

## On degenerate numbers and polynomials related to the Stirling numbers and the Bell polynomials

Cheon Seoung Ryoo

*Department of Mathematics,  
Hannam University, Daejeon 306-791, Korea.*

### Abstract

In this paper, we consider the degenerate numbers  $R_n(\lambda)$  and polynomials  $R_n(x, \lambda)$  related to the Stirling numbers and the Bell polynomials. We also obtain some explicit formulas for degenerate numbers  $R_n(\lambda)$  and polynomials  $R_n(x, \lambda)$ .

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### 1. Introduction

Recently, many mathematicians have studied the area of the Stirling numbers, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the degenerate Bernoulli numbers and polynomials, the degenerate Euler numbers and polynomials (see [1, 2, 5, 6, 7, 8, 9]). In [2], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.* [7] studied the partially degenerate Bernoulli polynomials of the first kind in  $p$ -adic field. In [3, 4], we introduced the polynomials  $R_n(x)$  and numbers  $R_n$  related to the Stirling numbers and the Bell polynomials. Also, from this polynomials, we obtained some relation between the Stirling numbers, the Bell numbers, the  $R_n$  and  $R_n(x)$ . In this paper, we establish some interesting properties for degenerate numbers  $R_n(\lambda)$  and polynomials  $R_n(x, \lambda)$ . Throughout this paper, we always make use of the following notations:  $\mathbb{N} = \{1, 2, 3, \dots\}$  denotes the set of natural numbers and  $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$  denotes the set of nonnegative integers,  $\mathbb{Z}$  denotes the set of integers.

Let us define the numbers  $R_n$  and polynomials  $R_n(x)$  as follows:

$$\left( \frac{2}{e^{e^t-1} + 1} \right) = \sum_{n=0}^{\infty} R_n \frac{t^n}{n!}, \quad (1.1)$$

$$\left(\frac{2}{e^{e^t-1} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} R_n(x) \frac{t^n}{n!}. \quad (1.2)$$

Observe that if  $x = 0$ , then  $R_n(0) = R_n$ . For more theoretical properties of the numbers  $R_n$  and polynomials  $R_n(x)$ , the readers may refer to [3, 4]. We recall that the classical Stirling numbers of the first kind  $S_1(n, k)$  and  $S_2(n, k)$  are defined by the relations(see [9])

$$(x)_n = \sum_{k=0}^n S_1(n, k)x^k \text{ and } x^n = \sum_{k=0}^n S_2(n, k)(x)_k,$$

respectively. Here  $(x)_n = x(x-1) \cdots (x-n+1)$  denotes the falling factorial polynomial of order  $n$ . The numbers  $S_2(n, m)$  also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!}. \quad (1.3)$$

We also have

$$\sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \quad (1.4)$$

The generalized falling factorial  $(x|\lambda)_n$  with increment  $\lambda$  is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1} (x - \lambda k) \quad (1.5)$$

for positive integer  $n$ , with the convention  $(x|\lambda)_0 = 1$ ; we may also write

$$(x|\lambda)_n = \sum_{k=0}^n S_1(n, k) \lambda^{n-k} x^k. \quad (1.6)$$

Note that  $(x|\lambda)$  is a homogeneous polynomials in  $\lambda$  and  $x$  of degree  $n$ , so if  $\lambda \neq 0$  then  $(x|\lambda)_n = \lambda^n (\lambda^{-1}x|1)_n$ . Clearly  $(x|0)_n = x^n$ . We also need the binomial theorem: for a variable  $x$ ,

$$(1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}. \quad (1.7)$$

As well known definition, the Bell polynomials are defined by Bell(1934) as below

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = e^{(e^t-1)x}. \quad (1.8)$$

Also, let  $S_2(n, k)$  be denote the Stirling numbers of the second kind. Then

$$B_n(x) = \sum_{k=0}^n S_2(n, k)x^k. \quad (1.9)$$

In the special case,  $B_n(1) = B_n$  are called the  $n$ -th Bell numbers.

## 2. On the degenerate numbers $R_n(\lambda)$ and polynomials $R_n(x, \lambda)$

In this section, we define the degenerate numbers  $R_n(\lambda)$  and polynomials  $R_n(x, \lambda)$ , and we obtain explicit formulas for them. For a variable  $t$ , we consider the degenerate polynomials  $R_n(x, \lambda)$  which are given by the generating function to be

$$\frac{2}{e^{(1+\lambda t)^{1/\lambda}-1} + 1} (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!}. \tag{2.1}$$

When  $x = 0$ ,  $R_n(0, \lambda) = R_n(\lambda)$  are called the degenerate numbers  $R_n(\lambda)$ . Note that  $(1 + \lambda t)^{1/\lambda}$  tends to  $e^t$  as  $\lambda \rightarrow 0$ .

From (2.1) and (1.2), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} R_n(x, \lambda) \frac{t^n}{n!} &= \lim_{\lambda \rightarrow 0} \frac{2}{e^{(1+\lambda t)^{1/\lambda}-1} + 1} (1 + \lambda t)^{x/\lambda} \\ &= \sum_{n=0}^{\infty} R_n(x) \frac{t^n}{n!}. \end{aligned}$$

Thus, we get

$$\lim_{\lambda \rightarrow 0} R_n(x, \lambda) = R_n(x), \quad (n \geq 0).$$

From (2.1) and (1.7), we have

$$\begin{aligned} \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!} &= \frac{2}{e^{(1+\lambda t)^{1/\lambda}-1} + 1} (1 + \lambda t)^{x/\lambda} \\ &= \left( \sum_{m=0}^{\infty} R_m(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (x|\lambda)_l \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} R_l(\lambda) (x|\lambda)_{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

**Theorem 2.1.** For  $n \geq 0$ , we have

$$R_n(x, \lambda) = \sum_{l=0}^n \binom{n}{l} R_l(\lambda) (x|\lambda)_{n-l}.$$

Applying binomial theorem and Cauchy’s rule for product of two power series will

yield

$$\begin{aligned}
 2 &= (e^{(1+\lambda t)^{1/\lambda}-1} + 1) \sum_{n=0}^{\infty} R_n(\lambda) \frac{t^n}{n!} \\
 &= (e^{(1+\lambda t)^{1/\lambda}-1}) \sum_{n=0}^{\infty} R_n(\lambda) \frac{t^n}{n!} + \sum_{n=0}^{\infty} R_n(\lambda) \frac{t^n}{n!} \\
 &= \left( \sum_{n=0}^{\infty} \frac{1}{e} \sum_{k=0}^{\infty} (k|\lambda)_n \frac{1}{k!} \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} R_m(\lambda) \frac{t^m}{m!} \right) + \sum_{n=0}^{\infty} R_n(\lambda) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} \frac{1}{e} \sum_{k=0}^{\infty} (k|\lambda)_l \frac{1}{k!} R_{n-l}(\lambda) + R_n(\lambda) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.3}$$

By comparing of the coefficients  $\frac{t^n}{n!}$  on the both sides of (2.3), we have the following theorem.

**Theorem 2.2.** For  $n \in \mathbb{Z}_+$ , we have

$$\frac{1}{e} \sum_{l=0}^n \binom{n}{l} \sum_{k=0}^{\infty} (k|\lambda)_l \frac{1}{k!} R_{n-l}(\lambda) + R_n(\lambda) = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

By (2.1), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} 2(x|\lambda)_n \frac{t^n}{n!} &= 2(1 + \lambda t)^{x/\lambda} \\
 &= (e^{(1+\lambda t)^{1/\lambda}-1} + 1) \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!} \\
 &= (e^{(1+\lambda t)^{1/\lambda}-1}) \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!} + \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!} \\
 &= \left( \sum_{n=0}^{\infty} \frac{1}{e} \sum_{k=0}^{\infty} (k|\lambda)_n \frac{1}{k!} \frac{t^n}{n!} \right) \left( \sum_{m=0}^{\infty} R_m(x, \lambda) \frac{t^m}{m!} \right) + \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} \frac{1}{e} \sum_{k=0}^{\infty} (k|\lambda)_l \frac{1}{k!} R_{n-l}(x, \lambda) + R_n(x, \lambda) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{2.4}$$

Comparing the coefficients of  $\frac{t^n}{n!}$  on the both sides of (2.4), we have the following theorem.

**Theorem 2.3.** For  $n \in \mathbb{Z}_+$ , we have

$$\sum_{l=0}^n \binom{n}{l} \sum_{k=0}^{\infty} (k|\lambda)_l \frac{1}{k!} R_{n-l}(x, \lambda) + R_n(x, \lambda) = 2e(x|\lambda)_n.$$

By (2.1) and (1.7), we get

$$\begin{aligned} \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!} &= 2 \sum_{l=0}^{\infty} (-1)^l e^{-l} e^{l(1+\lambda t)^{1/\lambda}} (1 + \lambda t)^{x/\lambda} \\ &= 2 \sum_{l=0}^{\infty} (-1)^l e^{-l} \sum_{k=0}^{\infty} (1 + \lambda t)^{k/\lambda} \frac{l^k}{k!} (1 + \lambda t)^{x/\lambda} \\ &= 2 \sum_{l=0}^{\infty} (-1)^l e^{-l} \sum_{k=0}^{\infty} l^k (1 + \lambda t)^{(k+x)/\lambda} \frac{1}{k!} \\ &= \sum_{n=0}^{\infty} \left( 2 \sum_{l=0}^{\infty} (-1)^l e^{-l} \sum_{k=0}^{\infty} l^k (k + x|\lambda)_n \frac{1}{k!} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.5}$$

By comparing of the coefficients  $\frac{t^n}{n!}$  on the both sides of (2.5), we have the following theorem.

**Theorem 2.4.** For  $n \in \mathbb{Z}_+$ , we have

$$R_n(x, \lambda) = 2 \sum_{l=0}^{\infty} (-1)^l e^{-l} \sum_{k=0}^{\infty} (k + x|\lambda)_n \frac{l^k}{k!}.$$

From (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} R_n(x + y, \lambda) \frac{t^n}{n!} &= \frac{2}{e^{(1+\lambda t)^{1/\lambda}-1} + 1} (1 + \lambda t)^{(x+y)/\lambda} \\ &= \frac{2}{e^{(1+\lambda t)^{1/\lambda}-1} + 1} (1 + \lambda t)^{x/\lambda} (1 + \lambda t)^{y/\lambda} \\ &= \left( \sum_{n=0}^{\infty} R_n(x, \lambda) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} (y|\lambda)_n \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} R_l(x, \lambda) (y|\lambda)_{n-l} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.6}$$

Therefore, by (2.6), we have the following theorem.

**Theorem 2.5.** For  $n \in \mathbb{Z}_+$ , we have

$$R_n(x + y, \lambda) = \sum_{k=0}^n \binom{n}{k} R_k(x, \lambda) (y|\lambda)_{n-k}.$$

By replacing  $t$  by  $\frac{e^{\lambda t} - 1}{\lambda}$  in (2.1), we obtain

$$\begin{aligned} \left( \frac{2}{e^{e^t-1} + 1} \right) e^{xt} &= \sum_{n=0}^{\infty} R_n(x, \lambda) \left( \frac{e^{\lambda t} - 1}{\lambda} \right)^n \frac{1}{n!} \\ &= \sum_{n=0}^{\infty} R_n(x, \lambda) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \lambda^m \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left( \sum_{n=0}^m R_n(x, \lambda) \lambda^{m-n} S_2(m, n) \right) \frac{t^m}{m!}. \end{aligned} \quad (2.7)$$

Thus, by (2.7) and (1.2), we have the following theorem.

**Theorem 2.6.** For  $n \in \mathbb{Z}_+$ , we have

$$R_m(x) = \sum_{n=0}^m \lambda^{m-n} R_n(x, \lambda) S_2(m, n).$$

By replacing  $t$  by  $\log(1 + \lambda t)^{1/\lambda}$  in (1.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} R_n(x) (\log(1 + \lambda t)^{1/\lambda})^n \frac{1}{n!} &= \frac{2}{e^{(1+\lambda t)^{2/\lambda} - 1} + 1} (1 + \lambda t)^{x/\lambda} \\ &= \sum_{m=0}^{\infty} R_m(x, \lambda) \frac{t^m}{m!}, \end{aligned} \quad (2.8)$$

and

$$\sum_{n=0}^{\infty} R_n(x) (\log(1 + \lambda t)^{1/\lambda})^n \frac{1}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^m R_n(x) \lambda^{m-n} S_1(m, n) \right) \frac{t^m}{m!}. \quad (2.9)$$

Thus, by (2.8) and (2.9), we have the following theorem.

**Theorem 2.7.** For  $n \in \mathbb{Z}_+$ , we have

$$R_m(x, \lambda) = \sum_{n=0}^m \lambda^{m-n} R_n(x) S_1(m, n).$$

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