\,(|x| + |y|)\,,\ \left|g\left(t, x', y'\right)\right| \leq h_2(t)\left(\left|x'\right| + \left|y'\right|\right),$$

where $x, y \in \mathbb{R}$, and there exists positive constants M , N such that:

$$\left(\int_0^1 \Psi\left(t, s, \alpha, \overline{\alpha}\right)^{(1-r)q} h_1^q(s)\, ds\right)^{\frac{1}{q}} \leq M,$$

$$\left(\int_0^1 \Psi\left(t, s, \beta, \overline{\beta}\right)^{(1-\overline{r})q} h_2^q(s)\, ds\right)^{\frac{1}{q}} \leq N,$$

(S_3) $\alpha \leq \beta$, $\mu \leq \nu$, and $\alpha - \nu \leq \beta - \mu$

(S_4) $\Delta = \max\left(K_f, L_f, K_g, L_g\right) \max\left\{\psi_{(\alpha,\nu)}, \psi_{(\beta,\mu)}\right\} < 1,$
Then the system $(1)_{(\alpha,\beta)}$ is stable.

Proof. Lemma 5 implies that

$$\begin{aligned}
u(t) - \overline{u}(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left[f\left(s, v(s), D^{\mu}v(s)\right) - f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right] ds \\
&- \int_0^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} \left[f\left(s, v(s), D^{\mu}v(s)\right) - f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right] ds \\
&+ \int_0^t \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right] f\left(s, v(s), D^{\mu}v(s)\right) ds \\
&- \int_0^1 \left[\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right] f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right) ds
\end{aligned}$$

Then,

$$\begin{aligned}
\left|u(t) - \overline{u}(t)\right| \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\left(s, v(s), D^{\mu}v(s)\right) - f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right| ds \\
&+ \int_0^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} \left|f\left(s, v(s), D^{\mu}v(s)\right) - f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right| ds \\
&+ \int_0^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| \left|f\left(s, v(s), D^{\mu}v(s)\right)\right| ds \\
&+ \int_0^1 \left|\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| \left|f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right| ds \\
= & \; I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left|f\left(s, v(s), D^{\mu}v(s)\right) - f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right| ds \\
&+ \int_0^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} \left|f\left(s, v(s), D^{\mu}v(s)\right) - f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right| ds,
\end{aligned}$$

and

$$\begin{aligned}
I_2 = & \int_0^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| \left|f\left(s, v(s), D^{\mu}v(s)\right)\right| ds \\
&+ \int_0^1 \left|\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| \left|f\left(s, \overline{v}(s), D^{\mu}\overline{v}(s)\right)\right| ds.
\end{aligned}$$

According to (S_1) , we can have

$$\begin{aligned} I_1 \;\; \leq \;\; & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(K_f \left|v(s) - \overline{v}(s)\right| + L_f \left|D^\mu v(s) - D^\mu \overline{v}(s)\right|\right) ds \\ & \int_0^t \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} \left(K_f \left|v(s) - \overline{v}(s)\right| + L_f \left|D^\mu v(s) - D^\mu \overline{v}(s)\right|\right) ds. \end{aligned}$$

Now, by using (S_2), we can write

$$\begin{aligned} I_2 \leq & \int_0^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| h_1(s) \left(|v(s)| + \left|D^\mu v(s)\right|\right) ds \\ & + \int_0^1 \left|\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| h_1(s) \left(|v(s)| + \left|D^\mu v(s)\right|\right) ds \\ & \leq C \left\{ \int_0^t \left|\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| h_1(s)\, ds \right. \\ & \quad \left. + \int_0^1 \left|\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| h_1(s)\, ds \right\}, \end{aligned}$$

by Minkowski's inequality, we obtain

$$\begin{aligned} \int_0^1 \Psi(t,s,\alpha,\overline{\alpha})\, h(s)\, ds & \leq \int_0^1 (\Psi(t,s,\alpha,\overline{\alpha}))^r (\Psi(t,s,\alpha,\overline{\alpha}))^{1-r} h(s)\, ds \\ & \leq \left(\int_0^1 (\Psi(t,s,\alpha,\overline{\alpha}))^{rp}\right)^{\frac{1}{p}} \left(\int_0^1 (\Psi(t,s,\alpha,\overline{\alpha}))^{(1-r)q} h_1^q(s)\, ds\right)^{\frac{1}{q}} \\ & \leq M \left(\int_0^1 (\Psi(t,s,\alpha,\overline{\alpha}))^{rp} ds\right)^{\frac{1}{p}}, \end{aligned}$$

if we set $rp = 1$, then

$$\begin{aligned} \int_0^1 \Psi(t;s,\alpha,\overline{\alpha})\, h(s)\, ds & = M \left(\int_0^1 \left|\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right|^{rp} ds\right)^{\frac{1}{p}} \\ & = M \left(\int_0^1 \left|\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| ds\right)^{r}. \end{aligned}$$

Now, we use the mean value theorem. For this, we use

$$\left\|\varphi_1(\alpha)(s) - \varphi_1(\overline{\alpha})(s)\right\|_1 \leq \sup_{1<\alpha<2} \left\|\varphi_1'(\alpha)(s)\right\|_1 . |\alpha - \overline{\alpha}|, \tag{10}$$

where,

$$\varphi_1(\alpha)(s) = \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)},$$

if we put $x = t(1-s)$, then

$$\varphi_1(\alpha)(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)},$$

therefore,

$$\varphi_1'(\alpha)(x) = \frac{x^{\alpha-1}\Gamma(\alpha)\ln x - \psi(\alpha)\Gamma(\alpha)x^{\alpha-1}}{\Gamma(\alpha)^2}$$

(where, $\psi(\xi) = \frac{d}{d\xi}(\log\Gamma(\xi))$: diagamma function)

$$= \frac{(t(1-s))^{\alpha-1}\ln(t(1-s))}{\Gamma(\alpha)} - \frac{\psi(\alpha)(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}.$$

Thus,

$$\begin{aligned}
\left\|\varphi_1'(\alpha)(s)\right\|_1 &= \int_0^1 \left|\varphi_1'(\alpha)(s)\right| ds \\
&= \int_0^1 \left\{-\frac{(t(1-s))^{\alpha-1}\ln(t(1-s))}{\Gamma(\alpha)} + \frac{\psi(\alpha)(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}\right\} ds \\
&= \frac{1}{\Gamma(\alpha)}(k_1 + k_2),
\end{aligned}$$

now, integrating by parts, we obtain

$$k_1 = -\int_0^1 \frac{(t(1-s))^{\alpha-1}\ln(t(1-s))}{\Gamma(\alpha)} ds = -\frac{t^{\alpha-1}\ln t}{\alpha} + \frac{t^{\alpha-1}}{\alpha^2},$$

and

$$k_2 = \psi(\alpha)\int_0^1 (t(1-s))^{\alpha-1} ds = \frac{t^{\alpha-1}}{\alpha}\psi(\alpha).$$

So,

$$\begin{aligned}
\|\varphi_1{}'(\alpha)(s)\|_1 &= \frac{1}{\Gamma(\alpha)}\left\{-\frac{t^{\alpha-1}\ln t}{\alpha} + \frac{t^{\alpha-1}}{\alpha^2} + \frac{\psi(\alpha)t^{\alpha-1}}{\alpha}\right\} \\
&\leq c_1 \quad (c_1 := c_{(1,\alpha)})
\end{aligned}$$

The inequality (10) becomes:

$$\begin{aligned}
(\|\varphi_1(\alpha)(s) - \varphi_1(\overline{\alpha})(s)\|_1)^r &= \left(\int_0^1 \left|\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-s))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})}\right| ds\right)^r \\
&\leq c_{(1,\alpha)}|\alpha - \overline{\alpha}|^r \quad \left(c_{(1,\alpha)} := c_1^r\right)
\end{aligned}$$

Simillary, we put $x = \dfrac{s}{t}$ as follows,

$$\int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})} \right| ds = \int_0^t \left| \frac{\left(t\left(1-\frac{s}{t}\right)\right)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\left(t\left(1-\frac{s}{t}\right)\right)^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})} \right| ds$$

and

$$\left(\int_0^1 \left| \frac{(t(1-x))^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t(1-x))^{\overline{\alpha}-1}}{\Gamma(\overline{\alpha})} \right| dx \right)^r \leq c_{(2,\overline{\alpha})} |\alpha - \overline{\alpha}|^r, \ \left(c_{(2,\overline{\alpha})} := c_2^r\right)$$

this implies that,

$$I_2 \leq M \left\{ c_{(1,\alpha)} + c_{(2,\overline{\alpha})} \right\} |\alpha - \overline{\alpha}|^r = K |\alpha - \overline{\alpha}|^r .$$

Then, we have

$$\begin{aligned} |u(t) - \overline{u}(t)| \leq K |\alpha - \overline{\alpha}|^r &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(K_f |v(s) - \overline{v}(s)| + L_f \left| D^\mu v(s) - D^\mu \overline{v}(s) \right| \right) ds \\ &+ \int_0^1 \frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} \left(K_f |v(s) - \overline{v}(s)| + L_f \left| D^\mu v(s) - D^\mu \overline{v}(s) \right| \right) ds. \end{aligned}$$

Now,

$$\begin{aligned} D^\nu u(t) &= D^\nu (I^\alpha f\left(t, v(t), D^\mu v(t)\right) - t^{\alpha-1} I^\alpha \ f\left(t, v(t), D^\mu v(t)\right)\Big|_{t=1} \\ &= I^{\alpha-\nu} f\left(t, v(t), D^\mu v(t)\right) - I^\alpha \ f\left(1, v(1), D^\mu v(1)\right)\Big|_{t=1} D^\nu t^{\alpha-1} \\ &= \int_0^t \frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} f\left(s, v(s), D^\mu v(s)\right) ds \\ &- \int_0^1 \frac{(t(1-s))^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} (1-s)^\nu \ f\left(s, v(s), D^\mu v(s)\right) ds. \end{aligned}$$

By the use of (S_1), (S_2) and the fact that $0 < (1-s)^\nu < 1$, we obtain

$$\begin{aligned} \left| D^\nu u(t) - D^\nu \overline{u}(t) \right| \leq &\int_0^t \frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} \left(\left| f\left(s, v(s), D^\mu v(s)\right) - f\left(s, \overline{v}(s), D^\mu \overline{v}(s)\right) \right| \right) ds \\ &+ \int_0^t \left| \frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} - \frac{(t-s)^{\overline{\alpha}-\nu-1}}{\Gamma(\overline{\alpha}-\nu)} \right| \left| f\left(s, \overline{v}(s), D^\mu \overline{v}(s)\right) \right| ds \\ &+ \int_0^1 \frac{(t(1-s))^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} \left(\left| f\left(s, v(s), D^\mu v(s)\right) - f\left(s, \overline{v}(s), D^\mu \overline{v}(s)\right) \right| \right) ds \\ &+ \int_0^1 \left| \frac{(t(1-s))^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)} - \frac{(t(1-s))^{\overline{\alpha}-\nu-1}}{\Gamma(\overline{\alpha}-\nu)} \right| \left| f\left(s, \overline{v}(s), D^\mu \overline{v}(s)\right) \right| ds, \end{aligned}$$

$$\begin{aligned}
&\left|D^{\nu}u(t)-D^{\nu}\overline{u}(t)\right| \\
&\quad\leq K\left|\alpha-\overline{\alpha}\right|^{r}+\int_0^t\frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\left|f\left(s,v(s),D^{\mu}v(s)\right)-f\left(s,\overline{v}(s),D^{\mu}\overline{v}(s)\right)\right|ds \\
&\qquad+\int_0^1\frac{(t(1-s))^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\left|f\left(s,v(s),D^{\mu}v(s)\right)-f\left(s,\overline{v}(s),D^{\mu}\overline{v}(s)\right)\right|ds \\
&\quad\leq K\left|\alpha-\overline{\alpha}\right|^{r}+\int_0^t\frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\left(K_f\left|v(s)-\overline{v}(s)\right|+L_f\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|\right)ds \\
&\qquad+\int_0^1\frac{(t(1-s))^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\left(K_f\left|v(s)-\overline{v}(s)\right|+L_f\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|\right)ds.
\end{aligned}$$

In the same way, we can bound $\left|v(t)-\overline{v}(t)\right|$ and $\left|D^{\mu}v(t)-D^{\mu}\overline{v}(t)\right|$.

Now, we apply the Gronwall theorem, and we obtain,

$$\begin{aligned}
&\max_{t\in[0,1]}\left[\left|u(t)-\overline{u}(t)\right|+\left|D^{\nu}u(t)-D^{\nu}\overline{u}(t)\right|\right]\leq K\left|\alpha-\overline{\alpha}\right|^{r} \\
&\quad+\max\left(K_f,L_f\right)\int_0^1\left(\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t(1-s))^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\right) \\
&\quad\times\max_{s\in[0,1]}\left(\left|v(s)-\overline{v}(s)\right|+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|\right)ds \\
&\quad+\max\left(K_f,L_f\right)\int_0^t\left[\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\right)\left(\left|v(s)-\overline{v}(s)\right|\right.\right. \\
&\quad\left.\left.+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|\right)\right]ds,
\end{aligned}$$

and

$$\begin{aligned}
&\max_{t\in[0,1]}\left(\left|v(t)-\overline{v}(t)\right|+\left|D^{\mu}v(t)-D^{\mu}\overline{v}(t)\right|\right)\leq L\left|\beta-\overline{\beta}\right|^{\overline{r}} \\
&\quad+\max\left(K_g,L_g\right)\int_0^1\left(\frac{(t(1-s))^{\beta-1}}{\Gamma(\beta)}+\frac{(t(1-s))^{\beta-\mu-1}}{\Gamma(\beta-\mu)}\right) \\
&\quad\times\max_{s\in[0,1]}\left(\left|u(s)-\overline{u}(s)\right|+\left|D^{\nu}u(s)-D^{\nu}\overline{u}(s)\right|\right)ds \\
&\quad+\int_0^t\left[\left(\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}+\frac{(t-s)^{\beta-\mu-1}}{\Gamma(\beta-\mu)}\right)\right. \\
&\quad\left.\times\ \max\left(K_g,L_g\right)\left(\left|u(s)-\overline{u}(s)\right|+\left|D^{\mu}u(s)-D^{\mu}\overline{u}(s)\right|\right)\right]ds.
\end{aligned}$$

Let us define the function δ by

$$\begin{aligned}
\left|\delta(t)\right|=\delta(t):=&\left|u(t)-\overline{u}(t)\right|+\left|D^{\nu}u(t)-D^{\nu}\overline{u}(t)\right| \\
&+\left|v(t)-\overline{v}(t)\right|+\left|D^{\mu}v(t)-D^{\mu}\overline{v}(t)\right|.
\end{aligned}$$

According to (S_3) and (S_4), we have

$$
\begin{aligned}
|\delta(t)| &\leq \max_{t\in[0,1]}\left[|u(t)-\overline{u}(t)|+\left|D^{\nu}u(t)-D^{\nu}\overline{u}(t)\right|\right]+\max_{t\in[0,1]}\left[|v(t)-\overline{v}(t)|+\left|D^{\mu}v(t)-D^{\mu}\overline{v}(t)\right|\right]\\
&\leq K|\alpha-\overline{\alpha}|^{r}+L\left|\beta-\overline{\beta}\right|^{\overline{r}}+\int_0^t\left[\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\right)\max\left(K_f,L_f,K_g,L_g\right)\right.\\
&\quad\left.\left(|v(s)-\overline{v}(s)|+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|+|u(s)-\overline{u}(s)|+\left|D^{\mu}u(s)-D^{\mu}\overline{u}(s)\right|\right)\right]ds\\
&\quad+\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\nu+1)}\right)\max\left(K_f,L_f\right)\max\left(|v(s)-\overline{v}(s)|+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|\right)\\
&\quad+\left(\frac{1}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta-\mu+1)}\right)\max\left(K_g,L_g\right)\max\left(|u(s)-\overline{u}(s)|+\left|D^{\nu}u(s)-D^{\nu}\overline{u}(s)\right|\right),
\end{aligned}
$$

setting $\Lambda_{K,L}=\max\left(K_f,L_f,K_g,L_g\right)$, we obtain,

$$
\begin{aligned}
|\delta(t)| &\leq K|\alpha-\overline{\alpha}|^{r}+L\left|\beta-\overline{\beta}\right|^{\overline{r}}+\Lambda_{K,L}\int_0^t\left[\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\right)\right.\\
&\quad\left(|v(s)-\overline{v}(s)|+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|+|u(s)-\overline{u}(s)|\right.\\
&\quad\left.\left.+\left|D^{\mu}u(s)-D^{\mu}\overline{u}(s)\right|\right)\right]ds\\
&\quad+\Delta\left[\max_{s\in[0,1]}\left(|v(s)-\overline{v}(s)|+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|\right)\right.\\
&\quad\left.+\max_{s\in[0,1]}\left(|u(s)-\overline{u}(s)|+\left|D^{\nu}u(s)-D^{\nu}\overline{u}(s)\right|\right)\right]
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&(1-\Delta)\left[\max_{t\in[0,1]}\left(|v(t)-\overline{v}(t)|+\left|D^{\mu}v(t)-D^{\mu}\overline{v}(t)\right|\right)\right.\\
&\quad\left.+\max_{t\in[0,1]}\left(|u(t)-\overline{u}(t)|+\left|D^{\nu}u(t)-D^{\nu}\overline{u}(t)\right|\right)\right]\\
&\leq K|\alpha-\overline{\alpha}|^{r}+L\left|\beta-\overline{\beta}\right|^{\overline{r}}+\Lambda_{K,L}\int_0^t\left[\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\right)\right.\\
&\left.\left(|v(s)-\overline{v}(s)|+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|+|u(s)-\overline{u}(s)|+\left|D^{\mu}u(s)-D^{\mu}\overline{u}(s)\right|\right)\right]ds
\end{aligned}
$$

$$
\begin{aligned}
|\delta(t)| &\leq \max_{t\in[0,1]}\left(|v(t)-\overline{v}(t)|+\left|D^{\mu}v(t)-D^{\mu}\overline{v}(t)\right|\right)\\
&\quad+\max_{t\in[0,1]}\left(|u(t)-\overline{u}(t)|+\left|D^{\nu}u(t)-D^{\nu}\overline{u}(t)\right|\right)\\
&\leq \frac{K|\alpha-\overline{\alpha}|^{r}}{(1-\Delta)}+\frac{L\left|\beta-\overline{\beta}\right|^{\widetilde{r}}}{(1-\Delta)}+\frac{\Lambda_{K,L}}{(1-\Delta)}\int_0^t\left[\left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}\right)\right.\\
&\quad\left.\left(|v(s)-\overline{v}(s)|+\left|D^{\mu}v(s)-D^{\mu}\overline{v}(s)\right|+|u(s)-\overline{u}(s)|+\left|D^{\mu}u(s)-D^{\mu}\overline{u}(s)\right|\right)\right]ds.
\end{aligned}
$$

Thus,

$$|\delta(t)| \leq \frac{K|\alpha-\overline{\alpha}|^{r}}{(1-\Delta)} + \frac{L\left|\beta-\overline{\beta}\right|^{\overline{r}}}{(1-\Delta)} + \frac{2\Lambda_{K,L}}{(1-\Delta)}\int_0^t \frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}|\delta(s)|\,ds,$$

we set,

$$c := \max\left(\frac{K}{1-\Delta}, \frac{L}{1-\Delta}\right) \text{ and } \lambda = \frac{2\Lambda_{K,L}}{(1-\Delta)}.$$

So,

$$\delta(t) \leq c\left(|\alpha-\overline{\alpha}| + \left|\beta-\overline{\beta}\right|\right) + \lambda\int_0^t \frac{(t-s)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}|\delta(s)|\,ds,$$

which means that,

$$0 \leq \delta(t) \leq c\left(|\alpha-\overline{\alpha}| + \left|\beta-\overline{\beta}\right|\right) E_{\alpha-\nu}(\lambda) < \epsilon.$$

Finally, the defintion 9 implies the stability of solutions to the system $(1)_{(\alpha,\beta)}$. ■

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