Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 15, Number 5 (2019), pp. 733-741

http://www.ripublication.com/gjpam.htm

Partial orderings on $k-$idempotent fuzzy matrices

K.Muthu Guru Packiam[1] **and K.S.Krishna Mohan** [2]

[1,2] *Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur - 5, South India.*

Abstract

Aim of this article is to characterize the partial orderings for the $k-$idempotent fuzzy matrices. Also, using the generalized inverses, we discuss some properties for the $T-$ordering and minus ordering in $k-$idempotent fuzzy matrices.
Keywords: $k-$idempotent fuzzy matrix, $T-$ordering, minus ordering, space ordering and inverses.

1. INTRODUCTION

Partial ordering plays an important role in fuzzy matrices. The concept of fuzzy matrix was first introduced by Thomosan [2] in 1977 and it has further developments by various researchers. Jian Miao Chen [3] initiated the partial orderings on fuzzy matrices which is the analogue of the star ordering on complex matrices. After that, a lot of works have been done using this notion. A.R. Meenachi [1] characterizes the minus ordering on matrices in terms of their generalized inverses. Furthermore, she defines space ordering [6] on fuzzy matrices, which is a partial ordering on the set of all idempotent matrices in F_n . A crisp binary relation $R(X, X)$ that is reflexive, anti-symmetric and transitive is called partial ordering [5]. The common symbol $\leq$ is suggestive of the properties of this class of relations. Thus $x \leq y$ denotes $\langle x, y\rangle \in R$ and signifies that x precedes y. The inverse partial ordering $R^{-1}(X, X)$ is suggested by the symbol $\geq$. If $y \leq x$ indicating that $\langle x, y\rangle \in R^{-1}$, then we say that y succeeds x. If we need to distinguish several partial orderings such as P, Q and R, we use the symbols $\leq^P$, $\leq^Q$ and $\leq^R$ respectively. In this paper, these orderings are characterized in $k-$idempotent fuzzy matrices [7]. We discuss these ordering related with some inverses on $k-$idempotent fuzzy matrices [8].

2. $T-$ORDERING ON $K-$IDEMPOTENT FUZZY MATRICES

Definition 2.1. *For $A, B \in F_{m,n}$, the $T-$ordering $A \leq^T B$ is defined as $A \leq^T B \Leftrightarrow A^T A = A^T B$ and $AA^T = BA^T$*

Theorem 2.1. *Let $A, B \in F_{n\times n}$ and K be the associated permutation matrix of k, then $A \leq^T B \Leftrightarrow KA \leq^T KB \Leftrightarrow AK \leq^T BK$*

Proof. $A \leq^T B \Leftrightarrow A^T A = A^T B$ and $AA^T = BA^T$

$\Leftrightarrow A^T KKA = A^T KKB$ and $KAA^T K = KBA^T K$

$\Leftrightarrow (KA)^T KA = (KA)^T KB$ and $KA(KA)^T = KB(KA)^T$

$\Leftrightarrow KA \leq^T KB$

Similarly, $A \leq^T B \Leftrightarrow A^T A = A^T B$ and $AA^T = AA^T = BA^T$

$\Leftrightarrow KA^T AK = KA^T BK$ and $AKKA^T = BKKA^T$

$\Leftrightarrow (KA)^T AK = (KA)^T BK$ and $AK(AK)^T = BK(AK)^T$

$\Leftrightarrow AK \leq^T BK$

Theorem 2.2. *Let $A, B \in F_{m,n}$ are $k-$idempotent, then $A \leq^T B$ if and only if $A^2 \leq^T B^2$*

Proof. Assuming that $A \leq^T B$, then we have

(i) $A^T A = A^T B$ and (ii) $AA^T = BA^T$

Pre multiplying by K, we have from (i)

$$KA^TAK = KA^TBK$$

$$KA^TKKAK = KA^TKKBK$$

$$(A^T)^2(A)^2 = (A^T)^2(B)^2 \quad (1)$$

From (ii), $KAA^TK = KBA^TK$

$$KAKKA^TK = KBKKA^TK$$

$$(A)^2(A^T)^2 = (B)^2(A^T)^2 \quad (2)$$

From (1) and (2) , we have $A^2 \leq^T B^2$

Conversely, if we assume that $A^2 \leq^T B^2$

$$KA^2 \leq^T KB^2$$

$$KA^2K \leq^T KB^2K$$

$$A \leq^T B$$

Remark 2.1. *If Moore − Penrose inverse of A and B exists and $A \leq^T B$, then we have $AB_k^+A = A = B_K^+AB$ and $A_k^+ = B_K^+BB_K^+$.*

Theorem 2.3. *If $A \leq^T B$ and B is $k-$idempotent, then A is also $k-$idempotent.*

Proof. $KA^2K = KAAK$

$$= KAB_k^+BBB_k^+AK$$

$$= AB_k^+KB^2KB_k^+A$$

$$= AB_k^+BB_k^+A$$

$$= AA_k^+ A$$

$$= A$$

Hence A is $k-$idempotent

3. MINUS ORDERING ON $K-$IDEMPOTENT FUZZY MATRICES

Definition 3.1. *For $A \in F_{m,n}$ and $B \in F_{m,n}$, the minus ordering denoted as $\overline{<}$ is defined as $A \overline{<} B \Leftrightarrow A_k^- A = A_k^- B$ and $AA_k^- = BA_k^-$.*

Theorem 3.1. *If $A \overline{<} B$ With B as $k-$idempotent, then $KBK \in A\{1\}$*

Proof. Since B is $k-$idempotent and B is regular,

then B itself a $g-$inverse of B.

Hence $B \in B\{1\}$

Since $A \overline{<} B \Rightarrow B\{1\} \subseteq A\{1\}$ [6]

Therefore $B \in A\{1\}$

Since $B^2 = KBK$ and B is $k-$idempotent, KBK $\in A\{1\}$

Example 3.1. *Let A =* $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ *and B =* $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Here B is $k-$idempotent and regular

Also, B itself a $g-$inverse of B$\{1\}$

Now, $B = BA = AB$

Hence $A \lessapprox B$ with respect to $B = B_k^- \in A\{1\}$

Remark 3.1. *For $A \in F_{m,n}$ and $B \in F_{m,n}$, $A \lessapprox B \Leftrightarrow A = AA_k^- B = BA_k^- A = BA_k^- B$*
.

Theorem 3.2. *For $A \in F_{m,n}$ and $B \in F_{m,n}$ with $A \lessapprox B$, if B is $k-$idempotent then A is also $k-$idempotent.*

Proof. $KA^2K = KAAK$

$$= K\{AA_k^- B\}\{BA_k^- A\}K$$

$$= AA_k^- KB^2KAA_k^-$$

$$= \{AA_k^- B\}A_k^- A$$

$$= AA_k^- A$$

$$= A$$

Hence A is $k-$idempotent.

Remark 3.2. *$If A \lessapprox B$ with A as $k-$ idempotent then B need not be a $k-$idempotent.*

Consider A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *and B =* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Here $A \lessapprox B$ respect to $A_k^- = A$

But B is not $k-$idempotent,

Remark 3.3. *For $A \in F_{m,n}$ and $B \in F_{m,n}$, the following are equivalent :*
(i) $A \lessapprox B$ and (ii) $A = AA_k^- B = BA_k^- A = BA_k^- B$

Theorem 3.3. *Let A and B are $k-$idempotent fuzzy matrices, then $A \lessapprox B$ if and only if $A^2 \lessapprox B^2$.*

Proof. Assume that $A \lessdot B$, $A = AA_k^- B = BA_k^- A = BA_k^- B$

$$\Leftrightarrow KAK = KAA_k^- BK = KBA_k^- AK = KBA_k^- BK$$

$$\Leftrightarrow A^2 = A^2 KAA_k^- KB^2 = B^2 KA_k^- KA^2 = B^2 KA_k^- KB^2$$

$$\Leftrightarrow A^2 = A^2 (A^2)_k^- B^2 = B^2 (A^2)_k^- A^2 = B^2 (A^2)_k^- B^2$$

Therefore $A^2 \lessdot B^2$.

Theorem 3.4. *If K is the associated permutation matrix of K, then for* $A, B \in F_{n \times n}$, $A \lessdot B \Leftrightarrow KA \lessdot KB \Leftrightarrow AK \lessdot BK$

Proof. Assuming that $A \lessdot B$, $A_k^- A = A_k^- B$ and $AA_k^- = BA_k^-$

$$\Leftrightarrow KA_k^- AK = KA_k^- BK \text{ and } KAA_k^- K = KBA_k^- K$$

$$\Leftrightarrow (AK)_k^- AK = (AK)_k^- BK \text{ and } AK(AK)_k^- = BK(AK)_k^-$$

$$\Leftrightarrow AK \lessdot BK$$

Similarly, $A \lessdot B$, $A_k^- KKA = A_k^- KKB$ and $AKKA_k^- = BKKA_k^-$

$$\Leftrightarrow (KA)_k^- KA = (KA)_k^- KB \text{ and } KA(KA)_k^- = KB(KA)_k^-$$

$$\Leftrightarrow KA \lessdot KB$$

4. $K-$SPACE ORDERING

Space ordering is a partial ordering on the set of all idempotent matrices in F_n

Definition 4.1. *Let A and B are $k-$idempotent fuzzy matrices, $k-$space ordering $A \leq_k B$ is defined as $A \leq_k B \Leftrightarrow R(A) \subseteq R(B)$and $\rho(A) \subseteq \rho(B)$.*

Theorem 4.1. *Let $\{A_i\}, i \in N$ be a finite set of $k-$idempotent matrices. Then*

$A = A_1A_2...A_k$ *is* $k-$*idempotent if* $A_iA_j = A_jA_i$

Proof. Consider $A_1A_2 = A_2A_1 = B_1^2$

Simlarly, $A_2A_3 = A_3A_2 = B_2^2$ and so on.

Since $A = A_1A_2...A_k$ are $k-$idempotent,

Therefore $B_1^2, B_2^2, ..., B_{\frac{m}{2}}^2$ are also $k-$idempotent.

Hence $A = B_1^2 + B_2^2 + ... + B_{\frac{m}{2}}^2$ is also $k-$idempotent.

Theorem 4.2. *Let* $\{A_i\}, i \in N$ *be a finite set of* $k-$*idempotent matrices and if either* $R(A_i) \subseteq R(A_j)$ *or* $\rho(A_i) \subseteq \rho(A_j)$*, then* $A = A_1A_2...A_k$ *is* $k-$*idempotent.*

Proof. Taking $R(A_i) \subseteq R(A_j)$

$$\Rightarrow R(A_1) \subseteq R(A_2), R(A_2) \subseteq R(A_3), ..., R(A_{k-1}) \subseteq R(A_k)$$

$$\Rightarrow R(A_1) \subseteq R(A_2) \subseteq R(A_3) \subseteq ... \subseteq R(A_{k-1}) \subseteq R(A_k)$$

$$\Rightarrow A = A_1$$

Hence A is $k-$idempotent.

Now $\rho(A_i) \subseteq \rho(A_j)$

$$\Rightarrow \rho(A_1) \subseteq \rho(A_2), \rho(A_2) \subseteq \rho(A_3), ..., \rho(A_k - 1) \subseteq \rho(A_k)$$

$$\Rightarrow \rho(A_1) \subseteq \rho(A_2) \subseteq \rho(A_3) \subseteq ... \subseteq \rho(A_k - 1) \subseteq \rho(A_k)$$

$$\Rightarrow A = A_k$$

Therefore, A is $k-$idempotent.

Example 4.1. *Consider* $A_1 = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \end{bmatrix}$ *and* $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Both are being idempotent.

Hence $\rho(A_1) \subseteq \rho(A_2)$

Here $A = A_1$*, which is* $k-$*idempotent.*

Theorem 4.3. *Let* $\{A_i\}, i \in N$ *be a finite set of* $k-$*idempotent matrices and* $A_i \leq A_j$*, then* $A = A_1A_2...A_k$ *is* $k-$*idempotent.*

Proof. $A_i \leq A_j \Rightarrow A_iA_j \leq A_j$ and $A_i \leq A_jA_i$

$\Rightarrow K(A_iA_j)^2K \leq A_iA_j$ and

$$K(A_iA_j)^2K = A_i(A_iA_j)A_j \geq A_iA_j$$

$\Rightarrow K(A_iA_j)^2K = A_iA_j$

$\Rightarrow A_iA_j$ is k$-$idempotent.

Hence $A = A_1A_2...A_k$ is $k-$idempotent.

5. CONCLUSION

Ordering principles are crucial for categorizing and ranking real world problems. The comparability relation on fuzzy matrices is a partial ordering. Minus ordering is a partial ordeing in the set of all regular fuzzy matrices. We have introduced ordering on $k-$ idempotent fuzzy matrices and developed the theory of fuzzy matrix partial ordering. The minus ordering and $k-$space ordering are identical for $k-$ idempotent matrices.

REFERENCES

[1] Meenakshi.A.R., and Inbam.C, *The minus partial order in Fuzzy matrices* , The Journal of Fuzzy matrices, **Vol.12 (3)** (2004),PP 695 - 700.

[2] M.G. Thomason, *Convergence of posets of a fuzzy matrix* , Journal of Mathl.Anal. Appl., **Vol 57** (1977), PP 3 - 15.

[3] Jian Miao Chen (1982), *Fuzzy matrix partial orderings and generalized inverses*, Fuzzy sets sys **105 : 453 – 458.**

[4] Kim J.B., *Idempotents and inverses in Fuzzy matrices* , Malaysian Math **6(2)** , 1983, 57 – 61.

[5] George J. Klir and Bo yuan,*Fuzzy sets and Fuzzy logic* , PHI, 2013

[6] Meenakshi.A.R., *Fuzzy matrix – Theory and its applications* , MJP Publishers (2008)

[7] K. Muthugurupackiam and K.S.Krishnamohan, *Generalisation of Idempotent fuzzy matrices*, International Journal of Applied Engineering Research, **13(13)** (2018), 11087–11090.

[8] K. Muthugurupackiam and K.S.Krishnamohan, *Some inverses on Generalised idempotent fuzzy matrices*, Journal of Applied Science and Computations, **VI(II)** (2019), 249-254.