Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3609–3619
© Research India Publications
http://www.ripublication.com/gjpam.htm

Mathematical Models of Single Population

Eugeny Petrovich Kolpak

Saint Petersburg State University,
Faculty of Applied Mathematics and
Control Process, Russian Federation,
199034 Saint Petersburg,
Universitetskaya nab., 7/9.

Mariia Vladimirovna Stolbovaia

Saint Petersburg State University,
Faculty of Applied Mathematics and Control Process,
Russian Federation, 199034 Saint Petersburg,
Universitetskaya nab., 7/9

Inna Sergeevna Frantsuzova

Saint Petersburg State University,
Faculty of Applied Mathematics and Control Process,
Russian Federation, 199034 Saint Petersburg,
Universitetskaya nab., 7/9.

Abstract

The paper presents a mathematical model for a single population on a line represented as boundary value problem for a nonlinear differential equation in partial derivatives. The steady equation is solved by quadratures. The paper suggests an algorithm offering a numerical solution to the nonlinear boundary linear problem. It outlines the results of a study on how various parameters affect the behavior of solutions.

AMS subject classification:
Keywords: Population, boundary value problem, mathematical modeling.

1. Introduction

The theoretical basis underpinning the majority of contemporary mathematical models of interacting populations was shaped by Lotka and Volterra [1]. Since the 1960s, numerous models describing interaction between populations have been formulated. They are mostly based on different kinetic functions than those offered by Volterra [2–7]. This approach is used in the predator–prey models [8–15] and the competition models [16–22].

Most of the models focusing on the dynamics of interacting populations are represented as a Cauchy problem for a system of ordinary differential equations. Point models assume that the habitat is homogeneous and the density of population does not depend on spatial coordinates. In reality populations occupy limited territories and space with varying environmental properties. Heterogeneous properties of the environment and internal social structure of the population shaped in the evolutionary process driven by survival may develop, at least in certain species, the need for displacement. Their dispersal is, as a rule, directed to territories with lower concentration of species [14, 23–28]. In this case mathematical models are designed as a set of equations in partial derivatives [29–s31].

Research in spontaneous dispersal of zoological species has focused mainly on the migration of small mammals [25, 28]. The displacement of species can happen at two different points in time: before or after the habitat is saturated with species. Displacement before the saturation is typical for species sensitive to population density. It shows their strive to occupy areas with lower density. When the territory is saturated with species, displacement concerns doomed sedentary species with a lower social status and younger species in search of trophic resource. Thus, we can regard territorial dispersal of population triggered by these reasons as random individual or group displacement of species.

The paper introduces a generalized model of a single logistic population on unlimited trophic resource placed on a linear habitat (a straight line). The dynamics of population growth is described by an evolutionary equation.

2. Generalized logistic population model

Generalized logistic population model. P. F. Verhulst was the first to introduce a mathematical model describing evolutionary processes within an isolated population with limited growth. A more general elaboration of the Verhulst model is a generalized model of logistic population [1, 32], whose growth is described by the equation

$$\frac{du}{dt} = f(u), \tag{1}$$

where the continuous and twice differentiable function satisfies the conditions

$$\begin{gathered} f(0) = f(K) = 0 \quad (0 < K < +\infty), \\ f'(0) = \mu > 0, \\ f'' \leq 0 \quad for \quad 0 < u < K. \end{gathered} \tag{2}$$

The function $f(u)$ is called a local rate of population growth, μ is a constant that is determined by the growth rate of the population. K is the environmental capacity [1, 2].

The condition $f(0) = 0$ is natural as population cannot emerge out of zero species, the condition $0 < f'(0)$ provides growth of the emerged population, whereas the condition $f(K) = 0$ is population size bounded above. The critical point $u = 0$ is unstable, unlike the stable $u = K$. Therefore, all the solutions to the equation (1) satisfying the conditions (2) are monotone increasing on the interval $[0, K]$, originate from point $u(t = 0) = u_0$ and tend to $u = K$, where $t \to \infty$. Hereinafter, the environmental capacity is taken as the unit of measurement of population size, i.e. $K = 1$.

Models of logistic population are exemplified by a logistic model – $f(u) = \mu u(1-u)$ and the model by M. Rosenzweig – $f(u) = \mu u(1 - u^\gamma)$. The Yu. M. Svirizhev models $f(u) = \mu u^2(1-u^\gamma)$, and their generalizations $f(u) = \mu u^\beta(1-u^\gamma)$ $(0 < \beta, 0 < \gamma)$ do not satisfy the $f'(0) > 0$ condition. However, as is in the case with logistic population the critical point $u = 0$ is unstable, whereas $u = 1$ is stable.

3. Diffusion model

Roadsides, edges of fields, pipe lines, rivers, etc. are natural examples of linear habitats home to a wide range of flora and fauna. This model regards the dispersal of population as a dispersal of population along a straight line. In this case the dispersal of spices is described by the evolutionary equation [33–35]

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \tag{3}$$

where x are Cartesian coordinates and the function $f(u)$ is a local rate of population growth. The parameter D defines the mobility of species. The model assumes that mobility of species on a given territory is random.

Equation (3) for a habitat of length l requires initial and boundary conditions. Initial conditions are determined by the function $u = u(t, x)$ at the initial instant. Hence $u(x) = u_0(x)$. There are two ways to define boundary conditions.

First – in points $x = 0$ and $x = l$

$$\frac{\partial u}{\partial x} = 0 \tag{4}$$

and second

$$at \quad x = 0: \quad u = 0; \qquad at \quad x = l: \quad \frac{\partial u}{\partial x} = 0. \tag{5}$$

If the function u tends to vanish at the boundary line, then there cannot possibly be any population at this point. If this condition holds true for the derivative $\partial u/\partial x$ (the condition of the environmental capacity [33]), then the environment can show certain independent growth of population.

If we change initial conditions for equation (3) and consider $u_0(x) = u^*$, where $u = u^*$ are the roots of the equation $f(u) = 0$, the function $u(t, x) = u^*$ becomes solution to this equation for boundary conditions (4). The equation $f(u) = 0$ has several solutions, whereas the equation (3) for boundary conditions (4) only has a unique solution.

4. Stationary solution

The equation (3), where $f(u) = 0$ becomes a diffusion equation and its solution for stationary boundary conditions gradually converges to the solution of a stationary problem disregarding the value of the required function at the initial instant [12]. Considering the boundary conditions (4) and (5), this provides a trivial solution. The result maybe different for the non-linear equation (3).

The stationary solution of the non-linear equation (3) must satisfy the differential equation

$$D\frac{\partial^2 u}{\partial x^2} + f(u) = 0, \tag{6}$$

when $f(0) = 0$ for boundary conditions (4) and (5), the equation gets a trivial solution. If the equation (6) is multiplied by du/dx and integrated, we get a quadrature [36]

$$\frac{D}{2}\left(\frac{du}{dx}\right)^2 = \Phi(u_*) - \Phi(u), \qquad \Phi(u) = \int_0^u f(u)d\,u, \tag{7}$$

where $u_* \neq 0$ is the value of the function u at point $x = l$. The integration constant u_* satisfies the boundary condition at $x = l$ for (4) and (5). If the function $f(u)$ is non-negative on the interval $[0, u_*]$, which holds true for a generalized logistic population, then $\Phi(u)$ is monotone increasing with an extreme value at point $u = u_*$ (for $x = l$). Therefore, the equation $\Phi(u_*) - \Phi(u) = 0$ only has one root on the interval $[0, u_*]$ and the first condition in (4), as follows from (7), is satisfied only if we get $u(x) \equiv u_*$. In this case the solutions to equation (6), for boundary conditions (4), are the roots of the equation $f(u) = 0$. For the generalized logistic population we get $u = 0$ and $u = 1$.

As the function $f(u)$ on the interval $[0, u_*]$ takes on positive values, the solution $u = u((x)$ of equation (6) for boundary conditions (5) at point $x = l$ gets an extreme value. In addition, the condition $0 \leq u(x) \leq u_*$ must be satisfied on the interval $[0, l]$. Accordingly, from (7) follows the dependence $u = u(x)$, expressed as an integral

$$\frac{1}{2D}\int_0^u \frac{du}{\sqrt{\Phi(u_*) - \Phi(u)}} = x.$$

This solution will satisfy the boundary condition $u(0) = 0$, if

$$\frac{1}{2D}\int_0^{u_*} \frac{du}{\sqrt{\Phi(u_*) - \Phi(u)}} = l, \tag{8}$$

where u_* is the unknown quantity. Therefore, (9) is an equation that defines the integration constant u_*.

The integration element in (8) has a singularity at point $u = u_*$, as the function $\Psi(u) = \Phi(u_*) - \Phi(u)$ tends to vanish. If its derivative set to $-f(u)$ also tends to vanish at this point, then gets a multiple root and the integer on the left side of equation (8) diverges. Therefore, in solutions of equation (7) u_* cannot equal the roots of $f(0) = 0$. For the generalized logistic population the function $f(u)$ tends to vanish when $u = 1$. Hence, equation (8) cannot be solved for $u_* = 1$. Therefore, the solution of equation (6) on boundary conditions (3) is subject to the condition $0 \leq u(x) \leq 1$.

The solution for equation (6) for boundary conditions (4) is similar to that of the problem of nonlinear deformation of a thin rectilinear rod [36], described by the equation

$$\frac{d^2u}{dx^2} + \beta^2 \sin u = 0.$$

In continuum mechanics, if β is less than a certain critical value, there is only a trivial solution for this equation; whereas if this value is bigger than a critical value, then a trivial as well as non-trivial solutions are possible. In the latter case the trivial solution will be unstable [36, 37].

5. Stability of a solution

Equation (6) for boundary conditions (4) and (5) is satisfied by the solution $u = u_* = 0$, for boundary conditions and the solution is $u = u_* = 1$. Assume that along with this solution there is a similar solution $u = u_* + \delta u$, where $\|\delta u\| << 1$. Then from equation (3) there follows a highly accurate equation for δu

$$\frac{\partial \delta u}{\partial t} = D\frac{\partial^2 \delta}{\partial x^2} + f'_u(u_*)\delta u \tag{9}$$

with an initial condition

$$\delta u(t = 0, x) = \delta u_0(x), \quad (\|\delta u_0\|\| << 1).$$

The solution to equation (9), for boundary conditions (4) for δu shows up as trigonometric series [12]

$$\delta u = \sum_{k=1}^{\infty} C_k(t) \sin \mu_k x \quad (\mu_k = (k - 1/2)\pi/l).$$

Coefficients $C_k(t)$ should satisfy the equations

$$\frac{dC_k}{dt} = -(D\mu_k^2 - f'_u(u_*))C_k.$$

Initial conditions for $C_k(t)$ are obtained from the expansion of the function in a Fourier series

$$\delta u_0(x) = \sum_{k-1}^{\infty} C_k(0) \sin \mu_k x.$$

If $u'(u_*) < 0$, all C_k as a function of time will tend to zero for any $C_k(0)$. In this case, the solution $u = u_*$ will be stable. If $u'(u_*) > 0$, the solution will be stable only if $u'(u_*) < D\mu_1^2$. On the infinite line, equation () is satisfied by the function

$$\delta u(t,x) = \int_{-\infty}^{\infty} \frac{\delta u_0(t=0,z)}{2\sqrt{\pi D_1 t}} \exp\Big(\gamma t - \frac{(x-z)^2}{4Dt}\Big) dz, \quad (\gamma = u'(u_*)).$$

Whichever point of the straight line is taken, this solution will represent an increasing function of time irrespective of the values of the coefficient D. Assuming that at the initial instant $\delta u_0 = u_0 \delta(0)$ ($\delta(x)$ is delta function), we get

$$\delta u(t,x) = \frac{u_0}{2\sqrt{\pi Dt}} \exp\Big(-\frac{(x-\nu t)(x+\nu t)}{4Dt}\Big), \quad (\nu = 4D\gamma).$$

It follows that at the initial stage of growth the function $\delta(t,x)$ represents a traveling wave, distributed on the infinite line in both directions at the speed $\nu = 2\sqrt{D\gamma}$ [12].

6. Numerical experiment

Numerical solutions were used to numerically solve

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \mu u^\beta (1 - u^\gamma), \tag{10}$$

for boundary conditions (5) on the line $l = 1$. The equation (10) was approximated by finite differences [38-44] on a uniform grid for spatial variables with a step $h = l/n$ and step τ for time variables ($i = 2, 3, \dot{;} n-1$)

$$u_i(t) = u_i(t-\tau) + D\frac{\tau}{h^2}(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) + f(u_i(t)), \tag{11}$$

for boundary conditions (4): $u_2 - u_1 = 0$, $\quad u_n - u_{n-1} = 0$, for boundary conditions (5): $u_1 = 0$, $\quad u_n - u_{n-1} = 0$, where $u_i(t)$ – is the value of function in node i at an instant t, n – is the number of segments in the integration interval, τ is the integration interval for time variables. Simple iteration was used to solve the set of equations (11)–(13) for each interval of time [38-44]. For grids with $n = 500$ and $n = 100$ and the integration step $\tau = 0.1h^2/D$ for time variables the results matched with an accuracy of 1%. The iteration converges in 2–3 iterations with 0.1% accuracy set for maximum relative divergence for all the nodes on the grid.

The results of basic numerical experiments are depicted in Figure 1 and Figure 2. Figure 1 shows the variation of function $u = u(x)$ for boundary conditions (4) at different instants ($t = 0.04$, $t = 0.06$, $t = 0.08$, $t = 0.10$) for $\mu = 1000$, $D = 1$, $\beta = 1$, $\gamma = 1$. The initial value of function u was set to $u(t=0,x) = 0.002 \sin \pi x/2$. Figure 2 represents the solution of equation (10) for a boundary condition $u(t, x=0) = 1$ and an initial condition $u(t=0,x) = 0$ with $0 < x \le 1$ at instances $t = 0.001$, $t = 0.002$, $t = 0.003$, $t = 0.004$ ($\mu = 1000$, $\beta = 1$, $\gamma = 1$).

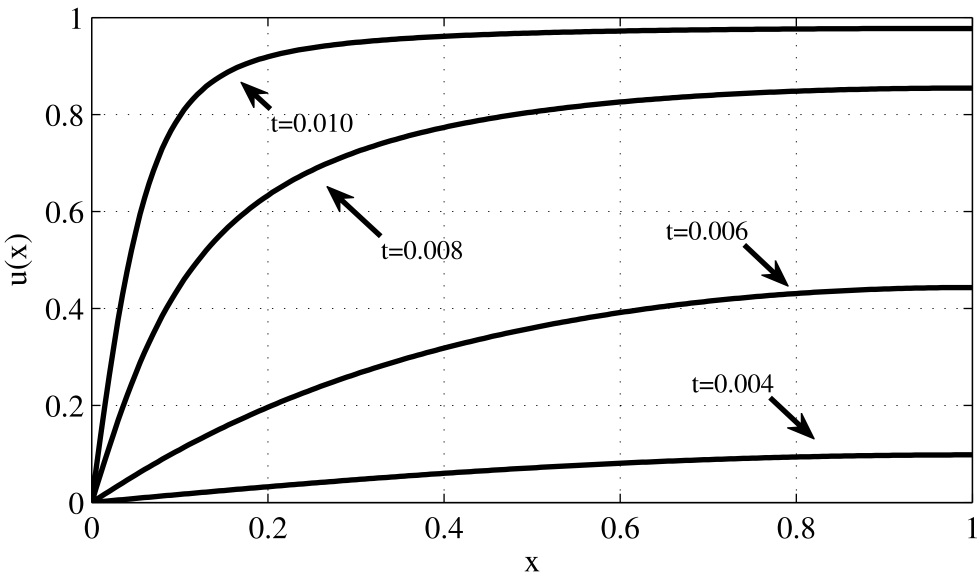

Figure 1: Solution to equation (10) at different instances for boundary conditions (4), $\mu = 1000$, $\beta = 1$, $\gamma = 1$

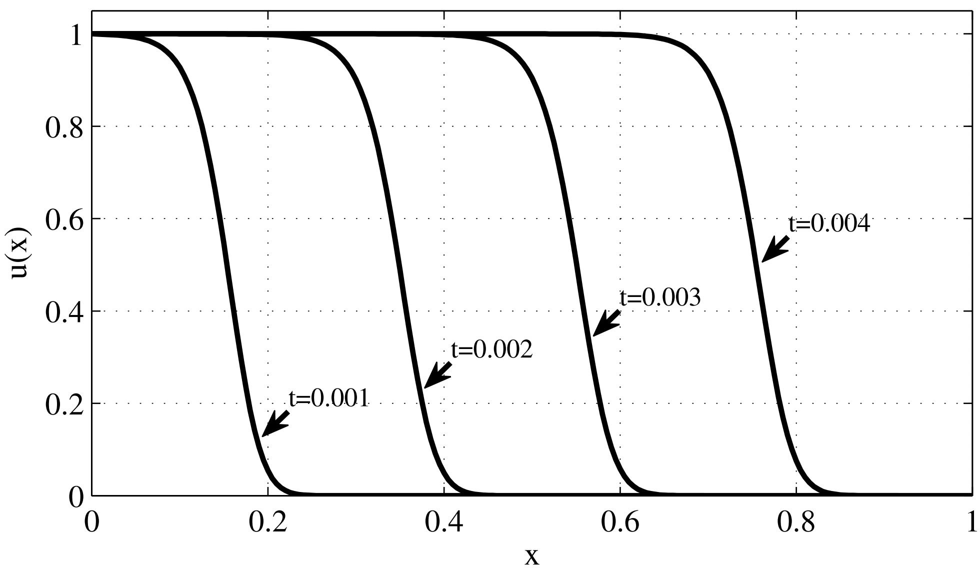

Figure 2: Solution to equation (10) at different instances for the boundary condition $u(x = 0) = 1$ and the initial condition $u(t = 0, x) = 0$ with $0 < x \leq 1$, $\mu = 10000$, $\beta = 1$, $\gamma = 1$

Our numerical experiments reveal that a logistic population showing low mobility along line with high birth rate ($D << \mu$) will gradually disperse along the whole length of the line irregardless of its germination coordinates. The dispersal can show a wave pattern, moving at the speed close to the theoretical value $2\sqrt{\mu D}$ [12, 24, 31].

7. Conclusion

The study obtained different results for diffusion and point models of single populations. A trivial solution is the only stable solution for cases when mobility exceeds critical value in a generalized logistic population on a line with limited growth on boundary areas with high mobility of species.

References

[1] Svirezhev, Y. M., 2008, "Nonlinearities in mathematical ecology: Phenomena and models. Would we live in Volterra's world?," Ecological Modelling, 216(2), pp. 89–101.

[2] Geritz, S. A. H., A. H., Kisdi, E., 2012, "Mathematical ecology: why mechanistic models?," Journal of Mathematical Biology, 65 (6–7), pp. 1411–1415. doi 10.1007/s00285-011-0496-3.

[3] Cobbold, C. A., Lutscher, F., 2014, "Mean occupancy time: linking mechanistic movement models, population dynamics and landscape ecology to population persistence," Journal of Mathematical Biology, 68(3), 549–579. doi 10.1007/s00285-013-0642-1.

[4] Isaev, A. S., Soukhovolsky, V. G., Ovchinnikova, T. M., 2008, "Phenomenological models of the forest plantations growth," Zhurnal Obshchei Biologii, 69(1), pp. 3–9.

[5] Kolesin, I. D., 2007, "A mathematical model of the development of an epidemic process with aerosol mechanism of infection," Biofizika, 52(1), pp. 147–150. PubMed ID: 17348408.

[6] Srinivas, P. M., Chaturvedi, A., 2015, "Codimension one bifurcation for a multispecies nonlinear system in a toxic environment," Global Journal of Pure and Applied Mathematics, 11(6), pp. 4297–4314.

[7] Kolesin, I. D., 2008, "Self-organization and formation of small groups, Journal of Computer and Systems Sciences International," 47(2), pp. 252–259. doi: 10.1134/S1064230708020123.

[8] Birch, D. A., Young, W. R., 2006, "A master equation for a spatial population model with pair interactions," Theoretical Population Biology, 70, pp. 26–42.

[9] Pusawidjayanti, K., Suryanto1, A., and Wibowo, R. B. E., 2015, "Dynamics of a Predator-Prey Model Incorporating Prey Refuge, Predator Infec-

tion and Harvesting," Applied Mathematical Sciences, 9(76), pp, 3751–3760. http://dx.doi.org/10.12988/ams.2015.54340.

[10] Almanza-Vasquez, E., Ortiz-Ortiz, R. D.,. Marin-Ramirez, A. M., 2015, "Dynamic Consequences of Lotka-Volterra Predation Model when the Area Inhabited by the Preys and Predators Decreases," Applied Mathematical Sciences, 9 (37–40), pp. 1961–1970. http://dx.doi.org/10.12988/ams.2015.5172.

[11] Andayani P., Kusumawinahyu, W. M., 2015, "Global Stability Analysis on a Predator-Prey Model with Omnivores," Applied Mathematical Sciences, 9(36), pp. 1771–1782. http://dx.doi.org/10.12988/ams.2015.5120.

[12] Zhukova, I. V., Kolpak, E. P., Balykina, Yu. E., 2014, "Mathematical Model of Growing Tumor," Applied Mathematical Sciences, 8(29–32), pp. 1455–1466. http://dx.doi.org/10.12988/ams.2014.4135.

[13] Sieber, M., Hilker, F. Y., 2012, "The hydra effect in predator–prey models," Journal of Mathematical Biology, 54, pp. 341–360. doi 10.1007/s00285-011-0416-6.

[14] Hasik, K., 2010, "On a predator–prey system of Gause type," Journal of Mathematical Biology, 60, pp. 59–74. doi 10.1007/s00285-009-0257-8.

[15] Kuwamura, M., Nakazawa, T., Ogawa, T., 2009, "A minimum model of prey-predator system with dormancy of predators and the paradox of enrichment," Journal of Mathematical Biology, 58, pp. 459–479. doi 10.1007/s00285-008-0203-1.

[16] Gasratova, N. A., Gasratov, M. G., 2015, "A network model of inventory control in the case of a quantitative competition," Journal of Applied and Industrial Mathematics, 9(2), pp. 165–178.

[17] Kolpak, E. P., Kabrits, S. A., Bubalo, V., 2015, "The follicle function and thyroid gland cancer," Biology and Medicine, 7(1), BM060.15.

[18] Balykina, Y. E., Kolpak, E. P., Kotina, E. D., 2014, "Mathematical model of thyroid function," Middle East Journal of Scientific Research, 19, pp. 429–433.

[19] Mindlin, Y. B., Kolpak, E. P., Gasratova, N. A., 2016, "Clusters in system of instruments of territorial development of the Russian Federation," International Review of Management and Marketing, 6(1), pp. 245–249.

[20] Shiryaev, D. V., Litvinenko, I. L., Rubtsova, N. V., Kolpak, E. P., Blaginin, V. A., Zakharova, E. N., 2016, "Economic Clusters as a form of Self-organization of the Economic System," International Journal of Economics and Financial Issues, 6(1S), pp. 284–288.

[21] Grigorenko, O. V., Klyuchnikov, D. A., Gridchina, A. V., Litvinenko, I. L., Kolpak, E. P., 2016, "The Development of Russian-Chinese Relations: Prospects for Cooperation in Crisis," International Journal of Economics and Financial Issues, 6(1S), pp. 256–260.

[22] Kolpak, E. P., Frantsuzova, I. S., Alexandrovich, K. S., 2016, "A mathematical model of thyroid tumor," Global Journal of Pure and Applied Mathematics, 12(1), pp. 55–66.

[23] Liao, J., Li, Z., Hiebeler, D.E., El-Bana, M., Deckmyn, G., Nijs, I., 2013, "Modelling plant population size and extinction thresholds from habitat loss and habitat fragmentation: Effects of neighbouring competition and dispersal strategy," Ecological Modelling, 268, pp. 9–17.

[24] Schofield, P., 2001, "Spatial explicit models of Turelli-Hoffmann Wolbachia invasive wave fronts," J. Theor. Biol., 212, pp. 121–131.

[25] Sharapova, T. A., 2010, "Spatial structure of zooperiphyton in a small river (West Siberia)," Inland Water Biology, 3(2), pp. 149–154.

[26] Burskii, O. V., Demidova, E. Iu., Morkovin, A. A., 2014, "Population dynamics of thrushes and seasonal resource partition," Zhurnal obshchei biologii, 75(4), pp. 287–301.

[27] Parfenyuk, S. B., Khrenov, M. O., Novoselova, T. V., Glushkova, O. V., Lunin, S. M., Fesenko, E. E., Novoselova, E. G., 2010, "Stressful effects of chemical toxins at low concentrations," Biophysics, 55(2), pp. 317–323.

[28] Railkin, A. I., Dobretsov, S. V., 1994, "Effect of bacterial repellents and narcotising substances on marine macrofouling," Russian Journal of Marine Biology, 20, 16.

[29] Kolpak, E. P., Maltseva, L. S., Ivanov, S. E., Kabrits, S. A., 2016, "Non-Linear Vibrations of Rubber Membrane," Applied Mathematical Sciences, 10(35), pp. 1797–1810. http://dx.doi.org/10.12988/ams.2016.6388.

[30] Kolpak, E. P., Ivanov, S. E., 2016, "On the three-dimensional Klein-Gordon equation with a cubic nonlinearity," International Journal of Mathematical Analysis, 10(13), pp. 611–622. http://dx.doi.org/10.12988/ijma.2016.611.

[31] Bressloff, P. C., 2013, "Propagation of CaMKII translocation waves in heterogeneous spiny dendrites," J. Math. Biol., 66, pp. 1499–1525. doi 10.1007/s00285-012-0542-9.

[32] Markov, A. V., Korotayev, A. V., 2007, "The dynamics of Phanerozoic marine animal diversity agrees with the hyperbolic growth model," Zhurnal Obshchei Biologii, 68(1), pp. 3–18.

[33] Murray, J. D, Mathematical biology, Springer-Verlag, New York, 2002.

[34] Kareiva, P. M., 1983, "Local movement in herbivorous insects: applying a passive diffusion model to mark-recapture field experiments," Oecologia, 57, pp. 322–327.

[35] Kot, M., "Elements of mathematical ecology," Cambridge University Press, 2001.

[36] Kolpak, E. P., Maltseva, L. S., Ivanov, S. E., 2015, "On the Stability of Compressed Plate," Contemporary Engineering Sciences, 8(20), pp. 933–942. http://dx.doi.org/10.12988/ces.2015.57213.

[37] Bondarenko, L. A., Zubov, A. V., Zubova, A. F., Zubov, S. V., Orlov, V. B., 2015, "Stability of quasilinear dynamic systems with after effect," Biosciences Biotechnology Research Asia, 12(1), pp. 779–788. doi: 10.13005/bbra/1723.

[38] Bondarenko, L. A., Zubov, A. V., Orlov, V. B., Petrova, V. A., Ugegov, N. S., 2016, "Application in practice and optimization of industrial information systems," Journal of Theoretical and Applied Information Technology, 85(3), pp. 305–308.

[39] Mickens, R. E., 2005, "A nonstandard finite difference scheme for a PDE modeling combustion with nonlinear advection and diffusion," Mathematics and computers in simulation, 69, pp. 439–446.

[40] Kolpak, E. P., Maltseva, L. S., 2015, "Rubberlike Membranes at Inner Pressure, Contemporary Engineering Sciences," 8(36), pp. 1731–1742. http://dx.doi.org/10.12988/ces.2015.510289.

[41] Sebastian, E., Victor, P., 2016, "Stability analysis of a discrete-time prey-predator model with allee effect in the predator using center manifold theory," Global Journal of Pure and Applied Mathematics, 12(1), pp. 73–77.

[42] Kabrits, S. A., Terent'ev, V. F., 1984, "Numerical solution of one-dimensional nonlinear statics problems for elastic rods and shells in the presence of rigid constraits," Soviet Applied Mechanics, 20(7), pp. 672–675. doi: 10.1007/BF00891729.

[43] Mickens, R. E., Washington, T. M., 2013, "NSFD discretizations of interacting population models satisfying conservation laws," Computers and Mathematics with Applications, 66(11), pp. 2307–2316. doi: 10.1016/j.camwa.2013.06.011.

[44] Chapwanya, M., Lubuma, J. M., Mickens, R. E., 2014, "Positivity-preserving nonstandard finite difference schemes for cross-diffusion equations in biosciences," Computers and Mathematics with Applications, 68(9), pp. 1071–1082. doi: 10.1016/j.camwa.2014.04.021.