

Solving Fractional Proportional Delay Integro Differential Equations of First Order by Reproducing Kernel Hilbert Space Method

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Abstract

In this paper, the reproducing kernel Hilbert space method (RKHSM) is applied to find approximate solution of fractional delay integro differential equations satisfying initial conditions. The fractional derivatives are considered in the Caputo sense. Numerical examples are provided to demonstrate the efficiency and accuracy of the presented method. It is shown that RKHSM gives excellent results when applied to both fractional linear and nonlinear delay integro differential equations.

AMS subject classification:

Keywords: Reproducing Kernel Hilbert Space Method (RKHSM), Fractional Delay Integro Differential Equation, Caputo Derivative, and Numerical Solution.

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1. Introduction

In this paper, we consider the following fractional delay integro differential equation

$$D^\alpha u(x) = G(x, u(q_1x), u(q_2x)) + F(x, \int_a^{q_1x} K_1(x, t, u(q_1t), u(q_2t)) dt, \int_a^{q_2x} K_2(x, t, u(q_1t), u(q_2t)) dt) + H(x) \quad (1)$$

subject to the condition

$$u(a) = \gamma \quad (2)$$

Where G , H are a given functions with appropriate domain of definition, $a \leq x \leq b$, $0 < \alpha \leq 1$, $q_1, q_2 \in [0, 1]$.

Fractional calculus has occupied a central place in many areas such as science, control, signal processing, and engineering due to its potential in solving differential and integral equations [1, 2, 3, 4, 5].

In real world systems, Integro differential equations with fractional derivatives play an important role in the describing of a variety of phenomena in physics, technology, biology, physiology, and many areas of applied science [6, 7, 8, 9], so they have received significant attention from many researchers by developing some methods to find approximate solutions for fractional integro differential equations such as: collocation method [10], fractional differential transform method [11] and Adomian decomposition method [12]. However, Fractional delay integro differential equations provide a valuable tools for modeling physical phenomena by giving explanations behave like the real process, so the interest for them will keep growing [13, 14, 15, 16]. Computational difficulties that face researchers in solving many of fractional delay integro differential equations, in addition to most of them do not have analytical solutions have led to use approximation and numerical methods. Therefore different studies [17, 18, 19, 20, 21, 22] have been done to build up methods for giving approximate solutions.

Reproducing kernel theory has important applications in different mathematical areas, such as, probability and statistics, numerical analysis, and differential equations [24, 25, 26]. This theory enables researchers to get approximate solutions for many types of problems, for example, nonlinear system of singular and regular boundary and initial value problems, nonlinear operator equations, nonlinear Fredholm-Volterra integro-differential equations and integro-differential equations of fractional order, non-local fractional boundary value problems and etc.[27, 28, 29, 30, 31, 32, 33, 34].

Our aim in this paper is to use the reproducing kernel Hilbert space method to find approximate solution for both fractional linear and nonlinear proportional delay integro differential equations of first order.

The paper is organized as follows. In section 2 of this paper a brief review of some basic definitions that used; in section 3 we illustrated a reproducing kernel Hilbert spaces with their kernels and the algorithm of reproducing kernel Hilbert space is presented; in section 4 the numerical examples are discussed. Finally, a brief conclusion of this paper is given in section 5.

2. Basic Definitions

Fractional derivatives have many definitions [35] but the most used of these definitions are Riemann-Liouville, Grunwald-Letnikov and Caputo. Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena using fractional differential equations. Therefore, we will introduce a modified fractional differential operator proposed by Caputo's work on the theory of viscoelasticity [36].

Definition 2.1. A function $f(x)$, $x > 0$, is said to be in the space $C_\mu \in R$ if there exists a real $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in [0, \infty)$. Clearly $C_\mu < C_\beta$ if $\beta < \mu$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space $C_\mu^m \in N \cup \{0\}$ if $f^{(m)} \in C_\mu$.

Definition 2.3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \quad x > a$$

with $J^0 f(x) = f(x)$. Properties of the operator J^α can be found in [37, 38].

Definition 2.4. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt,$$

for $m-1 < \alpha \leq m$, $m \in N$, $x > a$, $f \in C_{-1}^m$.

Lemma 2.5. Let $\alpha > 0$, $n = [\alpha]$ and $f(x) = x^c$ for some $c \geq 0$. Then,

$$D^\alpha f(x) = \begin{cases} 0 & \text{if } c \in \{0, 1, \dots, n-1\} \\ \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)} x^{c-\alpha} & \text{if } c \in N \text{ and } c \geq n \text{ or } c \text{ not in } N \text{ and } c > n-1 \end{cases}$$

Definition 2.6. Let M be a nonempty abstract set. A function $K : M \times M \rightarrow C$ is a reproducing kernel of the Hilbert space H if and only if

1. $\forall t \in M, K(\cdot, t) \in H$.
2. $\forall t \in M, \forall u \in H, \langle u(\cdot), K(\cdot, t) \rangle = u(t)$.

The condition (2) is called "the reproducing property", a Hilbert space which possesses a reproducing kernel is called a reproducing kernel Hilbert space.

Definition 2.7. $W_2^m[a, b] = \{u | u^{(j)}$ is absolutely continuous, $j = 1, 2, \dots, m - 1$ and $u^{(m)} \in L^2[a, b]\}$. The inner product and the norm is given by

$$\langle u, v \rangle_{W_2^m} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x) dx$$

with $\|u\|_{W_2^m} = \sqrt{\langle u, u \rangle_{W_2^m}}$.

3. Reproducing kernel Hilbert Space Method (RKHSM)

- The reproducing kernel space $W_2^1[a, b]$ is defined as $W_2^1[a, b] = \{u | u$ is absolutely continuous real value function, $u' \in L^2[a, b]\}$. The inner product and norm of $W_2^1[a, b]$, respectively, is given by

$$\langle u, v \rangle_{W_2^1} = \int_a^b (u(t)v(t) + u'(t)v'(t)) dt,$$

$$\|u\|_{W_2^1} = \sqrt{\langle u, u \rangle_{W_2^1}}$$

In [23], Li and Cui proved that $W_2^1[a, b]$ is a reproducing kernel Hilbert space and its reproducing kernel is given by

$$M(x, y) = \frac{1}{2 \sinh(b-a)} [\cosh(x+y-b-a) + \cosh|x-y|-b+a]$$

- The reproducing kernel space $W_2^2[a, b]$ is defined as $W_2^2[a, b] = \{u | u, u'$ are absolutely continuous real value functions, $u'' \in L^2[a, b]$, $u(a) = 0\}$. The inner product and norm of $W_2^2[a, b]$, respectively, is given by

$$\langle u, v \rangle_{W_2^2} = u(a)v(a) + u'(a)v'(a) + \int_a^b u''(t)v''(t) dt,$$

$$\|u\|_{W_2^2} = \sqrt{\langle u, u \rangle_{W_2^2}}$$

In [24], Cui and Lin proved that $W_2^2[a, b]$ is a reproducing kernel Hilbert space and its reproducing kernel is given by

$$S(x, y) = \begin{cases} \frac{1}{6}(y-a)(2a^2 - y^2 + 3x(2+y) - a(6+3x+y)) & \text{if } y \leq x \\ \frac{1}{6}(x-a)(2a^2 - x^2 + 3y(2+x) - a(6+3y+a)) & \text{if } y > x \end{cases}$$

The essential idea of solving fractional delay integro differential equations using RKHSM is offered as follows:

- Homogenization of the initial condition.
- construct a linear operator $T : W_2^2[a, b] \rightarrow W_2^1[a, b]$, such that $Tu(x) = D^\alpha u(x) - G(x, u(q_1x), u(q_2x))$. Obviously, T is bounded linear operator. After the previous steps, our problem becomes as follows:

$$Tu(x) = F(x, \int_a^{q_1x} K_1(x, t, u(q_1t), u(q_2t)) dt, \int_a^{q_2x} K_2(x, t, u(q_1t), u(q_2t)) dt) + H(x) \tag{3}$$

$$u(a) = 0 \tag{4}$$

Where, $x \in [a, b]$, $u(x) \in W_2^2[a, b]$ and

$$F\left(x, \int_a^{q_1x} K_1(x, t, u(q_1t), u(q_2t))dt, \int_a^{q_2x} K_2(x, t, u(q_1t), u(q_2t))dt\right) + H(x) \in W_2^1[a, b].$$

- Let $\{x_\tau\}_{\tau=1}^\infty$ be a countable dense set in $[a, b]$. Let $\omega_\tau(x) = M(x_\tau, x)$ and $\xi_\tau(x) = T^*\omega_\tau(x)$, where T^* is the adjoint operator of T .

By applying Gram-Schmidt process to $\{\xi_\tau(x)\}_{\tau=1}^\infty$ we can derived an orthonormal system $\{\bar{\xi}_\tau(x)\}_{\tau=1}^\infty$ of $W_2^2[a, b]$, where $\bar{\xi}_\tau(x) = \sum_{k=1}^\tau \beta_{\tau k} \xi_k(x)$, where $\beta_{\tau k}$ are orthogonal coefficients.

Theorem 3.1. Assume that the inverse operator T^{-1} exists. Then if $\{x_\tau\}_{\tau=1}^\infty$ is dense on $[a, b]$, then $\{\xi_\tau(x)\}_{\tau=1}^\infty$ is a complete system of $W_2^2[a, b]$.

For the proof, see [24].

Theorem 3.2. Let $\{x_\tau\}_{\tau=1}^\infty$ be a dense set on $[a, b]$ and the solution of (3-4) is unique on $W_2^2[a, b]$. Then the solution is given by $u(x) = \sum_{\tau=1}^\infty A_\tau \bar{\xi}_\tau(x)$, where

$$A_\tau = \sum_{k=1}^\tau \beta_{\tau k} \left(F\left(x_k, \int_a^{q_1x_k} K_1(x_k, t, u(q_1t), u(q_2t)) dt, \int_a^{q_2x_k} K_2(x_k, t, u(q_1t), u(q_2t)) dt\right) + H(x_k) \right).$$

Proof. Since $\{\bar{\xi}_\tau(x)\}_{\tau=1}^\infty$ is a complete orthonormal basis of $W_2^2[a, b]$.

$$\begin{aligned}
 u(x) &= \sum_{\tau=1}^{\infty} \langle u(x), \bar{\xi}_\tau(x) \rangle_{W_2^2} \bar{\xi}_\tau(x) = \sum_{\tau=1}^{\infty} \langle u(x), \sum_{k=1}^{\tau} \beta_{\tau k} \xi_k(x) \rangle_{W_2^2} \bar{\xi}_\tau(x) \\
 &= \sum_{\tau=1}^{\infty} \sum_{k=1}^{\tau} \beta_{\tau k} \langle u(x), \xi_k(x) \rangle_{W_2^2} \bar{\xi}_\tau(x) = \sum_{\tau=1}^{\infty} \sum_{k=1}^{\tau} \beta_{\tau k} \langle u(x), T^* \omega_k(x) \rangle_{W_2^2} \bar{\xi}_\tau(x) \\
 &= \sum_{\tau=1}^{\infty} \sum_{k=1}^{\tau} \beta_{\tau k} \langle Tu(x), \omega_k(x) \rangle_{W_2^1} \bar{\xi}_\tau(x) = \sum_{\tau=1}^{\infty} \sum_{k=1}^{\tau} \beta_{\tau k} \langle Tu(x), M(x_k, x) \rangle_{W_2^1} \bar{\xi}_\tau(x) \\
 &= \sum_{\tau=1}^{\infty} \sum_{k=1}^{\tau} \beta_{\tau k} Tu(x_k) \bar{\xi}_\tau(x) \\
 &= \sum_{\tau=1}^{\infty} \sum_{k=1}^{\tau} \beta_{\tau k} \left(F \left(x_k, \int_a^{q_1 x_k} K_1(x_k, t, u(q_1 t), u(q_2 t)) dt, \right. \right. \\
 &\quad \left. \left. \int_a^{q_2 x_k} K_2(x_k, t, u(q_1 t), u(q_2 t)) dt \right) + H(x_k) \right) \bar{\xi}_\tau(x) \\
 &= \sum_{\tau=1}^{\infty} A_\tau \bar{\xi}_\tau(x)
 \end{aligned}$$

By taking finitely many terms in the series representation of $u(x)$ one can has approximate solution $u_n(x) = \sum_{\tau=1}^n A_\tau \bar{\xi}_\tau(x)$.

Since $W_2^2[a, b]$ is a Hilbert space, then

$$\sum_{\tau=1}^{\infty} \sum_{k=1}^{\tau} \beta_{\tau k} \left(F \left(x_k, \int_a^{q_1 x_k} K_1(x_k, t, u(q_1 t), u(q_2 t)) dt, \right. \right. \\
 \left. \left. \int_a^{q_2 x_k} K_2(x_k, t, u(q_1 t), u(q_2 t)) dt \right) + H(x_k) \right) < \infty.$$

■

Theorem 3.3. The approximate solution $u_n(x)$ and its derivative $u'_n(x)$ are uniformly convergent.

For the proof, see [29].

4. Numerical Examples

In this section, the RKHSM is applied for some fractional delay integro differential equations. All computations for these numerical examples are performed by using Mathematica 10.0. Results of each example are compared with exact solution.

Example 4.1. [40] Consider the following Pantograph fractional delay integro differential equation:

$$D^\alpha u(x) = \frac{1}{2}u(x) + u\left(\frac{x}{4}\right) + \frac{1}{2} - \frac{x}{4}e^{\frac{x}{4}} + \frac{x^2}{32} + \frac{e^{3x}}{2} + e^{2x} + \int_0^x e^{(t+x)}u(t) dt + \int_0^{\frac{x}{4}} tu(t) dt$$

$$u(0) = 0$$

The exact solution for $\alpha = 1$ is $u(x) = e^x - 1$.

Solution: Using the RKHSM, taking $x_\tau = \frac{\tau}{n}$, $\tau = 1, 2, \dots, 100$, the numerical results are given in Table 1 and the graphs of the approximate solutions for different values of α are given in Figure 1.

Table 1: Numerical solution for Example 4.1.

x	Exact Solution	Approximate Solution	Absolute Error
0	0	0	0
0.1	0.105171	0.105169	$2.380712202 \times 10^{-6}$
0.2	0.221403	0.2213998	$4.705422046 \times 10^{-6}$
0.3	0.349859	0.349853	$6.131708729 \times 10^{-6}$
0.4	0.491825	0.491818	$6.447475122 \times 10^{-6}$
0.5	0.648721	0.648714	$6.775316183 \times 10^{-6}$
0.6	0.822119	0.822112	$7.155388168 \times 10^{-6}$
0.7	1.01375	1.01375	$7.673885816 \times 10^{-6}$
0.8	1.22554	1.22553	$8.485877928 \times 10^{-6}$
0.9	1.4596	1.45959	$9.888583331 \times 10^{-6}$
1.0	1.71828	1.71827	$1.24612491 \times 10^{-6}$

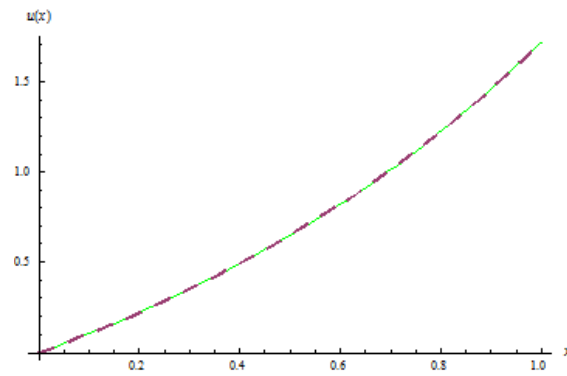
From the numerical results in Table 1 and Figure 1, it is clear that the approximate solutions are in good agreement with the exact solutions when $\alpha = 1$ and the solution continuously depends on the fractional derivative.

Example 4.2. [41] Consider the following fractional delay integro differential equation:

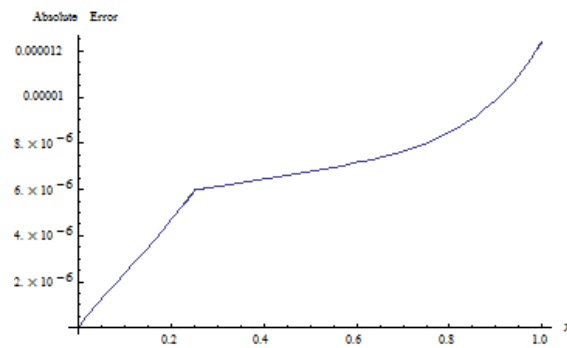
$$D^\alpha u(x) = u\left(\frac{x}{2}\right) + \frac{3}{2} \int_0^{\frac{x}{3}} u(x)u(t)u\left(\frac{t}{2}\right) dt$$

$$u(0) = 1$$

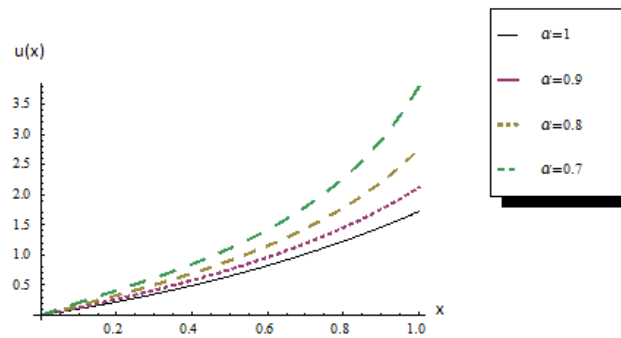
The exact solution for $\alpha = 1$ is $u(x) = e^x$.



(a) Comparison results for $u(x)$ and the exact solution



(b) Absolute Error for Example 4.1



(c) The comparison of approximate solution for $\alpha = 1, 0.9, 0.8, 0.7$ and the exact solution of Example 4.1

Figure 1: Graphical Results for Example 4.1.

Solution: Using the RKHSM, taking $x_\tau = \frac{\tau}{n}$, $\tau = 1, 2, \dots, 100$, the numerical results are given in Table 2 and the graphs of the approximate solutions for different values of α are given in Figure 2.

Table 2: Numerical solution for Example 4.2.

x	Exact Solution	Approximate Solution	Absolute Error
0	1	1	0
0.1	1.105171	1.10517	7.534209×10^{-7}
0.2	1.221402	1.221401	1.348448×10^{-6}
0.3	1.349859	1.349857	1.806692×10^{-6}
0.4	1.491825	1.491822	2.44178×10^{-6}
0.5	1.648721	1.648718	3.318686×10^{-6}
0.6	1.822119	1.822117	2.098947×10^{-6}
0.7	2.013753	2.013752	7.367289×10^{-7}
0.8	2.225541	2.225542	7.192589×10^{-7}
0.9	2.459603	2.459605	2.258516×10^{-6}
1.0	2.718282	2.718286	3.863248×10^{-6}

Example 4.3. [41] Consider the following fractional delay integro differential equation:

$$D^\alpha u(x) = 1 - \frac{1}{2}xu(x) + 2u(x) + 2 \int_0^x \left(u\left(\frac{t}{2}\right)\right)^2 dt$$

$$u(0) = 0$$

The exact solution for $\alpha = 1$ is $u(x) = xe^x$.

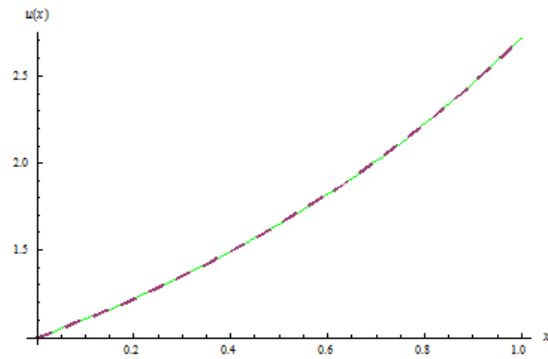
Solution: Using the RKHSM, taking $x_\tau = \frac{\tau}{n}$, $\tau = 1, 2, \dots, 100$, the numerical results are given in Table 3 and the graphs of the approximate solutions for different values of α are given in Figure 3.

Example 4.4. [40] Consider the following fractional delay integro differential equation:

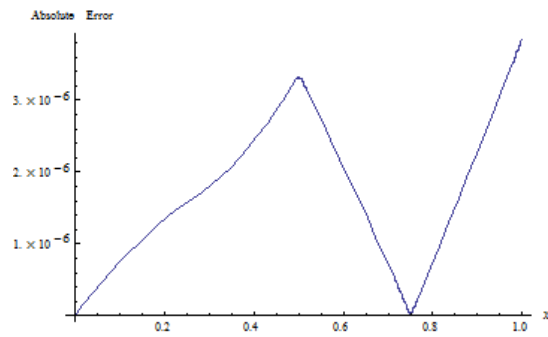
$$D^\alpha u(x) = u(x) - \frac{1}{2} \ln\left(1 + \frac{x}{2}\right) u\left(\frac{x}{2}\right) + \frac{1}{1+x} - \ln(1+x) \left(\frac{x}{2} \ln(1+x) + 1\right) \\ + \int_0^x \frac{x}{1+t} u(t) dt + \int_0^{\frac{x}{2}} \frac{1}{1+t} u(t) dt$$

$$u(0) = 0$$

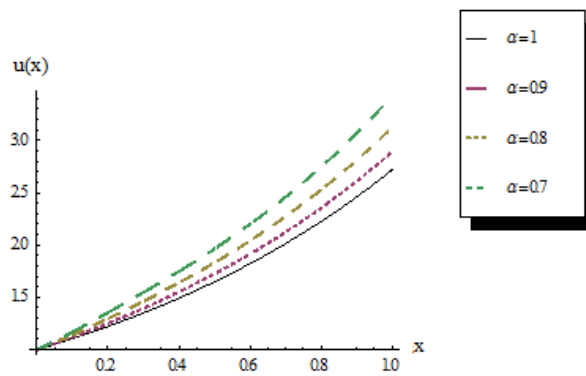
The exact solution for $\alpha = 1$ is $u(x) = \ln(1+x)$.



(a) Comparison results for $u(x)$ and the exact solution



(b) Absolute Error for Example 4.2



(c) The comparison of approximate solution for $\alpha = 1, 0.9, 0.8, 0.7$ and the exact solution of Example 4.2

Figure 2: Graphical Results for Example 4.2

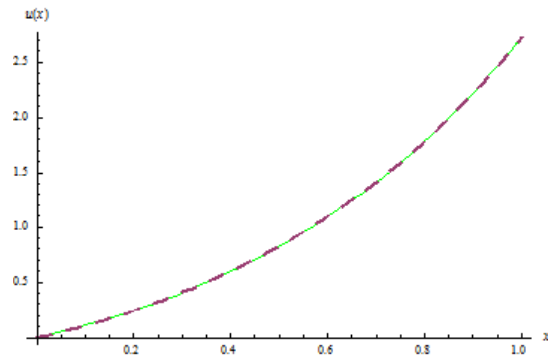
Table 3: Numerical solution for Example 4.3

x	Exact Solution	Approximate Solution	Absolute Error
0	0	0	0
0.1	0.110517	0.11054	$2.272490478 \times 10^{-5}$
0.2	0.244281	0.244286	$4.966069537 \times 10^{-6}$
0.3	0.404958	0.404946	$1.190127977 \times 10^{-5}$
0.4	0.59673	0.596702	$2.811077039 \times 10^{-5}$
0.5	0.824361	0.824317	$4.379312493 \times 10^{-5}$
0.6	1.09327	1.09321	$5.904012849 \times 10^{-5}$
0.7	1.40963	1.40955	$7.401542909 \times 10^{-5}$
0.8	1.78043	1.78034	$8.914361313 \times 10^{-5}$
0.9	2.21364	2.21354	$1.054170083 \times 10^{-4}$
1.0	2.71828	2.71816	$1.248829391 \times 10^{-4}$

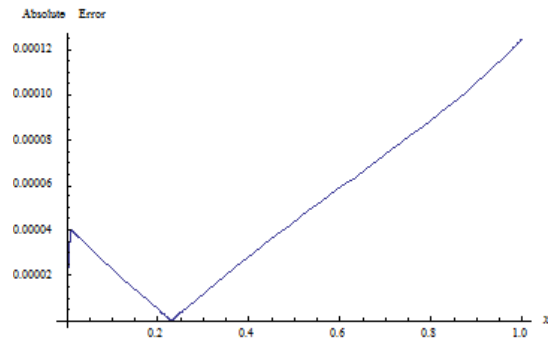
Solution: Using the RKHSM, taking $x_\tau = \frac{\tau}{n}$, $\tau = 1, 2, \dots, 100$, the numerical results are given in Table 4 and the graphs of the approximate solutions for different values of α are given in Figure 4.

Table 4: Numerical solution for Example 4.4

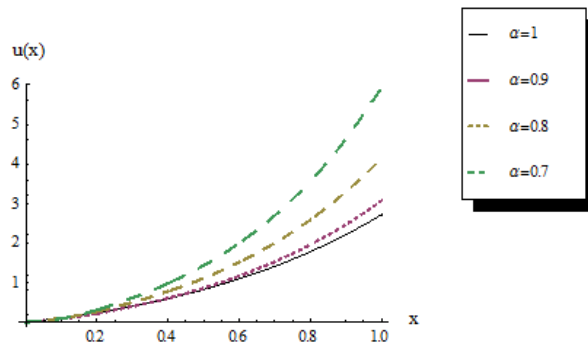
x	Exact Solution	Approximate Solution	Absolute Error
0	0	0	0
0.1	0.0953102	0.095313	$2.807703988 \times 10^{-6}$
0.2	0.182322	0.182327	$5.294432785 \times 10^{-6}$
0.3	0.262364	0.262372	$7.613572512 \times 10^{-6}$
0.4	0.336472	0.336482	$9.876593867 \times 10^{-6}$
0.5	0.405465	0.405477	$1.217317636 \times 10^{-5}$
0.6	0.470004	0.470018	$1.461429025 \times 10^{-5}$
0.7	0.530628	0.530645	$1.722895504 \times 10^{-5}$
0.8	0.587787	0.587807	$2.009678991 \times 10^{-5}$
0.9	0.641854	0.641877	$2.330358105 \times 10^{-5}$
1.0	0.693147	0.693174	$2.69441272 \times 10^{-5}$



(a) Comparison results for $u(x)$ and the exact solution

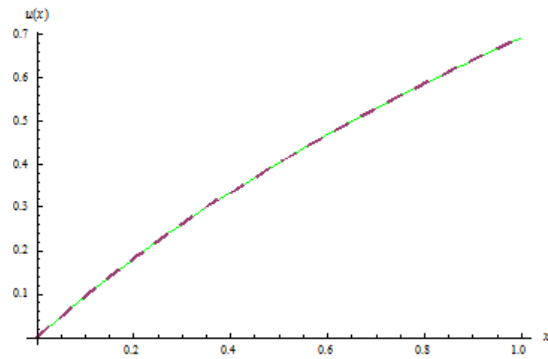


(b) Absolute Error for Example 4.3

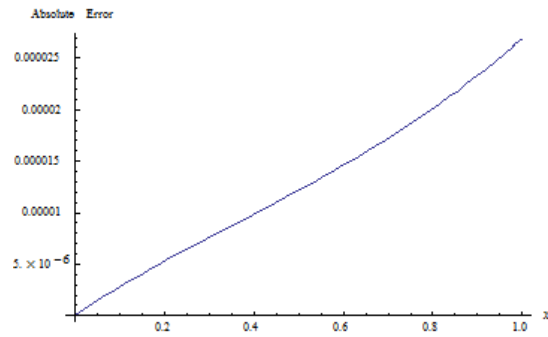


(c) The comparison of approximate solution for $\alpha = 1, 0.9, 0.8, 0.7$ and the exact solution of Example 4.3

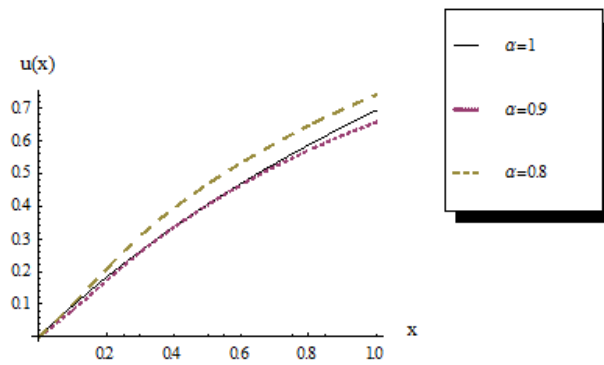
Figure 3: Graphical Results for Example 4.3



(a) Comparison results for $u(x)$ and the exact solution



(b) Absolute Error for Example 4.4



(c) The comparison of approximate solution for $\alpha = 1, 0.9, 0.8$ and the exact solution of Example 4.4

Figure 4: Graphical Results for Example 4.4

Example 4.5. Consider the following fractional delay integro differential equation:

$$D^{0.9}u(x) = \frac{\Gamma(4)x^{2.1}}{\Gamma(3.1)} - \frac{x^2}{5}e^x u(x) - \frac{(0.5x)^2}{5}u\left(\frac{x}{2}\right) + \int_0^x t e^x u(t) dt$$

$$+ \int_0^{\frac{x}{2}} t u(t) dt$$

$$u(0) = 0$$

The exact solution is $u(x) = x^3$.

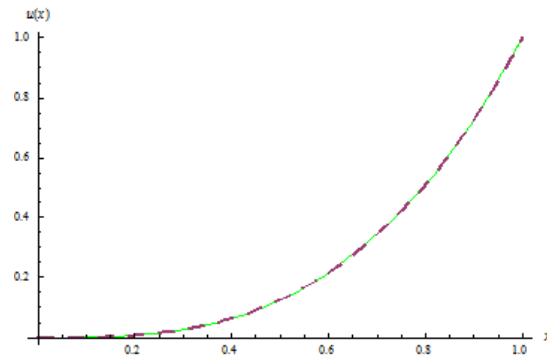
Solution: Using the RKHSM, taking $x_\tau = \frac{\tau}{n}$, $\tau = 1, 2, \dots, 20$, the numerical results are given in Table 5 and the graphs of the approximate solutions for different values of α are given in Figure 5.

Table 5: Numerical solution for Example 4.5

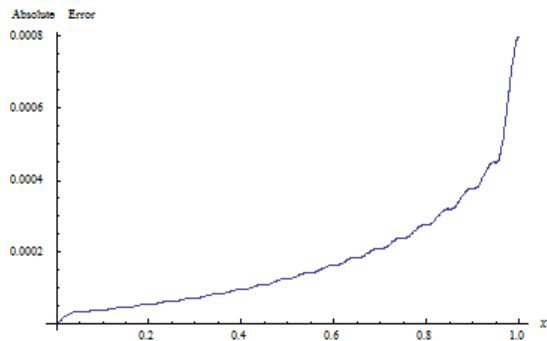
x	Exact Solution	Approximate Solution	Absolute Error
0	0	0	0
0.1	0.001	0.00103977	$3.977091251 \times 10^{-5}$
0.2	0.008	0.00805486	$5.485644081 \times 10^{-5}$
0.3	0.027	0.0270727	$7.268372792 \times 10^{-5}$
0.4	0.064	0.064095	$9.502448458 \times 10^{-5}$
0.5	0.125	0.125126	$1.259262531 \times 10^{-4}$
0.6	0.216	0.216162	$1.61550631 \times 10^{-4}$
0.7	0.343	0.343209	$2.085628645 \times 10^{-4}$
0.8	0.512	0.512273	$2.733701536 \times 10^{-4}$
0.9	0.729	0.729373	$3.728262884 \times 10^{-4}$
1.0	1.0	1.00079	$7.91487481 \times 10^{-4}$

5. Conclusion

In this paper, a method is proposed for finding approximate solutions for fractional delay integro differential equations of first order. It may be concluded from the obtained results that the major advantages of the presented method are; it can be used for both linear and nonlinear fractional delay integro differential equations; it gives approximate solutions with a high degree of accuracy and agreement between approximate and exact solutions. This confirms the validity of the present method and it is efficient, accurate and reliable for fractional delay integro differential equations.



(a) Comparison results for $u(x)$ and the exact solution



(b) Absolute Error for Example 4.5

Figure 5:

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