Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 4 (2016), pp. 3621–3629

http://www.ripublication.com/gjpam.htm

Signed chip-firing games on weighted graphs

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Abstract

In this paper, we consider a general version of the chip firing games, called the signed chip firing games on weighted graphs. After studying basic properties on the game, we characterize the score of the conventional chip firing games from the viewpoint of the signed chip firing games. We also provide an alternative proof of the well known fundamental theorem, called the *uniqueness of the final configuration* of chip firing games.

AMS subject classification: 05C57, 91A43.
Keywords: Chip firing game, final configuration.

1. Introduction

Chip-firing is a game played on a graph G with the following rule: We put some non-negative integral number of chips on each vertex of G. A vertex v is *ready* if it has at least as many chips as its degree. In that case we may *fire* it and the result is that it distributes one chip to each of its neighbors. This can cause another vertex to be ready, and so on. If there is no vertex which is ready, the game terminates. The game is used as a mathematical model of various phenomena on networks (cf. communication network model in which chips represent packets or jobs) and has recently been developed by many authors (cf.[1], [2], [3] and [4]).

The chip firing games can be generalized to those on *weighted* graphs by making a simple modification of its rule. If a vertex is fired, it distributes an amount of chips to each of its neighbors as much as the given weight of the edge between them. In this case, the amount of chips moved by firing is not an integral number but a positive real number. Throughout this paper, every chip-firing game is assumed to be the game on a weighted graph.

In the conventional chip firing games, it is assumed that the amount of chips on each vertex is nonnegative, but if we allow the negative amount of chips, it can be more widely used for modeling the phenomena on networks. For example, it can be used for modeling the economic network in which chips are considered as money and the negative number of chips are regarded as a debt.

In this paper, we consider a general version of the chip firing game, called the *signed chip firing game* or simply, the *signed game* which allows the negative amount of chips on each vertex. Our main concern is to investigate the relation between the conventional chip firing games and the signed games. We first characterize the *score* of the conventional chip firing games from the viewpoint of signed games and we provide an alternative proof of the well known fundamental theorem, called the *uniqueness of the final configuration* of chip firing games by using a reordering scheme which is developed in the signed chip firing games. The existing proof of the theorem (see, [2]) is difficult to understand, but the proof provided in this paper is intuitive and easy to understand.

2. Graph theoretic notions

In this section, we start with the graph theoretic notions frequently used throughout this paper.

By a *graph* $G = G(V, E)$ we mean a finite set $V(G)$ (or simply V) of *vertices* with a set $E(G)$ (or simply E), a subset of $V \times V$ whose elements are called *edges*. By $\{x, y\} \in E$ or $x \sim y$ we mean that two vertices x and y are joined by an edge. Conventionally used, we denote by $x \in V$ or $x \in G$ the fact that x is a vertex in G.

A *weight* on a graph $G(V, E)$ is a function $w : V \times V \to [0, \infty)$ satisfying

$$\begin{aligned}
&\text{(i) } w(x, x) = 0,\ x \in V,\\
&\text{(ii) } w(x, y) = w(y, x) \text{ if } x \sim y\ ,\\
&\text{(iii) } w(x, y) = 0 \text{ if and only if } \{x, y\} \notin E.
\end{aligned}$$

In particular, a weight function w satisfying

$$w(x, y) = 1, \qquad \text{if } x \sim y$$

is called the *standard* weight on G. A graph $G(V, E)$ associated with a weight w is said to be the *weighted graph* $G(V, E; w)$. The *degree* of a vertex x, denoted by $\deg_w x$, is defined to be

$$\deg_w x := \sum_{y \in V} w(x, y).$$

Throughout this paper, a function on a graph is understood as a function defined just on the set of vertices of the graph. The discrete *Laplacian* Δ_w of a function f on $G(V, E; w)$ is defined by

$$\Delta_w f(x) = \sum_{y \in V} w(x, y)\{f(y) - f(x)\},\ x \in V.$$

Considering a function on a graph as a vector, the Laplacian $-\Delta_w : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}$ is an linear operator, so that it can be regarded as a $|V| \times |V|$ matrix indexed by the vertices of G whose x, y entries are given by

$$-\Delta_w(x, y) = \begin{cases} \deg_w x & , \text{ if } x = y\ , \\ -w(x, y) & , \text{ if } x \neq y. \end{cases}$$

Here, $|V|$ denotes the number of elements in the set V.

3. Chip-firing games on weighted graphs

We start this section with recalling some definitions and notations related to the chip-firing game. Throughout this paper, graphs are assumed to be finite, simple, connected and weighted.

Suppose that an amount of chip is piled on each vertex in a graph G. By a *configuration*, we mean a vector whose components are given by the quantity of chips piled on each vertex. Let C_0 denote the initial configuration of a chip-firing game.

Playing a game following the rules of chip-firing game up to N-th configuration from some configuration, one obtains a *firing word*

$$\mathcal{F} = (\mathcal{F}(1) \to \mathcal{F}(2) \to \cdots \to \mathcal{F}(N))$$

where $\mathcal{F}(i)$, called a *letter*, corresponds to a vertex fired at $(i - 1)$-th configuration from the given configuration, for $i = 1, 2, \ldots, N$. To sum up, firing word means the order of vertices fired during the chip-firing game(See [3] for more details). For a given firing word $\mathcal{F}$, the function $C_{\mathcal{F}} : V \to \mathbb{R}^+ \cup \{0\}$, called the *configuration under* $\mathcal{F}$, denotes the configuration after playing a game under the firing word $\mathcal{F}$ from a given initial configuration. In particular, for a given initial configuration C_0, if there exists a firing word $\mathcal{F}$ such that there is no $x \in V$ satisfying $C_{\mathcal{F}}(x) \geq \deg_w x$, which means there is no vertex to be fired in the configuration under $\mathcal{F}$, then $C_{\mathcal{F}}$ is said to be a *final configuration* induced by C_0 under $\mathcal{F}$. If we get to the final configuration, the game is said to be *terminated*. Figure 1 shows an example of chip firing games.

For two firing words $\mathcal{F} = (\mathcal{F}(1) \to \mathcal{F}(2) \to \cdots \to \mathcal{F}(m))$ and $\mathcal{G} = (\mathcal{G}(1) \to \mathcal{G}(2) \to \cdots \to \mathcal{G}(n))$, we define $\mathcal{F} \vee \mathcal{G}$ by

$$\mathcal{F} \vee \mathcal{G} := (\mathcal{F}(1) \to \mathcal{F}(2) \to \cdots \to \mathcal{F}(m) \to \mathcal{G}(1) \to \mathcal{G}(2) \to \cdots \to \mathcal{G}(n)).$$

The *score* $f : V \to \mathbb{N}_0$ under $\mathcal{F}$ is the function such that $f(x)$ is the number of firing on the vertex x during the game followed by the firing word $\mathcal{F}$. The following results show that the score is a solution to the discrete Poisson equation with source $C_0 - C_{\mathcal{F}}$.

Lemma 3.1. Take any firing word $\mathcal{F}$. If x_0 is a vertex to be fired after performing the $\mathcal{F}$ then we have

$$-\Delta_w \delta_{x_0}(x), = C_{\mathcal{F}}(x) - C_{\mathcal{F} \vee (x_0)}(x), \quad x \in V, \tag{1}$$

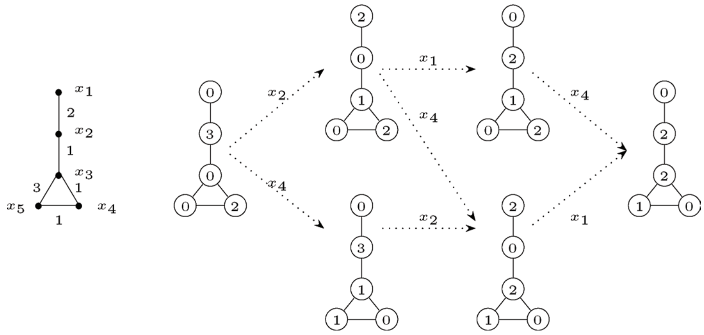

Figure 1: Example for chip-firing games on a weighted graph.

where

$$\delta_{x_0}(x) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since

$$\begin{aligned} -\Delta \delta_{x_0}(x) &= \sum_{y \in V} \left[\delta_{x_0}(x) - \delta_{x_0}(y)\right] w(x, y) \\ &= \delta_{x_0}(x) \deg_w x - \sum_{y \in V} \delta_{x_0}(y) w(x, y) \\ &= \begin{cases} -\deg_w x_0, & x = x_0 \\ w(x, x_0), & x \neq x_0, x \sim x_0 \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

we have the result. ■

The following theorem is easily proved from the above lemma.

Theorem 3.2. If f is the score under $\mathcal{F}$, then we have

$$-\Delta_w f(x) = C_0(x) - C_{\mathcal{F}}(x), \ x \in G. \tag{2}$$

When the chip-firing game terminates, we define the *length* of the game as $\sum_{x \in V} f(x)$ where f is a score under a firing word which induces the final configuration.

4. Signed chip firing games on weighted graphs

In this section, we introduce a general version of the chip firing game, called the *signed chip firing game* which allows the negative amount of chips on each vertex. As far as the authors' knowledge, this type of game has never seen in the literature.

A signed chip-firing game (or simply, *signed game*) is a chip-firing game without the rule that a vertex v is fired only if it has at least as much chip as the degree of v as one

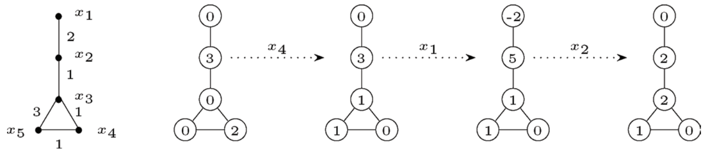

Figure 2: An example of the signed game.

can see in the Figure 2, for example. A firing word $\mathcal{F}$ generating a signed game is called a *signed firing word* and a configuration $S_{\mathcal{F}}$ under a signed firing word $\mathcal{F}$ is called a *signed configuration* under $\mathcal{F}$.

Remark 4.1. In the signed game, every vertex at any configuration can be fired. Therefore the set of scores coincides with $\mathbb{N}_0^N$ where N denotes the number of vertices and $\mathbb{N}_0$ denotes the set of nonnegative integers. Moreover, the final configuration is meaningless in this game.

Without any modification of the proof, it is shown that Theorem 3.2 still remains true for the signed games.

Theorem 4.2. Let $\mathcal{F}$ be a firing word generating a signed game on $G(V, E; w)$ with the initial configuration S_0. If f is the signed score under $\mathcal{F}$, then we have

$$-\Delta_w f(x) = S_0(x) - S_{\mathcal{F}}(x), \; x \in G. \tag{3}$$

Let $\mathcal{F}$ and $\mathcal{G}$ be the firing word for a signed game with the same initial configuration. It follows easily from the above theorem that if $\mathcal{F}$ and $\mathcal{G}$ induces the same score f, then we have $S_{\mathcal{F}} = S_{\mathcal{G}}$. Namely, the configuration is uniquely determined by the score.

For a word $\mathcal{F}$, we mean by its reordering word $\mathcal{F}'$ that $\mathcal{F}'$ is a word obtained by reordering the letters in $\mathcal{F}$. Unlike the conventional game, in the signed game, every reordering $\mathcal{F}'$ of a firing word $\mathcal{F}$ is also a firing word.

Using Theorem 4.2, we have the following result.

Theorem 4.3. Suppose an initial configuration S_0 is given. For a firing word $\mathcal{F}$ and any of its reordering $\mathcal{F}'$, the configurations under $\mathcal{F}$ and $\mathcal{F}'$ are the same.

Proof. Since $\mathcal{F}$ and $\mathcal{F}'$ have the same score, say f, it follows from Theorem 4.2 that

$$S_{\mathcal{F}} = -\Delta_w f + S_0 = S_{\mathcal{F}'}.$$

■

5. Main results

In this section, we study a reordering scheme of signed firing words and apply this scheme to prove two main results: a characterization of score and the uniqueness of finial configuration for conventional chip firing games.

Let a word $\mathcal{F} = (\mathcal{F}(1) \to \mathcal{F}(2) \to \cdots \to \mathcal{F}(m))$ be given. For a word $\mathcal{G}$, a reordering of $\mathcal{G}$ constructed by the following algorithm is said to be the *reordering of $\mathcal{G}$ associated to $\mathcal{F}$*, which is denoted by $\mathcal{G}_{\mathcal{F}}$.

Step 1. We first construct a word $\mathcal{H}$ as follows.

If there is no letter in $\mathcal{G}$ then $\mathcal{H}$ is an empty word and this step terminates.

If $\mathcal{F}(1)$ is a letter in $\mathcal{G}$, then take $\mathcal{F}(1)$ as the first letter of $\mathcal{H}$. Then delete the letter $\mathcal{F}(1)$ in $\mathcal{G}$ and denote it by $\mathcal{G}_1$. But if $\mathcal{F}(1)$ is not a letter in $\mathcal{G}$, then take $\mathcal{H}$ as an empty word and this step terminates.

Again, if $\mathcal{F}(2)$ is a letter in $\mathcal{G}_1$ then take $\mathcal{F}(2)$ as the second letter of $\mathcal{H}$. We obtain the word $\mathcal{G}_2$ by deleting the letter $\mathcal{F}(2)$ in $\mathcal{G}_1$. But if $\mathcal{F}(2)$ is not a letter in $\mathcal{G}$, then take $(\mathcal{F}(1))$ as $\mathcal{H}$ and this step terminates.

By the same manner, construct $\mathcal{H} = (\mathcal{F}(1) \to \cdots \to \mathcal{F}(n))$ until $\mathcal{F}(n+1)$ is not a letter in $\mathcal{G}_n$ and this step terminates.

Step 2. If the word $\mathcal{H}$ in Step 1 is an empty word, then we define

$$\mathcal{G}_{\mathcal{F}} := \mathcal{G}.$$

If the word $\mathcal{H}$ in Step 1 is $\mathcal{H} = (\mathcal{F}(1) \to \cdots \to \mathcal{F}(n))$ for some $n \in \mathbb{N}$, then we define

$$\mathcal{G}_{\mathcal{F}} := \mathcal{H} \vee \mathcal{G}_n.$$

For example, assume that we have two words

$$\mathcal{F} = (x_2 \to x_1 \to x_4 \to x_3 \to x_1 \to x_5 \to x_1)$$

and

$$\mathcal{G} = (x_1 \to x_3 \to x_3 \to x_2 \to x_4 \to x_5).$$

Following the step 1 of the above algorithm, we obtain $\mathcal{H} = (x_2 \to x_1 \to x_4 \to x_3)$ and $\mathcal{G}_4 = (x_3 \to x_5)$. Therefore the reordering of $\mathcal{G}$ associated to $\mathcal{F}$ is obtained as

$$\mathcal{G}_{\mathcal{F}} = (x_2 \to x_1 \to x_4 \to x_3 \to x_3 \to x_5).$$

Remark 5.1. In the above algorithm, if we define $\mathcal{G}_0 := \mathcal{G}$, then we can say that for each word $\mathcal{F}$ and $\mathcal{G}$, there exist a word (including an empty word) $\mathcal{H}$ and a nonnegative integer n such that the reordering of $\mathcal{G}$ associated to $\mathcal{F}$ can be represented as

$$\mathcal{G}_{\mathcal{F}} = \mathcal{H} \vee \mathcal{G}_n.$$

The following lemma plays an important role to prove the main results.

Lemma 5.2. Suppose an initial configuration C_0 and a firing word $\mathcal{F}$ for a conventional chip firing game is given on $G(V, E; w)$. For a given signed firing word $\mathcal{G}$, define a subset V_0 of V by

$$V_0 := \{x \in V \mid g(x) < f(x)\}.$$

If

$$V_0 \neq \emptyset \tag{4}$$

then there is a vertex $y \in V_0$ such that

$$S_{\mathcal{G}}(y) \geq \deg_w y.$$

Here, f and g denote the scores under $\mathcal{F}$ and $\mathcal{G}$, respectively.

Remark 5.3. In the above lemma, if $\mathcal{G}$ is also a firing word for a conventional game, then (5.2) means that $C_{\mathcal{G}}$ is not a final configuration.

Proof. Let $\mathcal{G}_{\mathcal{F}}$ be the reordering of $\mathcal{G}$ associated to $\mathcal{F}$. Then we have

$$\mathcal{G}_{\mathcal{F}} = \mathcal{H} \vee \mathcal{G}_n$$

where the words $\mathcal{H}$ and $\mathcal{G}_n$ are obtained following the previous algorithm. Then it is easy to see from the algorithm and (4) that there exists a nonempty normal word $\mathcal{F}'$ satisfying

$$\mathcal{F} = \mathcal{H} \vee \mathcal{F}'.$$

Now, let y be the first letter of $\mathcal{F}'$. Then y is in V_0 and $C_{\mathcal{H}}(y) \geq \deg_w y$. Also, it is easy to see from the algorithm that the word $\mathcal{G}_n$ does not contain y as its letter. Thus we have

$$C_{\mathcal{G}_{\mathcal{F}}}(y) \geq C_{\mathcal{H}}(y).$$

Then it follows from Theorem 4.3 that,

$$C_{\mathcal{G}}(y) = C_{\mathcal{G}_{\mathcal{F}}}(y),$$

which implies

$$C_{\mathcal{G}}(y) = C_{\mathcal{G}_{\mathcal{F}}}(y) \geq C_{\mathcal{H}}(y) \geq \deg_w y.$$

■

We now discuss the relation between signed games and the conventional chip firing games, which is one of the main results of this paper.

Theorem 5.4. Let an initial configuration C_0 be given on a weighted graph $G(V, E; w)$. A function f is a score of the conventional game which induces the final configuration if and only if

$$f(x) = \min_{g \in \mathcal{A}} g(x), \quad x \in G. \tag{5}$$

Here, $\mathcal{A}$ is the set of the signed scores g (with its firing word $\mathcal{G}$) satisfying

$$S_{\mathcal{G}}(x) < \deg_w x, \; x \in V.$$

Proof. ($\Rightarrow$) Since it is clear that $f \in \mathcal{A}$, (5) follows easily from Lemma 5.2.
($\Leftarrow$) Let f be a function given by (5) and $\mathcal{F}$ be a word which induces the score f. Then $C_{\mathcal{F}}(x) = -\Delta_w f(x) + C_0(x)$ for every $x \in V$. Now we show that

$$C_{\mathcal{F}}(x_0) < \deg_w x_0.$$

To the contrary, we suppose that there is a vertex x_0 in V such that $C_{\mathcal{F}}(x_0) \geq \deg_w x_0$. By (5), there is a function $h(\neq f)$ in $\mathcal{A}$ such that $f(x) \leq h(x)$ for every x in $V \backslash \{x_0\}$ and $f(x_0) = h(x_0)$. Then

$$-\Delta_w h(x_0) + C_0(x_0) \geq C_{\mathcal{F}}(x_0).$$

Thus

$$-\Delta_w h(x_0) + C_0(x_0) \geq \deg_w x_0.$$

This leads a contradiction. Now, we show that $C_{\mathcal{F}}(x) \geq 0$ for every x in V. To the contrary, we suppose that there is y_0 in V satisfying $C_{\mathcal{F}}(y_0) < 0$. Then there is $k_0 \in \mathbb{N}$ satisfying

$$0 \leq C_{\mathcal{F}}(y_0) + k_0 \deg_w y_0 < \deg_w y_0.$$

Put

$$h(x) = \begin{cases} f(x) & \text{, for every } x \in V \setminus \{y_0\} \\ f(y_0) - k_0 & \text{, for } y_0 \end{cases}$$

Then

$$-\Delta_w h(x) + C_0(x) = \begin{cases} C_{\mathcal{F}}(x) - k_0 w(y_0, x) & , x \in V \setminus \{y_0\} \\ C_{\mathcal{F}}(y_0) + k_0 \deg_w y_0 & \text{, at } y_0 \end{cases}$$

Thus $h \in \mathcal{A}$ but $h(y_0) < f(y_0)$, which leads a contradiction. Finally, it remains to show that that we can make a normal word by reordering $\mathcal{F}$. Let $\mathcal{G}$ be a normal word which induces a final configuration $C_{\mathcal{G}}$ and g be a score vector under $\mathcal{G}$. It is enough to show that $f = g$. Suppose that $f \neq g$. Since $g \in \mathcal{A}$, by (5), the set $V_0 = \{x \in V \mid f(x) < g(x)\}$ is nonempty. Thus by Lemma 5.2, there is a vertex x_0 in V_0 satisfying

$$C_{\mathcal{F}}(x_0) \geq \deg_w x_0.$$

It is a contradiction to the fact that $0 \leq C_{\mathcal{F}}(x) < \deg_w x$ for every x in S. This completes the proof. ∎

In [2], A. Björner, L. Lovász and P. W. Shor showed that if the chip-firing game on a graph terminates, then the final configuration of the chips does not depends on the order of the vertex fired during the chip-firing game on a graph but depends only on the given graph and the initial configuration. See Figure 1, for example.

Using the properties of the signed chip firing games, we provide an alternative proof of the above result on weighted graphs, which is more comprehensive than the original one.

Theorem 5.5. Suppose an initial configuration C_0 of (conventional) chip firing game on $G(V, E; w)$ is given. If the game terminates, then it has the unique final configuration.

Proof. Suppose $\mathcal{F}_1$ and $\mathcal{F}_2$ are the firing words of the given game, which induce final configurations $C_{\mathcal{F}_1}$ and $C_{\mathcal{F}_2}$. Applying Lemma 5.2 with $\mathcal{G} = \mathcal{F}_2$, we have

$$f_2(x) \geq f_1(x), \quad x \in V.$$

Changing the role of $\mathcal{F}_1$ and $\mathcal{F}_2$, we also have

$$f_1(x) \geq f_2(x), \quad x \in V.$$

Thus $f_1 \equiv f_2$ and it follows from Theorem 3.2 that $C_{\mathcal{F}_1} = C_{\mathcal{F}_2}$. ■

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2015R1D1A1A01058826).

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