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Performance Analysis of Half-sweep SOR Iteration with Rotated Nonlocal Arithmetic Mean Scheme for 2D Nonlinear Elliptic Problems

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Abstract

The aim of this paper is to examine the effectiveness of Half-Sweep Successive Over Relaxation (HSSOR) method with nonlocal discretization scheme which is derived based on the four-point rotated nonlocal arithmetic mean scheme in solving nonlinear elliptic boundary value problems. By using an approximate equation based on the second order finite difference scheme, the half-sweep approximation equation has been derived. Then, the nonlocal discretization scheme is applied to transform the system of nonlinear approximation equations into the corresponding system of linear equations. Throughout numerical results, it can be pointed out that the proposed HSSOR method was superior in terms of number of iterations, execution time and maximum error compared to Full-Sweep Successive Over-relaxation (FSSOR) and Half-Sweep Gauss-Seidel (HSGS).

Keywords: Nonlinear Elliptic Boundary Value Problems; Nonlocal Arithmetic Mean Scheme; Half-Sweep SOR iteration.

1. Introduction

Nonlinear elliptic boundary value problems occur in real time application such as numerical weather forecasting, radioactive transfer, optimal control and other areas of physics and engineering. Many numerical methods have been proposed to solve nonlinear two-dimensional elliptic problems such as non-polynomial spline scheme [1], Pade' approximation [2], collocation method [3], spline scheme [4], finite element methods [5], finite difference methods [6]and numerical integration method [7].

To get approximate solution of nonlinear elliptic boundary value problems, the paper deals with the finite difference method to discretize the proposed problem in order to develop a reliable algorithm. Based on the previous studies of the linear case, it can be observed that many researchers have also proposed and formulated high-order finite difference approximation equations, see in [8,9,10,11,12]. However in this paper, second order rotated finite difference approximation equations are used to construct the system of nonlinear equations. To get numerical solutions of nonlinear system iteratively, the nonlocal discretization scheme is imposed into the nonlinear system in order to develop the corresponding system of linear equations. Actually in the case of linear systems, many researchers proposed various iterative methods which are used to accelerate convergence rate in solving any linear systems. For instance, in year 1985 Evan proposed Group Explicit iterative method [13]. The, this method was extended by using half-sweep iteration concept [14,15,16,17,18]. As a matter of fact, the concept of half-sweep iteration is actually to reduce the computational complexities during iteration process. This is because of the implementation of the half-sweep iterations only considers nearly half of whole node points in a solution domain respectively. Due to the large scale of the generated linear system, the paper deals with the application of half-sweep iteration concept which is known as Half-sweep SOR method together with nonlocal discretization scheme for solving the two-dimensional nonlinear elliptic problems. Hence, the outlines of this paper were organized in following ways: Section 2 will show the formulation of nonlocal arithmetic mean schemes. Next, the explanation of FSGS, FSSOR and HSSOR iterative methods in Section 3 will be given and some numerical results will be shown in Section 4 to state the effectiveness of the proposed methods. Furthermore, the conclusion is mentioned in Section 5.

Now suppose that nonlinear two-dimensional elliptic boundary value problem is given in general form as

$$u_{xx} + u_{yy} = F(x, y, u, u_x, u_y) \quad (x, y) \in \Omega = [a,b] \times [a,b] \tag{1}$$

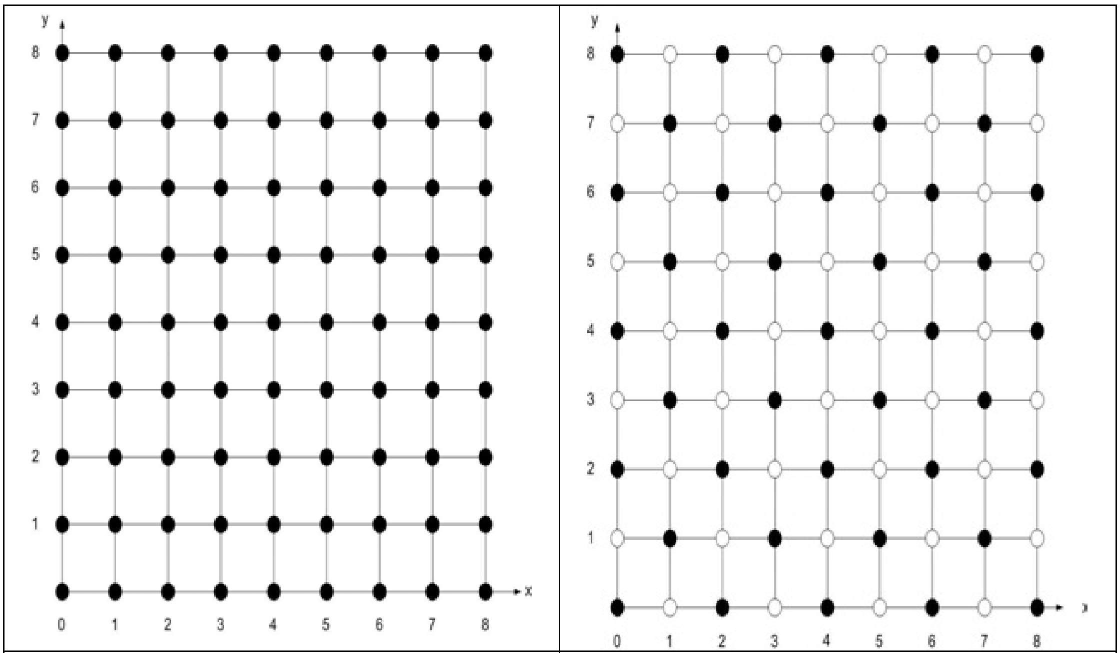
With

$$F(x, y, u, u_x, u_y) = -uu_x - uu_y + f(x, y, u) \tag{2}$$

subject to the boundary conditions

$$u(x, y) = g(x, y), \qquad \partial\Omega$$

where Ω is an arbitrary simply connected bounded region with smooth boundary $\partial\Omega$ and $f(x, y, u)$ and $g(x, y)$ are continuous functions in the respective domain. From Eq. (1), it can be observed that the function $F(x, y, u, u_x, u_y)$ is classified as nonlinear terms.



(a) (b)

Figure. 1: Finite grid networks for the full-sweep (a) and half-sweep (b) in case m=8

In formulating various iterative schemes such as full-and half-sweep cases, we need to build the finite grid network as a guide for development and implementation of the proposed methods. Figure 1 act as a guide to develop the proposed methods. Let us consider the finite rectangular grid network with spacing grid *h* as shown in Figure 1. Assume that the spacing grid *h* in which both directions with $x_i = x_0 + ih,\ y_j = y_0 + jh$ are defined as

$$h = \Delta x = \Delta y = \frac{(b-a)}{m} \tag{3}$$

where Δx and Δy represent the increment of *x* and *y* directions respectively while *m* is number of subintervals. Then let $U(x_i, y_i) = u_{i,j}$ indicates the approximation value of function *u* at point (x_i, y_i).

2. Formulation of Half-Sweep Nonlocal Arithmetic Mean Schemes

Before constructing the half-sweep finite difference approximation equation of problem (1), let us consider several nonlocal discretization schemes proposed by Moaddy *et al.* (2011) over one dimensional problems as follows:

$$U_i^2 \cong U_i U_{i+1} \tag{4}$$

$$U_i^2 \cong \left(\frac{U_{i-1} + U_{i+1}}{2}\right) U_i \tag{5}$$

$$U_i^3 \cong \left(\frac{U_{i-1}+U_{i+1}}{2}\right)U_i^2 \quad (6)$$

Actually, the formulation of nonlocal discretization scheme in Eq. (5) derived by using the concept of two-point arithmetic mean scheme which is defined as

$$U_i^2 \cong \frac{1}{2}\left(U_{i-1}+U_{i+1}\right) \quad (7)$$

To obtain an approximate value of U_{ij} for the two-dimensional problems, let us consider the approximate value of U_{ij} based on the four-point Nonlocal Arithmetic Mean (NAM) Scheme as follows:

$$U_{i,j} \cong \frac{1}{4}\left(U_{i-1,j}+U_{i+1,j}+U_{i,j-1}+U_{i,j+1}\right) \quad (8)$$

or Eq. (8) can be illustrated in stencil form as

$$U_{i,j} \cong \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} U_{ij} \quad (9)$$

Note that the Eq. (8) clearly considers a group of four-points and the value of U_{ij} is categorized as a reference point, then four neighbor points, $U_{i-1,j}, U_{i+1,j}, U_{i,j-1}$ and $U_{i,j+1}$ are used as a nonlocal approach to evaluate the approximate value of U_{ij}. Therefore, by using the approach of second-order centered finite difference discretization scheme, we obtain the following five-point full-sweep finite difference approximation equation of problems (1) as follows

$$\frac{U_{i-1,j}-2U_{i,j}+U_{i+1,j}}{h^2}+\frac{U_{i,j-1}-2U_{i,j}+U_{i,j+1}}{h^2}=f\left(x_i, y_j, U_{ij}, \frac{U_{i+1,j}-U_{i-1,j}}{2h}, \frac{U_{i,j+1}-U_{i,j-1}}{2h}\right) \quad (10)$$

Then, by imposing Eq. (10), we have the following linear approximation equations

$$U_{i-1,j}+U_{i+1,j}+U_{i,j+1}+U_{i,j-1}-4U_{i,j}-h^2 f_{ij}=0, \quad i, j=1,2,.\cdots,n \quad (11)$$

Where

$$f_{i,j}=f\left(x_i, y_j, \frac{1}{4}\left(U_{ij}\right), \frac{U_{i+1,j}-U_{i-1,j}}{2h}, \frac{U_{i,j+1}-U_{i,j-1}}{2h}\right) \quad (12)$$

Apart from the full-sweep approximation equation in Eq. (10), Abdullah [20] has introduced the half-sweep concept via the second-order rotated finite difference scheme to derive five-point rotated finite difference approximation equation for 2D linear elliptic problems. By adopting the same idea, we proposed the half-sweep iteration concept applied to the Eq. (8) by introducing a new formulation of four-point Nonlocal Arithmetic Mean (NAM) Scheme. To do this, by referring in Figure 1(b), we initiate a four-point rotated nonlocal arithmetic mean scheme as follows:

$$U_{i,j} \cong \frac{1}{4}\left(U_{i-1,j-1}+U_{i+1,j+1}+U_{i-1,j+1}+U_{i+1,j-1}\right) \quad (13)$$

or Eq. (15) can be illustrated in stencil form as

$$U_{i,j} \cong \frac{1}{4}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} U_{ij} \tag{14}$$

In similar way to derive the formulae of Eqs. (5) and (6) by using the four-point rotated nonlocal arithmetic mean scheme, we produce the following four-point rotated arithmetic mean schemes as:

$$U^2_{i,j} \cong \frac{1}{4}\left(U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1}\right)U_{i,j} \tag{15}$$

$$U^3_{i,j} \cong \frac{1}{4}\left(U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1}\right)U^2_{i,j} \tag{16}$$

Actually the concept of arithmetic mean approach has been widely used to derive several formulations in numerical methods such as Arithmetic Mean (AM) [21], 2-Point Block Arithmetic Mean [22] and Half-Sweep Arithmetic Mean (HSAM) [23] methods. Therefore, by using four-point rotated nonlocal arithmetic mean scheme, we obtain the following five-point rotated finite difference approximation equation of problems (1) as follows

$$U_{i-1,j-1} + U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} - 4U_{i,j} - 2h^2 f_{ij} = 0 \tag{17}$$

Where

$$f_{i,j} = f\left(x_i, y_j, U_{ij}, \frac{U_{i+1,j+1} - U_{i-1,j-1}}{2\left(\sqrt{2h}\right)}, \frac{U_{i-1,j+1} - U_{i+1,j-1}}{2\left(\sqrt{2h}\right)} \right) \tag{18}$$

Actually, Eq. (18) is called as the nonlinear term of the problem (1) for half-sweep case. To solve the nonlinear system in Eq. (17), the four-point rotated nonlocal arithmetic mean approach in Eq. (13) being imposed over the nonlinear function in (18). Therefore, the approximation equation (17) can be rewritten as

$$f_{i,j} = f\left(x_i, y_j, \frac{1}{4}\left(U_{i-1,j-1} + U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1}\right), \frac{U_{i+1,j+1} - U_{i-1,j-1}}{2\left(\sqrt{2h}\right)}, \frac{U_{i-1,j+1} - U_{i+1,j-1}}{2\left(\sqrt{2h}\right)} \right) \tag{19}$$

Then both of the approximation equations (17) and (18) can be used to construct the following linear system in matrix form or

$$A\underset{\sim}{U} = \underset{\sim}{f} \tag{20}$$

3. Formulation of Successive Over Relaxation Method

As mentioned in the previous section, Refer to the linear system (20), it can be seen that the characteristic of the coefficient matrix of the linear system has large scale and sparse. Therefore, the iterative methods are suitable option to solve the linear system [24]. Actually, the SOR iterative method was proposed by Young [25]. The standard SOR method is also named as Full-sweep SOR (FSSOR) method. According to that, we consider the application of HSSOR method as linear solver. Particularly, the HSSOR method is essentially the extension of the FSSOR iterative method. The main purpose of the half-sweep iteration is to reduce the computational complexities during iteration process. Let the linear system (21) be expressed as summation of the three matrices

$$A = D - L - V \tag{21}$$

where *D*, *L* and *V* are diagonal, lower triangular and upper triangular matrices respectively.

According to Eq. (13), the HSSOR iterative method can be defined generally as [26]:

$$\underset{\sim}{U}^{(k+1)} = (D - \omega L)^{-1}\left[(\omega V - (1-\omega L)D)\underset{\sim}{U}^{(k)}\right] + (D - \omega L)^{-1}\underset{\sim}{f} \tag{22}$$

Where ω and $U_{ij}^{(k)}$ represent as a relaxation factor and the kth the estimation for corresponding exact solutions respectively.

As taking $\omega = 1$, the HSSOR reduces to HSGS. In this paper, the FSGS iterative method will be used as a control method. In addition to that, a good choice for the value of the parameter ω can be used to accelerate the convergence rate of the iteration process. In practice, the optimal value of ω in range $1 \le \omega < 2$ will be obtained by implementing several computer programs and then the best approximate value of ω is chosen in which its number of iterations is the smallest. Also the implementation of the HSSOR iterative method may be described in Algorithm 1.

Algorithm 1 : HSSOR scheme

i. Initialize $U_i^{(0)} \leftarrow 0, \varepsilon \leftarrow 10^{-10}$

ii. Assign the optimal value of ω

iii. Calculate $U_i^{(k+1)}$ using

$$\underset{\sim}{U}^{(k+1)} = (D - \omega L)^{-1}\left[(\omega V - (1-\omega L)D)\underset{\sim}{U}^{(k)}\right] + (D - \omega L)^{-1}\underset{\sim}{f}$$

iv. Check the convergence test, $\left|U_i^{(k+1)} - U_i^{(k)}\right| \le \varepsilon = 10^{-10}$. If yes, go to step (v). Otherwise go back to step (iii).

v. Display approximate solutions.

4. Numerical Experiment

In order to investigate the performance of the proposed iterative methods together with the corresponding NAM, we consider three examples of nonlinear elliptic equations being used to demonstrate effectiveness of the HSSOR compared with HSGS and FSSOR iterative methods. In comparison, three criterions will be considered such as number of iterations, execution time (in seconds) and maximum absolute error. For implementation of these three iterative schemes, the convergence test considered the tolerance error, which is fixed as $\varepsilon = 10^{-10}$.

Examples 1[27]

$$u_{xx} + u_{yy} + uu_x + uu_y = \exp(u), \quad \Omega \tag{23}$$

where exact solution is defined by

$u(x,y) = x^2 + y, \quad \Omega$ (24)

Example 2 [27]

$u_{xx} + u_{yy} - uu_x - uu_y = \exp(u), \quad \Omega$ (25)

where the exact solution is defined by

$u(x,y) = \exp(-x)\sin \pi y, \quad \Omega$ (26)

Example 3 [27]

$u_{xx} + u_{yy} - uu_y = \exp(-x)\sin \pi y\left(1 - \pi^2 - \pi \exp-(x)\cos \pi y\right), \quad \Omega$ (27)

where the exact solution is defined by

$u(x,y) = \exp(-x)\sin \pi y, \quad \Omega$ (28)

All the simulations were implemented by C programming language. Results of numerical simulations, which were obtained from implementation of FSGS, FSSOR, and HSSOR iterative methods have been recorded in Table 1 at different values of mesh sizes, m = 32, 64, 128, 256 and 512.

Table 1: Comparison of Number Iterations (K), Execution Time (seconds) and Maximum Errors for The Iterative Methods Using Examples at grid size 32, 64, 128, 256 and 512.

EXAMPLE 1				EXAMPLE 2			EXAMPLE 3		
Number of iteration									
M	**FSGS**	**FSSOR**	**HSSOR**	**FSGS**	**FSSOR**	**HSSOR**	**FSGS**	**FSSOR**	**HSSOR**
32	1806	136 ω = 1.815600	97 ω = 1.748000	1893	135 ω =1.820792	91 ω =1.760000	1874	130 ω =1.824111	90 ω =1.757177
64	6688	266 ω =1.902690	194 ω =1.864400	7004	258 ω =1.908310	184 ω =1.871200	6933	259 ω =1.908173	182 ω =1.870220
128	24616	524 ω = 1.950120	387 ω = 1.929500	25717	522 ω =1.951567	365 ω =1.933100	25454	514 ω =1.953209	358 ω =1.932445
256	89975	1035 ω =1.974601	772 ω = 1.963800	93638	1026 ω =1.976000	725 ω =1.965800	92667	1025 ω =1.975365	713 ω =1.965414
512	326055	2058 ω =1.987200	1538 ω = 1.981500	327035	2050 ω =1.987960	1439 ω =1.982608	334019	2049 ω =1.987719	1416 ω =1.982419
Execution time (seconds)									
M	**FSGS**	**FSSOR**	**HSSOR**	**FSGS**	**FSSOR**	**HSSOR**	**FSGS**	**FSSOR**	**HSSOR**
32	0.77	0.11	0.06	0.56	0.11	0.05	0.65	0.11	0.05
64	8.85	0.39	0.33	5.33	0.34	0.12	4.21	0.34	0.26
128	124.43	2.73	1.21	73.48	1.95	1.23	48.28	1.30	0.70
256	1828.76	21.27	9.85	1117.96	8.12	6.80	691.48	8.67	3.83
512	13573.85	168.81	67.81	2893.71	62.80	48.26	2926.35	65.14	32.19
Maximum errors									
M	**FSGS**	**FSSOR**	**HSSOR**	**FSGS**	**FSSOR**	**HSSOR**	**FSGS**	**FSSOR**	**HSSOR**
32	9.6161e-02	9.6161e-02	5.8860e-02	2.8691e-04	2.8692e-04	2.4094e-02	3.24351-04	3.2436e-04	1.8003e-02
64	9.6274e-02	9.6274e-02	5.8717e-02	7.1708e-05	7.1746e-05	2.4164e-02	8.1132e-05	8.1170e-05	1.8115e-02
128	9.6298e-02	9.6298e-02	5.8709e-02	1.7792e-05	1.7937e-05	2.4166e-02	2.0146e-05	2.0293e-05	1.8152e-02
256	9.6304e-02	9.6305e-02	5.8699e-02	3.9149e-06	4.4848e-06	2.4171e-02	4.4965e-06	5.0710e-06	1.8160e-02
512	9.6304e-02	9.6307e-02	5.8698e-02	6.6923e-06	1.1212e-06	2.4172e-02	1.5640e-06	1.2673e-06	1.8162e-02

5. Conclusion

For the numerical solution of the two-dimensional nonlinear elliptic boundary value problems, this paper applied the HSSOR iterative method associated with the four-point rotated nonlocal arithmetic mean scheme. Based numerical results recorded in Table 1, for various mesh size m such as 32, 64,128,256 and 512 the number of iterations has declined approximately by 50% corresponds to the HSSOR iterative method as compared with FSGS and FSSOR methods with four-point nonlocal AM. Particularly in terms of execution time, implementations of HSSOR method are much faster than FSGS and FSSOR methods. It means that the HSSOR method requires the fewer amounts for number of iterations and computational time as compared with FSGS and iterative methods. In the aspect of accuracy, numerical solutions obtained for test nonlinear problems are comparable for all the tested iterative methods. Finally, it can be concluded that the HSSOR method is superior to FSGS and FSSOR methods. This is mainly because of the reduction of computational complexity in which the HSSOR method will only consider approximately half of all interior node points in a solution domain during iteration process.

References

[1] B. Attili, Numerical Treatment of Singularly Perturbed Two Point Boundary Value Problems Exhibiting Boundary Layers, Communication in Nonlinear Science and Numerical Simulation., 16 (2011), 3504–3511.

[2] Azizi, H & Loghmani, G.B, Numerical Approximation for Space-Fractional Diffusion Equations via Chebyshev Finite Difference Method, Journal of Fractional and Applications., 4(2) (2013), 303-311,.

[3] I.A. Tirmizi, Fazal-i-Haq and Siraj-ul-Islam, Non polynomial Spline Solution of Singularly Perturbed Boundary-value Problems, Applied Mathematics and Computation., 196 (2008), 6–16.

[4] M.K. Kadalbajoo and A.S.Yadaw, B-Spline Collocation Method For A Two-Parameter Singularly Perturbed Convection–Diffusion Boundary Value Problems, Applied Mathematics and Computation., 201 (2008), 504–513.

[5] A. Khan, I. Khan and T. Aziz, Sextic Spline Solution of a Singularly Perturbed Boundary-Value Problems, Applied Mathematics and Computation., 181 (2006), 432–439.

[6] J. Li and I.M. Navon, Uniformly Convergent Finite Element Methods For Singularly Perturbed Elliptic Boundary Value Problems I: Reaction-Diffusion Type, Computer Math. Applic., 35(3) (1998), 57-70.

[7] Y.N. Reddy and K. A. Reddy, Numerical Integration Method For General Singularly Perturbed Two Point Boundary Value Problems, Applied Mathematics and Computation., 133 (2002), 351–373.

[8] W.F. Spotz, "High-Order Compact Finite Difference Schemes for Computational Mechanics," Ph.D Dissertation, The University of Texas at Austin. (1995).

[9] M.M. Gupta, J. Kouatchou and J. Zhang, A Compact Multigrid Solver for Convection-Diffusion Equations, Journal of Computational Physics., 132(1) (1997), 123-129.

[10] M.M. Gupta, J. Kouatchou and J. Zhang, A Compact Multigrid Solver for Convection-Diffusion Equations, Journal of Computational Physics., 132(2) (1997), 226-232.

[11] R.H. Shariffudin and A.R. Abdullah, Hamiltonian Circuited Simulations of Elliptic Partial Differential Equations Using a Spark, Applied Mathematics Letter., 14 (2011), 413-418.

[12] Norhashidah Hj. Mohd Ali, Yuzaimi Yunus & Mohamed Othman, A New Nine Point Multigrid V-Cycle Algorithm,Sains Malaysiana., 31 (2002), 135-147.

[13] D.J. Evans, Group Explicit Iterative Methods For Solving Large Linear Systems, Int. J. Computer Maths., 17 (1985), 81-108.

[14] A. Ibrahim and A.R. Abdullah, Solving the Two-Dimensional Diffusion Equation by the Four Point Explicit Decoupled Group (EDG) Iterative Method, Int. J. Comp. Math., 58 (1995), 253-256.

[15] W.S. Yousif and D.J. Evans, Explicit Decoupled Group Iterative Methods and Their Implementations, Parallel Algorithms and Applications., 7 (1995), 53-71.

[16] M. Othman and A.R. Abdullah, The Half-Sweeps Multigrid Method as A Fast Multigrid Poisson Solver, In. J. Comp. Math., 69 (1998), 219-229.

[17] J. Sulaiman, M.K. Hasan and M. Othman, The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) Method for Diffusion Equations, Lecture Note on Computer Science (LNCS 3314)., (2004), 57–63.

[18] J. Sulaiman, M.K. Hasan, M. Othman and S.A.A. Karim, MEGSOR Iterative Method for the Triangle Element Solution Of 2D Poisson Equations, Procedia Computer Science., 1(1) (2010), 377-385.

[19] K. Moaddy, I. Hashim, A.K. Alomari and S.Momani, A New Hybrid Non-standard Finite Difference-Adomian Scheme for Solution of Nonlinear Equations. Sains Malaysiana., 40(5) (2011), 515-519.

[20] A.R. Abdullah, The Four Point Explicit Decouple Group (EDG) method. A Fast Poisson Solver, Int. J. Computer Maths., 38 (1991), 61-70.

[21] M. S. Muthuvalu and J. Sulaiman, The Arithmetic Mean Iterative Methods For Solving Dense Linear Systems Arise From First Kind Linear Fredholm Integral Equations. Proceeding of the Romanian Academy Series A., 13(3) (2012), 198-206.

[22] Muthuvalu, M. S. and Sulaiman, J. An Implementation of the 2-Point Block Arithmetic Mean Iterative Method for First Kind Linear Fredholm Integral Equations. World Journal of Modeling and Simulation (WJMS)., 8(4) (2012), 293-301.

[23] M. S. Muthuvalu and J. Sulaiman. Half-sweep Arithmetic Mean Method with Composite Trapezoidal Scheme for Solving Linear Fredholm Integral Equations. Applied Mathematics and Computation., 217 (2011), 5442-5448.

[24] D.M. Young, Iterative solution of large linear systems, Academic Press: London. (1971).

[25] D.M. Young, Iterative Method for Solving Partial Differential Equation Elliptic Type, Trans. Amer. Math. Soc., 76 (1954), 92-111.

[26] M.K.M. Akhir, M. Othman, Z.A. Majid, M. Suleiman and J. Sulaiman, Four point Explicit Decoupled Group Iterative Method Applied to Two-Dimensional Helmholtz Equation, Int. J. Mathemetical Analysis., 6 (2012), 963–974.

[27] M.K. Akhir, M. Othman, J. Sulaiman, Z.A. Majid and M. Suleiman, The four point-EDGMSOR iterative method for solution of 2D Helmholtz equations, In Communications in Computer and Information Science. Berlin: Springer-Verlag., 253 (2011), 218-227.

[28] P.C. Jain and M.K. Kadalbajoo, A Numerical Technique for Solving Nonlinear Elliptic Problems, Indian Journal Pure Applied Mathematics, 11(1) (1979), 20-32.