

# **Rational Block Method for the Numerical Solution of First Order Initial Value Problem II: A-Stability and L-Stability**

**Teh Yuan Ying\***

*School of Quantitative Sciences, UUM College of Arts and Sciences,  
Univertiti Utara Malaysia, 06010 UUM Sintok, Kedah Darul Aman, Malaysia.*

**Zurni Omar**

*School of Quantitative Sciences, UUM College of Arts and Sciences,  
Univertiti Utara Malaysia, 06010 UUM Sintok, Kedah Darul Aman, Malaysia.*

**Kamarun Hizam Mansor**

*School of Quantitative Sciences, UUM College of Arts and Sciences,  
Univertiti Utara Malaysia, 06010 UUM Sintok, Kedah Darul Aman, Malaysia.*

## **Abstract**

In this study, an  $A$ -stable 2-point explicit rational block method and a  $L$ -stable 2-point explicit rational block method are developed for the numerical solution of first order initial value problems. Local truncation error analyses showed that the  $A$ -stable 2-point explicit rational block method possesses second order of accuracy, while the  $L$ -stable 2-point explicit rational block method possesses first order of accuracy. The absolute stability analyses confirmed the absolute stability characteristics of the proposed methods. Several test problems are solved using the new methods and two existing rational methods via the variable step-size approach. Numerical results generated by the proposed explicit rational block method are promising in terms of numerical accuracy and computational cost.

## **1. INTRODUCTION**

Rational block methods (RBMs) are block multistep methods that are based on rational approximants. Like conventional block multistep methods, RBMs can be considered as a set of simultaneously applied rational multistep methods to obtain

several numerical approximations within each integration step. The concept of RBMs is introduced to overcome the stability drawbacks and at the same time, retain the advantages of conventional block multistep methods, which being less expensive in terms of functions evaluations. For excellence surveys and various perspectives of conventional block multistep methods, see, for example, Sommeijer et al. [1], Watanabe [2], Ibrahim et al. [3, 4], Chollom et al. [5], Majid et al. [6, 7], Mehrkanoon et al. [8], Akinfenwa et al. [9], Ehigie et al. [10], Ibijola et al. [11] and Majid and Suleiman [12].

The aim of this current study is to develop RBMs with tighter requirements of absolute stability. These requirements are the  $A$ -stability and the  $L$ -stability. It is well known that  $A$ -stability and  $L$ -stability are both demanding requirements, particularly for conventional linear multistep method [13]. Although a RBM possesses similar structure of a multistep method, it is easier to establish  $A$ -stability and  $L$ -stability in a RBM due to its nonlinearity formed by the rational approximant therein. For excellence surveys and various perspectives on how  $A$ -stability and  $L$ -stability can be achieved through rational approximants, one can refer to, for examples, Lambert [14], Lambert and Shaw [15], Luke et al. [16], Fatunla [17, 18], van Niekerk [19, 20], Ikhile [21, 22, 23], Ramos [24], Okosun and Ademiluyi [25, 26], Teh et al. [27, 28], Yaacob et al. [29], and Teh and Yaacob [30, 31].

When developing a RBM, we have the freedom to decide the structure of the underlying rational approximant, either the degree of numerator greater than the degree of denominator, the degree of denominator greater than the degree of numerator, or both numerator and denominator in equal degree. By adopting the concepts of  $A$ -acceptable and  $L$ -acceptable for one-step Runge-Kutta method to RBMs [13], we found out that an  $A$ -stable RBM can be developed by considering a rational approximant with both numerator and denominator in equal degree, whereas a  $L$ -stable RBM can be developed if the underlying rational approximant has the degree of denominator greater than the degree of numerator.

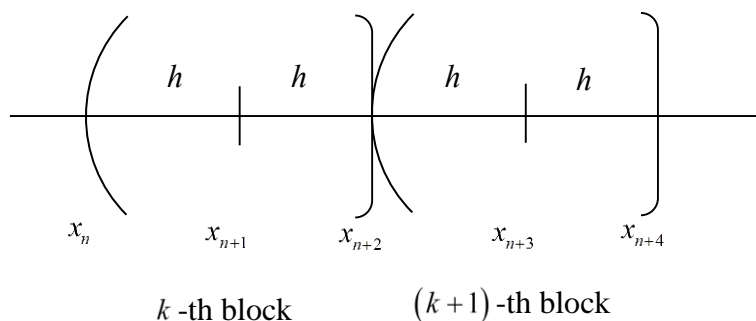
In Section 2 and Section 3, we demonstrate the developments of an  $A$ -stable 2-point explicit RBM and a  $L$ -stable 2-point explicit RBM, respectively. We also demonstrate the calculations of principal local truncation error terms for each of the developed 2-point RBM. The absolute stability analyses are included to verify the absolute stability characteristics for each of the developed RBM. Section 4 talks about the variable step-size strategy suited for the developed RBMs. In Section 5, numerical comparisons are made between the developed RBMs and some existing rational methods, and discussions are carried out based on the numerical findings. Finally, some useful conclusions are drawn.

## 2. DERIVATION OF $A$ -STABLE 2-POINT EXPLICIT RATIONAL BLOCK METHOD

The  $A$ -stable 2-point explicit rational block method, or in brief as 2-point ERBM-A, is formulated to solve the following first order initial value problem given by

$$y' = f(x, y), y(a) = \eta, \quad (1)$$

where  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $f(x, y)$  is assumed to satisfy all the required conditions such that problem (1) possesses a unique solution. Suppose that the interval of numerical integration is  $x \in [a, b] \subset \mathbb{R}$  and is divided into a series of blocks with each block containing two points as shown in Figure 1.



**Figure 1:** Two 2-point Consecutive Blocks.

From Figure 1, we observe that  $k$ -th block contains three points  $x_n$ ,  $x_{n+1}$  and  $x_{n+2}$ , and each of these points is separated equidistantly by a constant step-size  $h$ . The next  $(k + 1)$ -th block also contains three points. In the  $k$ -th block, we want to use the values  $y_n$  at  $x_n$  to compute the approximation values of  $y_{n+1}$  and  $y_{n+2}$  simultaneously. In the  $(k + 1)$ -th block, the previously computed value of  $y_{n+2}$  is used to generate the approximated values of  $y_{n+3}$  and  $y_{n+4}$ . The same computational procedure is repeated to compute the solutions for the next few blocks until the end-point i.e.  $x = b$  is reached. The evaluation information from the previous step in a block can be used for other steps of the same block.

Along the  $x$ -axis, we consider the points  $x_n$ ,  $x_{n+1}$  and  $x_{n+2}$  to be given by

$$x_n = x_0 + nh, \tag{2}$$

$$x_{n+1} = x_0 + (n + 1)h, \tag{3}$$

and

$$x_{n+2} = x_0 + (n + 2)h, \tag{4}$$

where  $h$  is the step-size. Let us assume that the approximate solution of (1) is locally represented in the range  $[x_n, x_{n+1}]$  by the rational approximant

$$R(x) = \frac{a_0 + a_1x}{b_0 + x}, \tag{5}$$

where  $a_0$ ,  $a_1$  and  $b_0$  are undetermined coefficients. We note that the numerator and denominator of rational approximant (5) are both degree 1 (i.e. numerator and denominator in equal degree). This rational approximant in equation (5) is required to

pass through the points  $(x_n, y_n)$  and  $(x_{n+1}, y_{n+1})$ , and moreover, must assume at these points the derivatives given by  $y' = f(x, y)$  and  $y'' = f'(x, y)$ . Altogether, there are four equations to be satisfied i.e.

$$R(x_n) = y_n, \quad (6)$$

$$R(x_{n+1}) = y_{n+1}, \quad (7)$$

$$R'(x_n) = y'_n, \quad (8)$$

and

$$R''(x_n) = y''_n, \quad (9)$$

where  $y'_n = f_n = f(x_n, y_n)$  and  $y''_n = f'_n = f'(x_n, y_n)$ . On using *MATHEMATICA 8.0*, the elimination of the three undetermined coefficients  $a_0$ ,  $a_1$  and  $b_0$  from equations (6) – (9) yields the one-step rational method

$$y_{n+1} = y_n + \frac{2h(y'_n)^2}{2y'_n - hy''_n}. \quad (10)$$

We note that equation (10) is exactly the one-step second order rational method proposed by Ramos [24]. Equation (10) is the formula to approximate  $y_{n+1}$  by using the information at the previous point  $(x_n, y_n)$ .

To approximate  $y_{n+2}$ , we have to assume that the approximate solution of (1) is locally represented in the range  $[x_n, x_{n+2}]$  by the same rational approximant given in equation (5). It is crucial to retain the same rational approximant in the same block. Now, we required the rational approximant (5) to pass through the points  $(x_n, y_n)$ ,  $(x_{n+1}, y_{n+1})$  and  $(x_{n+2}, y_{n+2})$ , and moreover, must assume at these points the derivative given by  $y' = f(x, y)$ . There are five equations to be satisfied i.e.

$$R(x_n) = y_n, \quad (11)$$

$$R(x_{n+1}) = y_{n+1}, \quad (12)$$

$$R(x_{n+2}) = y_{n+2}, \quad (13)$$

$$R'(x_n) = y'_n, \quad (14)$$

and

$$R'(x_{n+1}) = y'_{n+1}, \quad (15)$$

where  $y'_n = f_n = f(x_n, y_n)$  and  $y'_{n+1} = f_{n+1} = f(x_{n+1}, y_{n+1})$ . On using *MATHEMATICA 8.0*, the elimination of the four undetermined coefficients  $a_0$ ,  $a_1$ ,  $b_0$  and  $y'_n$  from equations (11) – (15) yields the following two-step rational method,

$$y_{n+2} = y_{n+1} + \frac{hy'_{n+1}(y_{n+1} - y_n)}{2(y_{n+1} - y_n) - hy'_{n+1}}. \quad (16)$$

We note that equation (16) is exactly the two-step second order rational method proposed by Lambert [32]. Equation (16) is the formula to approximate  $y_{n+2}$  by using the information at the previous points  $(x_n, y_n)$  and  $(x_{n+1}, y_{n+1})$ . Hence, the 2-point ERBM-A based on the rational approximant (5) consists of two formulae i.e. formulae (10) and (16).

The implementation of the 2-point ERBM-A is as follows: with  $y_n$  is known, compute the approximate solution  $y_{n+1}$  using formula (10); and then compute the approximate solution  $y_{n+2}$  using formula (16) with the value of  $y_{n+1}$  obtained from formula (10).

### 2.1 Local Truncation Errors and Order of Consistency of 2-point ERBM-A

To obtain the order of consistency of the 2-point ERBM-A, we would need to investigate the order of consistency of formulae (10) and (16) individually, which can be found by establishing the local truncation errors for both (10) and (16). Since formulae (10) and (16) are used in the same block to solve for the approximate solutions at  $x_{n+1}$  and  $x_{n+2}$ , we wish to have both formulae possess the same order of consistency. For formula (10), we can associate the following nonlinear operator defined by

$$L[y(x); h] = y(x+h) - y(x) - \frac{2hy'(x)^2}{2y'(x) - hy''(x)}, \quad (17)$$

where  $y(x)$  is an arbitrary function, continuously differentiable on  $[a, b]$ . From equation (17), on expanding  $y(x+h)$  in Taylor series about  $x$ , we obtain

$$L[y(x); h] = h^3 \left( -\frac{y''(x)^2}{4y'(x)} + \frac{y'''(x)}{6} \right) + O(h^4). \quad (18)$$

Equation (18) indicates that formula (10) has second order of consistency, and the local truncation error (LTE) for formula (10) can be written as

$$\text{LTE}_{(10)} = L[y(x_n); h] = h^3 \left( -\frac{(y''_n)^2}{4y'_n} + \frac{y'''_n}{6} \right) + O(h^4), \quad (19)$$

where  $y(x)$  is now taken to be the theoretical solution of problem (1).

For formula (16), the associate nonlinear operator is defined as follows

$$L[y(x);h] = y(x+2h) - y(x+h) - \frac{hy'(x+h)(y(x+h) - y(x))}{2(y(x+h) - y(x)) - hy'(x+h)}, \quad (20)$$

where  $y(x)$  is an arbitrary function, continuously differentiable on  $[a,b]$ . From equation (20), on expanding  $y(x+h)$ ,  $y(x+2h)$  and  $y'(x+h)$  in Taylor series about  $x$ , we obtain

$$L[y(x);h] = h^3 \left( -\frac{y''(x)^2}{2y'(x)} + \frac{y'''(x)}{3} \right) + O(h^4). \quad (21)$$

Equation (21) indicates that formula (16) has second order of consistency, and the LTE for formula (16) can be written as

$$\text{LTE}_{(16)} = h^3 \left( -\frac{(y_n'')^2}{2y_n'} + \frac{y_n'''}{3} \right) + O(h^4), \quad (22)$$

where  $y(x)$  is now taken to be the theoretical solution of problem (1).

From the local truncation errors given in equations (19) and (22), we can verify that both formulae (10) and (16) possess second order of consistency. This also indicates that the 2-point ERBM-A is effectively of order 2.

## 2.2 Absolute Stability Analysis of 2-point ERBM-A

To investigate the linear stability condition for formulae (10) and (16) in the same block, we need to combine both formulae and apply the Dahquist's test equation

$$y' = \lambda y, \quad y(a) = y_0, \quad \text{Re}(\lambda) < 0, \quad (23)$$

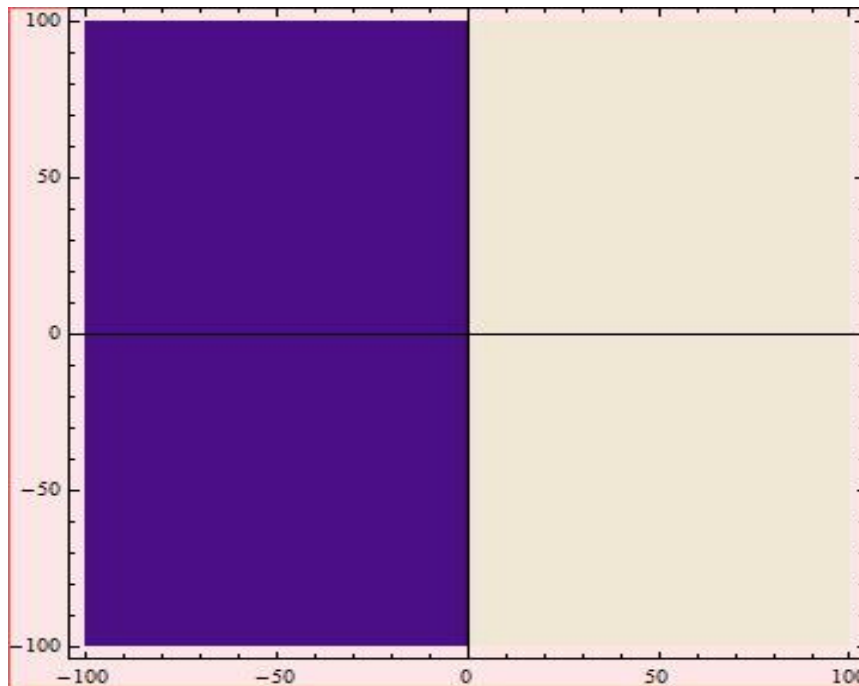
to both formulae. With  $y_{n+1}' = \lambda y_{n+1}$ ,  $y_n' = \lambda y_n$  and  $y_n'' = \lambda^2 y_n$ , we can obtain the following difference equation

$$y_{n+2} = \left( \frac{h\lambda + 2}{h\lambda - 2} \right)^2 y_n. \quad (24)$$

On setting  $h\lambda = z$ ,  $y_{n+2} = \zeta^2$  and  $y_n = \zeta^0 = 1$  in equation (24), then the stability polynomial for the 2-point ERBM-A is

$$\zeta^2 - \left( \frac{z+2}{z-2} \right)^2 = 0. \quad (25)$$

Here,  $\zeta$  can be interpreted as the roots of stability polynomial (25). By taking  $z = x + iy$  in the roots of equation (25), we have plotted the region of absolute stability of the 2-point ERBM-A in Figure 2.



**Figure 2:** Absolute stability region of the 2-point ERBM-A.

The shaded region in Figure 2 is the region of absolute stability of the 2-point ERBM-A. Hence, this shaded region can also be viewed as the ‘combined’ region of absolute stability of formulae (10) and (16). The shaded region is the place where the absolute value of each root of equation (25) is less than or equal to 1. From Figure 2, we can see that the region of absolute stability contains the whole left-hand half plane which shows that the 2-point ERBM-A is *A*-stable.

### 3. DERIVATION OF *L*-STABLE 2-POINT EXPLICIT RATIONAL BLOCK METHOD

The *L*-stable 2-point explicit rational block method, or in brief as 2-point ERBM-L, is formulated to solve problem (1). The formulation of 2-point ERBM-L is identical to the formulation of 2-point ERBM-A, which based on Figure 1 and also the assumptions from equations (2) – (4).

Let us assume that the approximate solution of (1) is locally represented in the range  $[x_n, x_{n+1}]$  by the rational approximant

$$R(x) = \frac{a_0}{b_0 + x}, \quad (26)$$

where  $a_0$  and  $b_0$  are undetermined coefficients. We note that, from the rational approximant in (26), the degree of the denominator is 1 while the degree of the numerator is 0 (i.e. degree of denominator greater than the degree of numerator). This rational approximant in equation (26) is required to pass through the points  $(x_n, y_n)$

and  $(x_{n+1}, y_{n+1})$ , and moreover, must assume at these points the derivative given by  $y' = f(x, y)$ . Altogether, there are three equations to be satisfied i.e.

$$R(x_n) = y_n, \quad (27)$$

$$R(x_{n+1}) = y_{n+1}, \quad (28)$$

and

$$R'(x_n) = y'_n, \quad (29)$$

where  $y'_n = f_n = f(x_n, y_n)$ . On using *MATHEMATICA 8.0*, the elimination of the two undetermined coefficients  $a_0$  and  $b_0$  from equations (27) – (29) yields the one-step rational method

$$y_{n+1} = \frac{(y_n)^2}{y_n - hy'_n}. \quad (30)$$

We note that equation (30) is exactly the one-step first order rational method presented in Fatunla [17]. Equation (30) is the formula to approximate  $y_{n+1}$  by using the information at the previous point  $(x_n, y_n)$ .

To approximate  $y_{n+2}$ , we have to assume that the approximate solution of (1) is locally represented in the range  $[x_n, x_{n+2}]$  by the same rational approximant given in equation (26). It is crucial to retain the same rational approximant in the same block. Now, we required the rational approximant (26) to pass through the points  $(x_n, y_n)$ ,  $(x_{n+1}, y_{n+1})$  and  $(x_{n+2}, y_{n+2})$ , and moreover, must assume at these points the derivative given by  $y' = f(x, y)$ . There are four equations to be satisfied i.e.

$$R(x_n) = y_n, \quad (31)$$

$$R(x_{n+1}) = y_{n+1}, \quad (32)$$

$$R(x_{n+2}) = y_{n+2}, \quad (33)$$

and

$$R'(x_n) = y'_n, \quad (34)$$

where  $y'_n = f_n = f(x_n, y_n)$ . On using *MATHEMATICA 8.0*, the elimination of the two undetermined coefficients  $a_0$  and  $b_0$  from equations (31) – (34) yields the following two-step rational method,

$$y_{n+2} = \frac{(y_n)^2 - hy'_n y_{n+1}}{y_{n+1} - 4hy'_n}. \quad (35)$$



To the best of our knowledge, equation (35) is not seen in the literature review before. Equation (35) is the formula to approximate  $y_{n+2}$  by using the information at the previous points  $(x_n, y_n)$  and  $(x_{n+1}, y_{n+1})$ . Hence, the 2-point ERBM-L based on the rational approximant (26) consists of two formulae i.e. formulae (30) and (25). The implementation of the 2-point ERBM-L is similar to the implementation of the 2-point ERBM-A: with  $y_n$  is known, compute the approximate solution  $y_{n+1}$  using formula (30); and then compute the approximate solution  $y_{n+2}$  using formula (35) with the value of  $y_{n+1}$  obtained from formula (30).

**3.1 Local Truncation Errors and Order of Consistency of 2-point ERBM-L**

To obtain the order of consistency of the 2-point ERBM-L, we would need to investigate the order of consistency of formulae (30) and (35) individually, which can be found by establishing the local truncation errors for both (30) and (35). Since formulae (30) and (35) are used in the same block to solve for the approximate solutions at  $x_{n+1}$  and  $x_{n+2}$ , we wish to have both formulae possess the same order of consistency. For formula (30), we can associate the following nonlinear operator defined by

$$L[y(x); h] = y(x+h) - \frac{y(x)^2}{y(x) - hy'(x)}, \tag{36}$$

where  $y(x)$  is an arbitrary function, continuously differentiable on  $[a, b]$ . From equation (36), on expanding  $y(x+h)$  in Taylor series about  $x$ , we obtain

$$L[y(x); h] = h^2 \left( -\frac{y'(x)^2}{y(x)} + \frac{y''(x)}{2} \right) + O(h^3). \tag{37}$$

Equation (37) indicates that formula (30) has first order of consistency, and the LTE for formula (30) can be written as

$$\text{LTE}_{(30)} = L[y(x_n); h] = h^2 \left( -\frac{(y'_n)^2}{y_n} + \frac{y''_n}{2} \right) + O(h^3), \tag{38}$$

where  $y(x)$  is now taken to be the theoretical solution of problem (1).

For formula (35), the associate nonlinear operator is defined as follows

$$L[y(x); h] = y(x+2h) - \frac{y(x)^2 - hy'(x)y(x+h)}{y(x+h) - 4hy'(x)}, \tag{39}$$

where  $y(x)$  is an arbitrary function, continuously differentiable on  $[a, b]$ . From equation (39), on expanding  $y(x+h)$  and  $y(x+2h)$  in Taylor series about  $x$ , we obtain

$$L[y(x);h] = h^2 \left( -\frac{5y'(x)^2}{y(x)} + \frac{5y''(x)}{2} \right) + O(h^3). \quad (40)$$

Equation (40) indicates that formula (35) has first order of consistency, and the LTE for formula (35) can be written as

$$\text{LTE}_{(35)} = h^2 \left( -\frac{5(y'_n)^2}{y_n} + \frac{5y''_n}{2} \right) + O(h^3), \quad (41)$$

where  $y(x)$  is now taken to be the theoretical solution of problem (1).

From the local truncation errors given in equations (38) and (41), we can verify that both formulae (30) and (35) possess first order of consistency. This also indicates that the 2-point ERBM-L is effectively of order 1.

### 3.2 Absolute Stability Analysis for 2-point ERBM-L

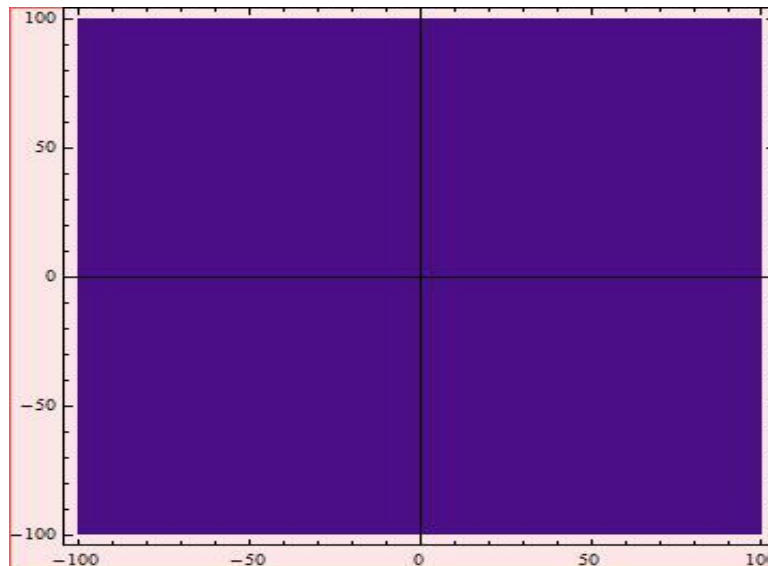
To investigate the linear stability condition for formulae (30) and (35) in the same block, we need to combine both formulae and apply the Dahquist's test equation (23) to both formulae. With  $y'_n = \lambda y_n$ , we can obtain the following difference equation

$$y_{n+2} = \frac{1}{1-2h\lambda} y_n. \quad (42)$$

On setting  $h\lambda = z$ ,  $y_{n+2} = \zeta^2$  and  $y_n = \zeta^0 = 1$  in equation (42), then the stability polynomial for the 2-point ERBM-L is

$$\zeta^2 - \frac{1}{1-2z} = 0. \quad (43)$$

Here,  $\zeta$  can be interpreted as the roots of stability polynomial (43). By taking  $z = x + iy$  in the roots of equation (43), we have plotted the region of absolute stability of the 2-point ERBM-L in Figure 3.



**Figure 3:** Absolute stability region of the 2-point ERBM-L.

The shaded region in Figure 3 is the region of absolute stability of the 2-point ERBM-L. Hence, this shaded region can also be viewed as the ‘combined’ region of absolute stability of formulae (30) and (35). The shaded region is the place where the absolute value of each root of equation (43) is less than or equal to 1. From Figure 3, we can see that the region of absolute stability contains the whole left-hand half plane which suggests that the 2-point ERBM-L is  $A$ -stable. In addition, the absolute value of each root of equation (43) approaches zero as  $\text{Re}(z) \rightarrow -\infty$ . This shows that 2-point ERBM-L is  $L$ -stable.

#### 4. VARIABLE STEP-SIZE STRATEGY FOR 2-POINT ERBM-A AND 2-POINT ERBM-L

Suppose that we have solved numerically the initial value problem (1) using the 2-point ERBM-A (or 2-point ERBM-L) up to a point  $x_n$ , and have obtained a value  $y_n$  as an approximation of  $y(x_n)$ , which is the theoretical solution of problem (1).

For every integration step, there is always a step-size, say  $h_s$  available to compute two approximations to the solution of problem (1), namely  ${}^{(s)}y_{n+1}$  using formula (10) (or (30)), and  ${}^{(s)}y_{n+2}$  using formula (16) (or (35)), where  $s$  represents the (current)  $s$ -th iteration. After that, integrate twice by halving the step-size  $h_s$  i.e.  $h_s/2$ , yields the values of  ${}^{(s)}\hat{y}_{n+1}$  using formula (10) (or (30)), and  ${}^{(s)}\hat{y}_{n+2}$  using formula (16) (or (35)). Then an estimate of the error for the less precise result is  $err = \left\| {}^{(s)}\hat{y}_{n+2} - {}^{(s)}y_{n+2} \right\|_{\infty}$ . It is important to note that the estimate error is always perform on the approximate

solution obtain by formula (16) (or (35)), not on the approximate solution obtain by formula (10) (or (30)). This error estimation is used to control the error of a block. We want this error estimation to satisfy

$$\left\| {}^{(s)}\hat{y}_{n+2} - {}^{(s)}y_{n+2} \right\|_{\infty} \leq TOL, \quad (44)$$

where  $TOL$  is the desired tolerance prescribed by the user. If the inequality (44) is satisfied, then the computed step is accepted and this also means that  ${}^{(s)}y_{n+1}$  is accepted as  $y_{n+1}$  and  ${}^{(s)}y_{n+2}$  is accepted as  $y_{n+2}$ . The value  $y_{n+2}$  is then used to start the computation of the next block. The current  $h_s$  is now used to advance to the next block.

If the inequality (44) is not satisfied by the current  $h_s$ , i.e.

$$\left\| {}^{(s)}\hat{y}_{n+2} - {}^{(s)}y_{n+2} \right\|_{\infty} > TOL, \quad (45)$$

then the computed values of  ${}^{(s)}y_{n+1}$ ,  ${}^{(s)}y_{n+2}$  and the current  $h_s$  are rejected. Following this, we need to start another iteration, say  $(s+1)$ -th iteration, with a new step-size, say  $h_{s+1}$ . The step-size  $h_{s+1}$  can be calculated using the following formulae:

$$h_{s+1} = h_s \times r, \quad (46)$$

and

$$r = \min \left( \max \left( 0.5, 0.9 \left( \frac{TOL}{err} \right)^{\frac{1}{p+1}} \right), 1.0 \right). \quad (47)$$

From equation (47), we note that  $p=2$  for 2-point ERBM-A, and  $p=1$  for 2-point ERBM-L, where  $p$  is the order of the underlying rational block method. At this point,  $h_{s+1}$  is not used to advance to the next block, but remain in the current block to obtain:  ${}^{(s+1)}y_{n+1}$  and  ${}^{(s+1)}y_{n+2}$  via the step-size  $h_{s+1}$ , and  ${}^{(s+1)}\hat{y}_{n+1}$  and  ${}^{(s+1)}\hat{y}_{n+2}$  by halving the step-size  $h_{s+1}$  i.e.  $h_{s+1}/2$ . Then, the validation processes take place again using the inequalities

$$\left\| {}^{(s+1)}\hat{y}_{n+2} - {}^{(s+1)}y_{n+2} \right\|_{\infty} \leq TOL,$$

or

$$\left\| {}^{(s+1)}\hat{y}_{n+2} - {}^{(s+1)}y_{n+2} \right\|_{\infty} > TOL.$$

The iterating process to recalculate the values of  $y_{n+1}$  and  $y_{n+2}$  in the current block is repeated, every time with a new adjusted step-size using equations (46) and (47) until the error estimation is less than the prescribed tolerance.

To prevent the step-size from exceeding the right boundary of the integration interval  $[a, b]$ , every time when a step-size  $h_s$  is calculated at any point of  $x$ , we must check whether  $x + 2h_s$  still lie in the interval  $[a, b]$  i.e.

$$x + 2h_s < b. \quad (48)$$

If (48) is satisfied, then the computation continues without any interruption. However, if  $x + 2h_s$  is found to coincide with or greater than the right boundary  $b$  i.e.

$$x + 2h_s \geq b, \quad (49)$$

then current step-size  $h_s$  is immediately rejected. The rejected current step-size  $h_s$  is then replaced by a final step-size, say  $h_b$  which can be obtained using the formula

$$h_b = \frac{b-x}{2}. \quad (50)$$

Then, the last integration is performed at the last block to obtain the last two numerical approximations using the new step-size  $h_b$  obtained from (50). We note that the  $h_s$  in equations (48) and (49) has to be multiplied by 2, whereas  $b-x$  has to be divided by 2 to obtain  $h_b$ . This is due to the nature of the 2-point ERBMs for able to obtain two approximate solutions i.e. each approximate solution for a step-size  $h_s$ .

## 5. NUMERICAL EXPERIMENTS AND COMPARISONS

In this section, we solved *Problem 1 – Problem 4* with the variable step-size strategy described in Section 4, using the 2-point ERBM-A, 2-point ERBM-L, and existing rational methods from Fatunla [17], and Ramos [24]. We denote:

- TOL as the user prescribed tolerance  $TOL$ ,
- METHOD as the various rational methods used in comparison,
- SSTEP as the total number of successful steps within the interval  $[a, b]$ ,
- FSTEP as the total number of rejected steps within the interval  $[a, b]$ , and
- MAXE as the maximum absolute relative error defined by  $\max_{0 \leq n \leq SSTEP} \{ |y(x_n) - y_n| \}$ ,

where  $y(x_n)$  and  $y_n$  represent the theoretical solution and numerical solution of a test problem at point  $x_n$ , respectively.

**Problem 1**

$$y'(x) = -2y(x) + 4x, \quad y(0) = 3, \quad x \in [0, 0.5].$$

The theoretical solution is  $y(x) = 4e^{-2x} - 1 + 2x$ .

**Problem 2 [17]**

$$y'(x) = -2000e^{-200x} + 9e^{-x} + xe^{-x}, \quad y(0) = 10, \quad x \in [0, 1].$$

The theoretical solution is  $y(x) = 10 - 10e^{-x} - xe^{-x} + 10e^{-200x}$ .

**Problem 3 [4]**

$$y_1'(x) = 198y_1(x) + 199y_2(x), \quad y_1(0) = 1, \quad x \in [0, 10];$$

$$y_2'(x) = -398y_1(x) - 399y_2(x), \quad y_2(0) = -1, \quad x \in [0, 10];$$

The theoretical solutions are  $y_1(x) = e^{-x}$  and  $y_2(x) = -e^{-x}$ .

**Problem 4 [33]**

$$y''(x) + 101y'(x) + 100y(x) = 0, \quad y(0) = 1.01, \quad y'(0) = -2, \quad x \in [0, 10].$$

The theoretical solution is  $y(x) = 0.01e^{-100x} + e^{-x}$ . *Problem 4* can be reduced to a system of first order differential equations, i.e.

$$y_1'(x) = y_2(x), \quad y_1(0) = 1.01, \quad x \in [0, 10];$$

$$y_2'(x) = -100y_1(x) - 101y_2(x), \quad y_2(0) = -2, \quad x \in [0, 10].$$

The theoretical solutions are  $y_1(x) = 0.01e^{-100x} + e^{-x}$  and  $y_2(x) = -e^{-100x} - e^{-x}$ .

Table 1 until Table 5 showed the numerical comparisons of various second order rational methods (including 2-point ERBM-A), while Table 6 until Table 10 showed the numerical comparisons of various first order rational methods (including 2-point ERBM-L), all obtained using the variable step-size strategy. We also provide the initial step-size ( $h_0$ ) for each problem being solved.

**Table 1:** Comparisons of various second order rational methods in solving *Problem 1* ( $h_0 = 0.1$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-2}$	Ramos [24]	5	0	1.44998(-02)
	2-point ERBM-A	3	1	1.21645(-02)
$10^{-3}$	Ramos [24]	20	8	1.33876(-03)
	2-point ERBM-A	13	5	1.32444(-03)
$10^{-4}$	Ramos [24]	67	13	1.33675(-04)
	2-point ERBM-A	127	9	1.33301(-04)

From Table 1, both second order rational methods are found to have comparable accuracy for the three prescribed tolerance  $10^{-2}$ ,  $10^{-3}$  and  $10^{-4}$ . However, 2-point ERBM-A seems to require smaller number of successful steps compared to the second order rational method of Ramos [24], except for the tolerance  $10^{-4}$ .

**Table 2:** Comparisons of various second order rational methods in solving *Problem 2* ( $h_0 = 0.0001$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-2}$	Ramos [24]	10001	0	3.72253(-04)
	2-point ERBM-A	5001	0	3.78696(-04)
$10^{-3}$	Ramos [24]	10001	0	3.72253(-04)
	2-point ERBM-A	5001	0	3.78696(-04)

The initial step-size of *Problem 2* is set to  $h_0 = 0.0001$  so that stability and convergence of numerical solution generated by all second order rational methods are guaranteed under specific prescribed tolerance. With this initial step-size, we observed from Table 2 that, the second order rational method of Ramos [24] required 10001 successful steps within the interval  $[0,1]$ ; meanwhile the 2-point ERBM-A only needs 5001 successful steps for the prescribed tolerances  $10^{-2}$  and  $10^{-3}$ . Hence, the generated maximum absolute relative errors for every prescribed tolerance are found to be identical. We can see that the 2-point ERBM-A is cheaper in computational cost compared to the second order rational method of Ramos [24].

**Table 3:** Comparisons of various second order rational methods in solving *Problem 3* ( $y_1(x)$ ) ( $h_0 = 0.1$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-2}$	Ramos [24]	607	3	9.11258(-03)
	2-point ERBM-A	517	6	1.23849(-02)
$10^{-3}$	Ramos [24]	926	4	2.35036(-04)
	2-point ERBM-A	646	6	2.60973(-04)
$10^{-4}$	Ramos [24]	996	11	1.32011(-04)
	2-point ERBM-A	796	8	1.23095(-04)

**Table 4:** Comparisons of various second order rational methods in solving *Problem 3* ( $y_2(x)$ ) ( $h_0 = 0.1$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-2}$	Ramos [24]	607	3	9.26506(-03)
	2-point ERBM-A	517	6	1.93711(-02)
$10^{-3}$	Ramos [24]	926	4	2.34970(-04)
	2-point ERBM-A	646	6	3.42722(-04)
$10^{-4}$	Ramos [24]	996	11	1.31996(-04)
	2-point ERBM-A	796	8	1.34045(-04)

From Table 3 and Table 4, and when the tolerance is  $10^{-2}$ , we have observed that the second order rational method of Ramos [24] is more accurate than the 2-point ERBM-A in computing the components  $y_1(x)$  and  $y_2(x)$ . Both methods are found to possess comparable accuracy when the prescribed tolerances are  $10^{-3}$  and  $10^{-4}$ . However, for all three prescribed tolerances, the 2-point ERBM-A is cheaper in computational cost.

**Table 5:** Comparisons of various second order rational methods in solving *Problem 4* ( $h_0 = 0.1$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-2}$	Ramos [24]	1650	5	4.76520(-04)
	2-point ERBM-A	774	7	6.40045(-04)
$10^{-3}$	Ramos [24]	5983	9	5.24244(-05)
	2-point ERBM-A	2554	7	6.37900(-05)
$10^{-4}$	Ramos [24]	20166	13	5.22558(-06)
	2-point ERBM-A	9578	11	5.61566(-06)

From Table 5, both rational methods are found to have comparable accuracy for the three prescribed tolerance  $10^{-2}$ ,  $10^{-3}$  and  $10^{-4}$ . However, 2-point ERBM-A seems to require smaller number of successful steps compared to the second order rational method of Ramos [24].



**Table 6:** Comparisons of various first order rational methods in solving *Problem 1* ( $h_0 = 0.1$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-1}$	Fatunla [17]	5	0	3.02063(-02)
	2-point ERBM-L	2	0	6.15770(-02)
$10^{-2}$	Fatunla [17]	12	6	1.99322(-02)
	2-point ERBM-L	12	6	1.99163(-02)

From Table 6, both first order rational methods are found to have comparable accuracy for the two prescribed tolerances  $10^{-1}$  and  $10^{-2}$ . Both first order methods used the same number of successful steps when the tolerance is  $10^{-2}$ . Although there is a slight difference between the number of successful steps used for both methods when the tolerance is  $10^{-1}$ , but the difference is less obvious and not significant.

**Table 7:** Comparisons of various first order rational methods in solving *Problem 2* ( $h_0 = 0.0001$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-1}$	Fatunla [17]	10001	0	8.19894(-02)
	2-point ERBM-L	5001	0	1.55306(-01)

From Table 7, the first order rational method of Fatunla [17] required 10001 successful steps within the interval  $[0,1]$ ; meanwhile the 2-point ERBM-L only needs 5001 successful steps for the prescribed tolerance  $10^{-1}$ . We can see that the 2-point ERBM-L is cheaper in computational cost compared to the first order rational method of Fatunla [17].

**Table 8:** Comparisons of various first order rational methods in solving *Problem 3* ( $y_1(x)$ ) ( $h_0 = 0.1$ )

TOL	METHOD	SSTEP	FSTEP	MAXE
$10^{-1}$	Fatunla [17]	1035	8	1.19285(-01)
	2-point ERBM-L	800	8	7.84739(-02)
$10^{-2}$	Fatunla [17]	1365	7	1.71785(-02)
	2-point ERBM-L	975	21	2.07879(-02)

**Table 9:** Comparisons of various first order rational methods in solving *Problem 3* ( $y_2(x)$ ) ( $h_0 = 0.1$ )

	<b>TOL</b>	<b>METHOD</b>	<b>SSTEP</b>	<b>FSTEP</b>	<b>MAXE</b>
$10^{-1}$		Fatunla [17]	1035	8	1.24326(-01)
		2-point ERBM-L	800	8	7.99934(-02)
$10^{-2}$		Fatunla [17]	1365	7	1.71785(-02)
		2-point ERBM-L	975	21	2.07879(-02)

From Table 8 and Table 9, and when the tolerance is  $10^{-1}$ , we have observed that the 2-point ERBM-L is more accurate than the first order rational method of Fatunla [17] in computing the components  $y_1(x)$  and  $y_2(x)$ . Both methods are found to possess comparable accuracy when the prescribed tolerance is  $10^{-2}$ . However, for both prescribed tolerances, the 2-point ERBM-L is cheaper in computational cost.

**Table 10:** Comparisons of various first order rational methods in solving *Problem 4* ( $h_0 = 0.1$ )

	<b>TOL</b>	<b>METHOD</b>	<b>SSTEP</b>	<b>FSTEP</b>	<b>MAXE</b>
$10^{-1}$		Fatunla [17]	551	3	1.37723(-02)
		2-point ERBM-L	534	5	1.22657(-02)
$10^{-2}$		Fatunla [17]	4631	10	2.57741(-03)
		2-point ERBM-L	4631	11	2.57741(-03)

From Table 10, both first order rational methods are found to have comparable accuracy for the two prescribed tolerances  $10^{-1}$  and  $10^{-2}$ . Both first order methods used the same number of successful steps when the tolerance is  $10^{-2}$ . Although there is a slight difference between the number of successful steps used for both methods when the tolerance is  $10^{-1}$ , but the difference is less obvious and not significant.

## CONCLUSIONS

The aim of this paper is to introduce the approaches used to develop rational block methods (RBMs) which possess  $A$ -stability and  $L$ -stability. In order to illustrate the approaches used, we have reported an  $A$ -stable 2-point explicit rational block method (2-point ERBM-A) and a  $L$ -stable 2-point explicit rational block method (2-point ERBM-L). Both proposed rational block methods are able to approximate two successive solutions at the points  $x_{n+1}$  and  $x_{n+2}$  defined in the same block (see Figure 1), within every single integration step. Every 2-point ERBM also contains two rational formulae, and both formulae are found to possess the same order of accuracy. The 2-point ERBM-A has second order of accuracy, while the 2-point ERBM-L has first order of accuracy. Figure 2 and Figure 3 showed the regions of absolute stability

for 2-point ERBM-A and 2-point ERBM-L, respectively. Numerical experiments showed that at most of the time, the proposed 2-point ERBMs generated converging numerical solution with comparable accuracy. In general, the proposed 2-point ERBMs are cheaper in computational cost due to fairly low number of successful steps.

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