Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 19, Number 2 (2023), pp. 229-254

https://dx.doi.org/10.37622/GJPAM/19.2.2023.229-254

Various Forms of Cubic Transmuted Rayleigh Distribution and Its Properties

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Abstract

In this research, we suggested many kinds of Cubic Transmuted Rayleigh Distributions (CTRD) provided by Granzotto et al. (2017), Kareema abed AL Kadim et al. (2017), Mahabubur Rahman et al. (2018), and Rahman et al. (2019). since numerous cubic transmuted maps are accessible in the literature. To identify them, we have designated this cubic transmuted Rayleigh distribution as CTRD-I, CTRD-II, CTRD-III, and CTRD-IV. Detailed explanations of the statistical properties of the suggested distributions are provided, as are parameter estimates and related reliability metrics. The order statistics for the suggested distributions are also computed. Real-world data sets are used to demonstrate the applicability of the presented models.

1. INTRODUCTION

Lifetime data modeling and analysis are crucial in many applied disciplines. A variety of lifetime distributions have been employed to model such data. The accuracy of statistical analysis is heavily dependent on the assumed probability model. As a result, significant effort has been undertaken to extend traditional probability distributions as well as appropriate statistical approaches. However, there are several significant issues where real-world data does not fit many standard probability models.

This article studied numerous cubic transmutation maps utilizing the Rayleigh distribution as a foundation distribution to create a unique cubic transmuted family of Rayleigh distributions.

The probability density function (*pdf)* of the Rayleigh distribution is

$$f(x;\sigma)=\frac{x}{\sigma}e^{\frac{-x^2}{(2\sigma^2)}}, \quad x\geq 0, \quad \sigma^2\geq 0 \tag{1}$$

The corresponding cumulative density function (*cdf*) of the Rayleigh distribution is

$$F(x;\sigma)=1-e^{\frac{-x^2}{(2\sigma^2)}}, \qquad x\geq 0 \tag{2}$$

Where, $\sigma\geq 0$ is the shape parameter.

Faton Merovci (2013) suggested a unique transmuted Rayleigh distribution utilizing a quadratic transmutation map provided by Shaw and Buckley (2009). They proposed the following quadratic transmuted map, and its *cdf* is given as

$$F(x)=(1+\lambda)G(x)-\lambda G^2(x)\ , \qquad \lambda\in[-1,1] \tag{3}$$

This transmutation map was used by several researchers to study various transmuted distributions. A cubic transmutation map is the extended form of the quadratic transmutation map. In this study, we employed four distinct kinds of cubic transmutation maps to create a new family of distributions based on the Rayleigh distribution.

I. **Granzotto et al. (2017)** used the cubic rank transmutation map to establish a new class of transmuted distributions. The following is the general form of *cdf*:

$$F(x)=\ \lambda_1 G(x)+(\lambda_2-\lambda_1)G^2(x)+(1-\lambda_2)G^3(x) \quad \lambda_1\in[0,1] \ \ and \ \ \lambda_2\in[-1,1]. \tag{4}$$

When $\lambda_1=\lambda_2=1$, It will provide the base distribution and $F(x)$ is the *cdf* of the baseline distribution.

Granzotto et al. (2017) used this transmutation map to generate the cubic rank transmuted Weibull distribution, as well as the cubic rank transmuted log-logistic distribution and its attributes. Celik N (2018) developed certain cubic rank transmuted distributions utilizing Gumbel, Gompertz, and Fréchet distributions as base distributions, and he investigated various features of the resultant distributions.

II. **Kareema abed AL kadim et al. (2017)** developed a new cubic transmuted distributional family. It is provided by

$$F(x)=(1+\lambda)G(x)-2\lambda G^2(x)+\lambda G^3(x) \qquad \lambda\in[-1,1] \tag{5}$$

When $\lambda=0$, the general form will provide the base distribution, thus $G(x)$ is the *cdf* of the base distribution.

III. **Mahabubur Rahman et al. (2018)** introduced a novel cubic-transmuted Pareto distribution. The *cdf* of the cubic transmuted family is denoted as

$$F(x)=(1+\lambda_1)G(x)+(\lambda_2-\lambda_1)G^2(x)-\lambda_2 G^3(x) \quad \lambda_1\in[-1,1],\ \lambda_2\in[-1,1] \tag{6}$$

Where the cubic transmuted distribution family shown above will expand any baseline distribution, making it more applicable. It can deal with data that is bi-modal.

IV. **Md. Mahabubur Rahman et al. (2019)** proposed a *cdf* for a distinct cubic transmuted distribution family It is supplied by

$$F(x)=(1-\lambda)G(x)+3\lambda G^2(x)-2\lambda G^3(x) \qquad \lambda\in[0,1] \tag{7}$$

When $\lambda=0$, the general form will reduce to the base distribution. Md. Mahabubur Rahman et al. (2019) explored the cubic transmuted uniform distribution utilizing this cubic transmutation map.

This article is structured as follows: Section 2 introduces cubic transmuted Rayleigh distribution. Section 3 has the moments of various forms of CTRD. Section 4 explored the likelihood approach for estimating the parameters of various forms of CTRD. Section 5 presents a reliability analysis. Section 6 contains order statistics for the CTRD distributions. Finally, real-time data was employed for the planned distribution as an application in section 7.

2. CUBIC TRANSMUTED RAYLEIGH DISTRIBUTIONS

The *cdf* and *pdf* of the various forms of CTRD are given below:

2.1 CTRD-I

The *cdf* and *pdf* of the CTRD-I is

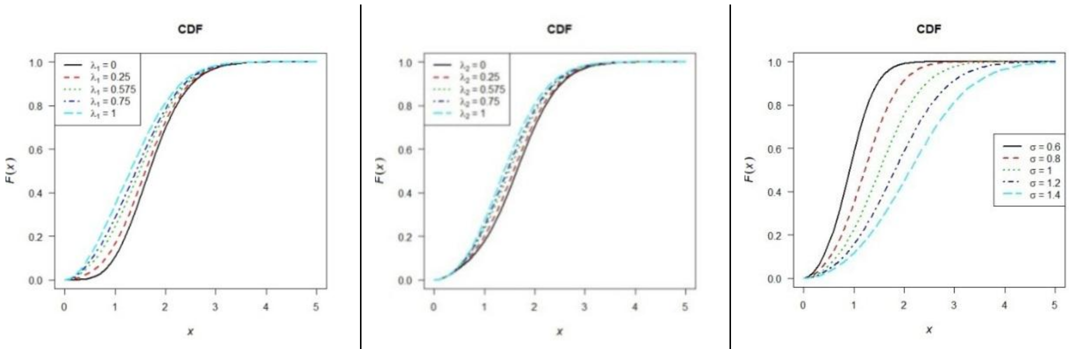
$$F(x)=\left(1-e^{\frac{-x}{2\sigma^2}}\right)\left[1+e^{\frac{-x^2}{2\sigma^2}}\left(\lambda_2+\lambda_1-2+e^{\frac{-x^2}{2\sigma^2}}-\lambda_2 e^{\frac{-x^2}{2\sigma^2}}\right)\right] \tag{8}$$

$$f(x)=\frac{-x}{\sigma^2}e^{\frac{-3x^2}{2\sigma^2}}\left[(\lambda_2+\lambda_1-3)e^{\frac{x^2}{\sigma^2}}+(-4\lambda_2-2\lambda_1+6)e^{\frac{x^2}{2\sigma^2}}+3\lambda_2-3\right] \tag{9}$$

Where, $\sigma>0,$ and $\lambda_1,\ \lambda_2\in[0,1].$

It should be noted that the different type of CTRD is a more advanced model for analyzing complicated data.

When $\lambda_1=\lambda_2=1,$ the Rayleigh distribution is a particular instance of the CTRD-I. The figures below depict some of the conceivable forms of the *cdf* and *pdf* of a CTRD-I for various parameter values.



a b c

FIGURE 1. Different shapes of *cdf* of CTRD-I for different parameter values.

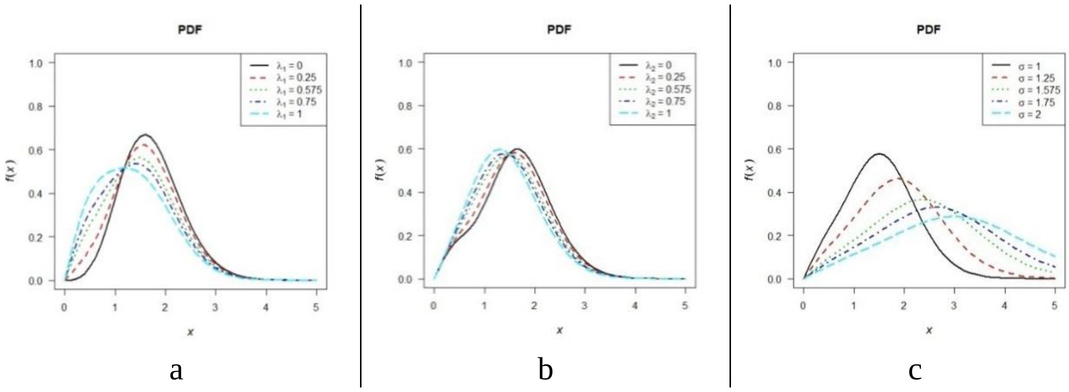


FIGURE 2. Different shapes of *pdf* of CTRD-I for different parameter values.

2.2 CTRD –II

The *cdf* and *pdf* of the CTRD-II are available below:

$$F(x,\sigma,\lambda)=1-e^{\frac{-x^2}{2\sigma^2}}+\lambda e^{\frac{-x^2}{\sigma^2}}-\lambda e^{\frac{-3x^2}{2\sigma^2}} \tag{10}$$

$$f(x,\sigma,\lambda)=\frac{x}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}\left[1-2\lambda e^{\frac{-x^2}{2\sigma^2}}+3\lambda e^{\frac{-x^2}{\sigma^2}}\right] \tag{11}$$

Where $x>0,\ \sigma>0,$ and $\lambda\in[-1,1]$. The figures below are shown some of the feasible structures of *cdf* and *pdf* forms of a CTRD-II for various parameter values.

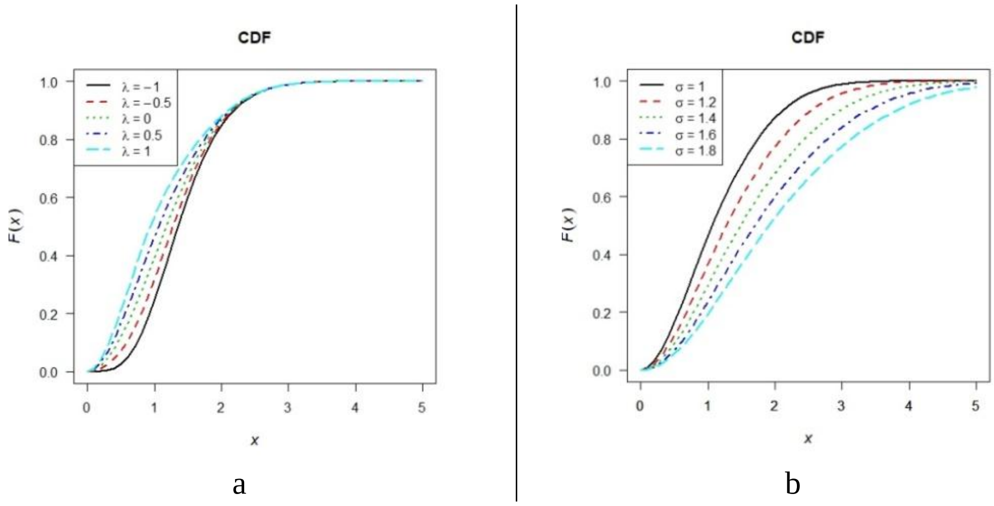
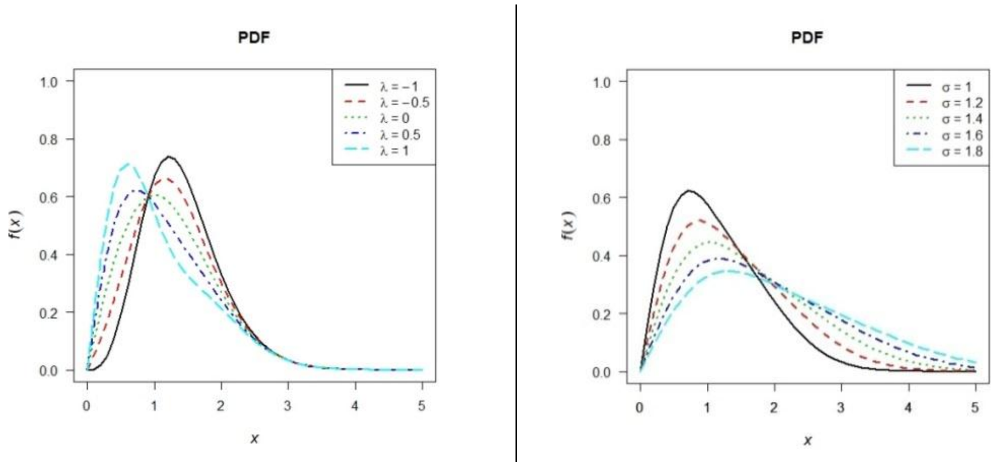


FIGURE 3. Different shapes of *cdf* of CTRD-II for different parameter values.



a b

FIGURE 4. Various shapes of *pdf* of CTRD-II for varying parameter values.

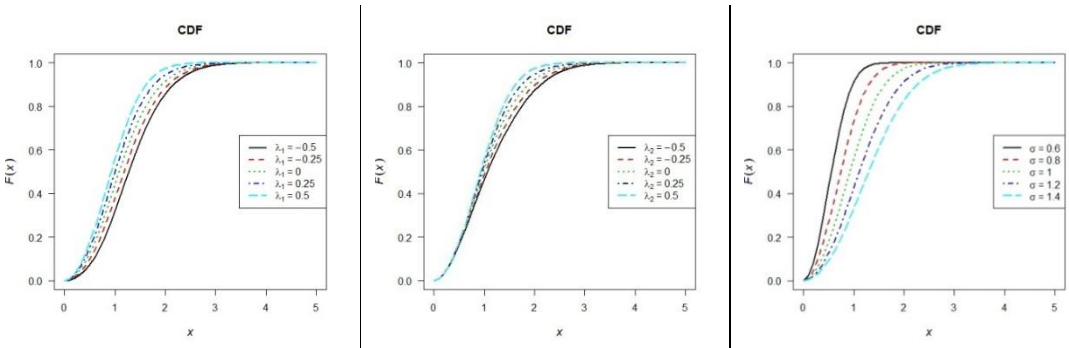
2.3 CTRD –III

The *cdf* and *pdf* of the CTRD-III are

$$F(x)=\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)\left[1+(\lambda_1+\lambda_2)e^{\frac{-x^2}{2\sigma^2}}-\lambda_2 e^{\frac{-x^2}{\sigma^2}}\right] \tag{12}$$

$$f(x)=\frac{-x}{\sigma^2}\left[(\lambda_1+\lambda_2-1)e^{\frac{-x^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-x^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3x^2}{2\sigma^2}}\right] \tag{13}$$

Where, $\sigma > 0,$ and $\lambda_{1,}, \lambda_2 \in [0,1]$. Figure 5 and 6 demonstrates some of the probable forms of a CTRD-III *cdf* and *pdf* for different parameter values.



a b c

FIGURE 5. Different shapes of CTRD-III *cdf* for different parameter values.

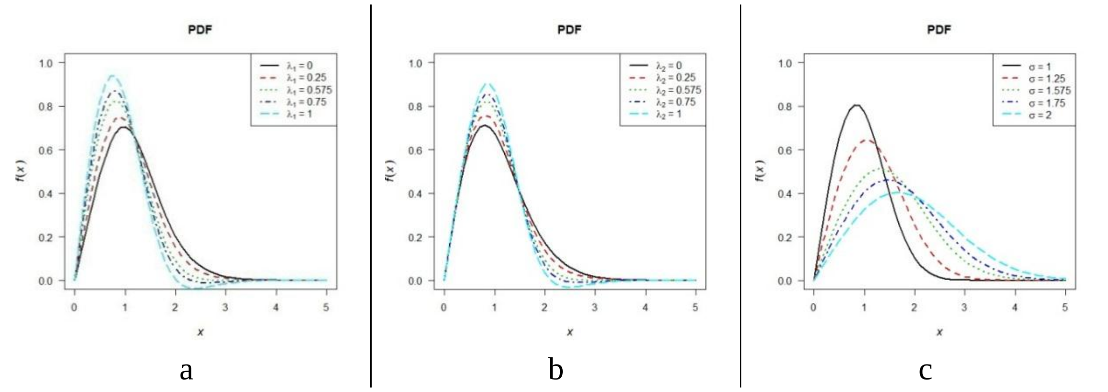


FIGURE 6. Different shapes of CTRD-III *pdf* for different parameter values.

2.4 CTRD –IV

The *cdf* and *pdf* of the CTRD-IV is

$$F(x,\sigma,\lambda)=1-e^{\frac{-x^2}{2\sigma^2}}-3\lambda e^{\frac{-x^2}{\sigma^2}}+2\lambda e^{\frac{-3x^2}{2\sigma^2}}+\lambda e^{\frac{-x^2}{2\sigma^2}} \tag{14}$$

$$f(x,\sigma,\lambda)=\frac{x}{\sigma^2}\left[(1-\lambda)e^{\frac{-x^2}{2\sigma^2}}+6\lambda e^{\frac{-x^2}{\sigma^2}}-6\lambda e^{\frac{-3x^2}{2\sigma^2}}\right] \tag{15}$$

Where $x>0, \sigma>0,$ and $\lambda\in[-1,1]$. The figures below depict some of the probable forms of the *cdf* and *pdf* of a CTRD-IV for particular parameter values.

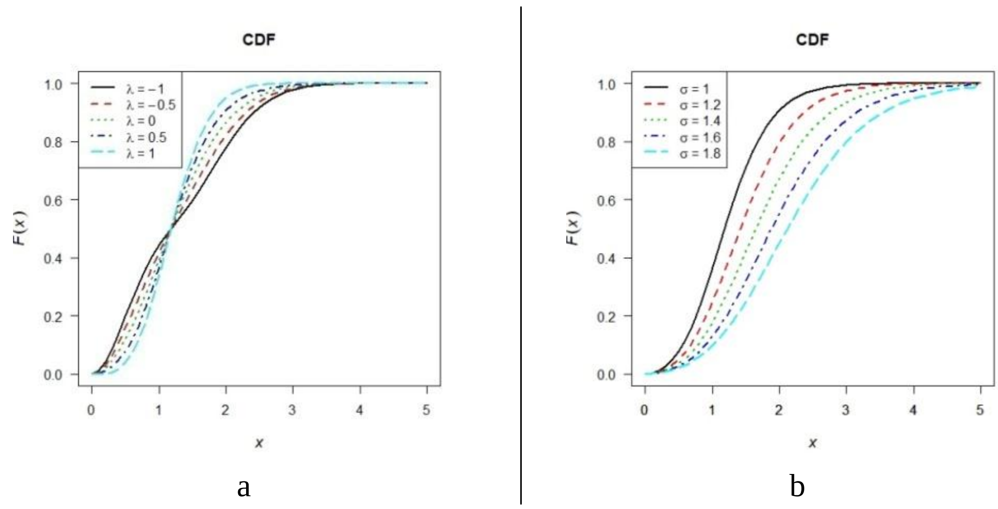


FIGURE 7. Different *cdf* shapes of CTRD-IV for different parameter values.

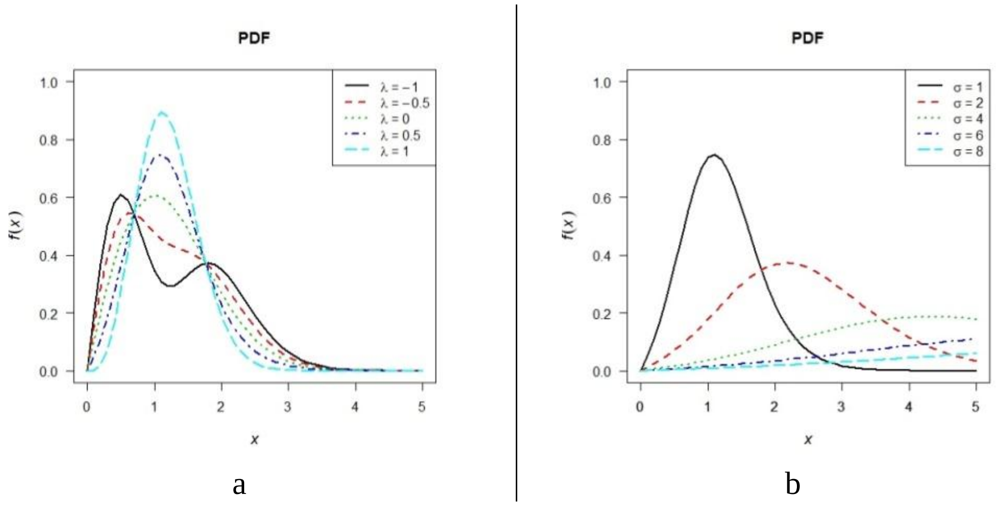


FIGURE 8. Various shapes of CTRD-IV *pdf* for varying parameter values**.**

3. SOME PROPERTIES OF THE PROPOSED DISTRIBUTIONS

3.1 Moments

The r^{th} moment of the CTRD distributions are given. We may obtain the distribution mean and variance. The r^{th} moment of various forms of CTRD random variable X is expressed as

$$E\left(X^r\right)=\int_0^\infty x^r f(x)dx \tag{16}$$

The first four moments of the CTRD distributions are listed below.

3.2 CTRD –I

The r^{th} moment of the CTRD-I is

$$E\left(X^r\right)=\frac{-\sigma^r}{3^{\frac{r}{2}}}\Gamma\left(\frac{r+2}{2}\right)\left[\left((\lambda_2+\lambda_1-3)2^{\frac{r}{2}}-(2\lambda_2-\lambda_1+3)3^{\frac{r}{2}}+2^{\frac{r}{2}}(\lambda_2-1)\right)\right] \tag{17}$$

The corresponding mean and variance of the CTRD-I is

$$E(X)=\frac{-\sigma}{\sqrt{3}}\frac{\sqrt{\pi}}{2}\left[\sqrt{2}\lambda_2\left(\sqrt{3}-\sqrt{6}+1\right)+\sqrt{3}\lambda_1\left(\sqrt{2}-1\right)+3\sqrt{3}-3\sqrt{6}-\sqrt{2}\right] \tag{18}$$

$$V(X)=\frac{-\sigma^2}{3}\left[3\lambda_1+2\lambda_2-11-\frac{\pi}{4}\left(\begin{array}{l}\sqrt{2}\lambda_2\left(\sqrt{3}-\sqrt{6}+1\right)+\sqrt{3}\lambda_1\left(\sqrt{2}-1\right)\\+3\sqrt{3}-3\sqrt{6}-\sqrt{2}\end{array}\right)^2\right] \tag{19}$$

$$E\left(X^3\right)=\frac{-\sigma^3}{2^{\frac{5}{2}}}\frac{\sqrt{\pi}}{3}\left[\left(4\sqrt{3}-92^{\frac{3}{2}}+36\right)\lambda_2+\left(36-9\sqrt{2}\right)\lambda_1-4\sqrt{3}+27\sqrt{2}-180\right] \quad (20)$$

$$E\left(X^4\right)=\frac{-\sigma^4}{9}\left(44\lambda_2+54\lambda_1-170\right) \quad (21)$$

3.3 CTRD –II

The r^{th} moment of the CTRD-II is

$$E\left(\mathrm{X}^{\mathrm{r}}\right)=\frac{\sigma^r}{3^{\frac{r}{2}}}\Gamma\left(\frac{r+2}{2}\right)\left[\left(2^{\frac{r}{2}}-\lambda\right)3^{\frac{r}{2}}+\lambda 2^{\frac{r}{2}}\right] \quad (22)$$

In particular,

$$\mu=E(X)=\frac{\sigma}{\sqrt{3}}\frac{\sqrt{\pi}}{2}\left(\sqrt{6}+\lambda\left(\sqrt{2}-\sqrt{3}\right)\right) \quad (23)$$

$$\sigma^2=V(X)=\frac{\sigma^2}{3}\left[6-\lambda-\left(\frac{\pi}{4}\left(\sqrt{6}+\lambda\left(\sqrt{2}-\sqrt{3}\right)\right)\right)^2\right] \quad (24)$$

$$E\left(X^3\right)=\frac{\sigma^3}{\sqrt{3}}\frac{\sqrt{\pi}}{4}\left(2^{\frac{3}{2}}3^{\frac{3}{2}}+\lambda\left(3^{\frac{3}{2}}-2^{\frac{3}{2}}\right)\right) \quad (25)$$

$$E\left(X^4\right)=\frac{\sigma^4}{9}\left(72-10\lambda\right) \quad (26)$$

3.4 CTRD –III

The r^{th} moment of the CTRD-III is

$$E\left(\mathrm{X}^{\mathrm{r}}\right)=\frac{-\sigma^r}{3^{\frac{r}{2}}}\Gamma\left(\frac{r+2}{2}\right)\left[\left(\left(\lambda_2+\lambda_1-1\right)2^{\frac{r}{2}}-2\lambda_2-\lambda_1\right)3^{\frac{r}{2}}+2^{\frac{r}{2}}\lambda_2\right] \quad (27)$$

In particular,

$$E(X)=\frac{-\sigma}{\sqrt{3}}\frac{\sqrt{\pi}}{2}\left[\sqrt{2}\lambda_2\left(\sqrt{3}-\sqrt{6}+1\right)+\sqrt{3}\lambda_1\left(\sqrt{2}-1\right)-\sqrt{3}\sqrt{2}\right] \quad (28)$$

$$V(X)=\frac{-\sigma^2}{3}\left[3\lambda_1+2\lambda_2-6-\frac{\pi}{4}\left(\sqrt{2}\lambda_2\left(\sqrt{3}-\sqrt{6}+1\right)+\sqrt{3}\lambda_1\left(\sqrt{2}-1\right)+\sqrt{3}\sqrt{2}\right)^2\right] \quad (29)$$

$$E\left(X^3\right)=\frac{-\sigma^3}{\sqrt{3}}\frac{\sqrt{\pi}}{4}\left[\left(\left(3*3^{\frac{3}{2}}-6\right)\sqrt{3}+2^{\frac{3}{2}}\right)\lambda_2+\left(3*2^{\frac{3}{2}}-3\right)\sqrt{3}\lambda_1-2^{\frac{3}{2}}3^{\frac{3}{2}}\right] \quad (30)$$

$$E\left(X^4\right)=\frac{-\sigma^4}{9}\left(44\lambda_2+54\lambda_1-72\right) \quad (31)$$

3.5 CTRD –IV

The r^{th} moment of the CTRD-IV is

$$E\left(X^{r}\right)=\frac{-\sigma^{r}}{3^{\frac{r}{2}}}\Gamma\left(\frac{r+2}{2}\right)\left[\left((\lambda-1)\left(2^{\frac{r}{2}}\right)-3\lambda\right)3^{\frac{r}{2}}+\lambda 2^{\frac{r+2}{2}}\right] \tag{32}$$

In particular,

$$E(X)=\frac{-\sigma}{\sqrt{3}}\frac{\sqrt{\pi}}{2}\left[\left(\sqrt{6}-3^{\frac{3}{2}}+2^{\frac{3}{2}}\right)\lambda-\sqrt{6}\right] \tag{33}$$

$$V(X)=\frac{-\sigma^{2}}{3}\left[6-\lambda+\frac{\pi}{4}\left(\lambda\left(\sqrt{6}-3^{\frac{3}{2}}+2^{\frac{3}{2}}\right)-\sqrt{6}\right)^{2}\right] \tag{34}$$

$$E\left(X^{3}\right)=\frac{-\sigma^{3}}{\sqrt{3}}\frac{\sqrt{\pi}}{4}\left[\left(2^{\frac{3}{2}}3^{\frac{3}{2}}-9\sqrt{3}+2^{\frac{5}{2}}\right)\lambda-2^{\frac{3}{2}}3^{\frac{3}{2}}\right] \tag{35}$$

$$E\left(X^{4}\right)=\frac{\sigma^{4}}{9}(72-34\lambda) \tag{36}$$

3.6 Moment Generating Function

The MGF of the CTRD random variable *X* is calculated below

$$M_{X}(t)=E\left(e^{tX}\right)=\int_{0}^{\infty}e^{tx}f(x)dx \tag{37}$$

The MGF of the four CTRD distributions are

$$CTRD-I:M_{X}(t)=\sum_{i=0}^{\infty}\frac{t^{i}}{i!}\frac{\sigma^{i}}{3^{\frac{i}{2}}}\Gamma\left(\frac{i+2}{2}\right)\left[-\left((\lambda_{2}+\lambda_{1}-3)2^{\frac{i}{2}}-2\lambda_{2}-\lambda_{1}+3\right)3^{\frac{i}{2}}-2^{\frac{i}{2}}(\lambda_{2}-1)\right] \tag{38}$$

$$CTRD-II:M_{X}(t)=\sum_{i=0}^{\infty}\frac{t^{i}}{i!}\frac{\sigma^{i}}{3^{\frac{i}{2}}}\Gamma\left(\frac{i+2}{2}\right)\left[\left(2^{\frac{i}{2}}-\lambda\right)3^{\frac{i}{2}}+\lambda 2^{\frac{i}{2}}\right] \tag{39}$$

$$CTRD-III:M_{X}(t)=\sum_{i=0}^{\infty}\frac{t^{i}}{i!}\frac{\sigma^{i}}{3^{\frac{i}{2}}}\Gamma\left(\frac{i+2}{2}\right)\left[\left((\lambda_{2}+\lambda_{1}-1)2^{\frac{i}{2}}-2\lambda_{2}-\lambda_{1}\right)3^{\frac{i}{2}}+2^{\frac{i}{2}}\lambda_{2}\right] \tag{40}$$

$$CTRD-IV:M_{X}(t)=\sum_{i=0}^{\infty}\frac{t^{i}}{i!}\frac{\sigma^{i}}{3^{\frac{i}{2}}}\Gamma\left(\frac{i+2}{2}\right)\left[-\left((\lambda-1)2^{\frac{i}{2}}-3\lambda\right)3^{\frac{i}{2}}-\lambda 2^{\frac{i+2}{2}}\right] \tag{41}$$

3.7 Characteristic Function

The characteristic function of the cubic transmuted Rayleigh distributed random variable X is

$$\Phi_X(t) = E\left(e^{itX}\right) = \int_0^\infty e^{itx} f(x)dx \tag{42}$$

The characteristic function of the four forms of CTRD are

$$CTRD-I : \Phi_X(t) = \sum_{i=0}^{\infty} \frac{it^i}{i!} \frac{\sigma^i}{3^{\frac{i}{2}}} \Gamma\left(\frac{i+2}{2}\right)\left[-\left((\lambda_2+\lambda_1-3)2^{\frac{i}{2}} - 2\lambda_2 - \lambda_1 + 3\right)3^{\frac{i}{2}} - 2^{\frac{i}{2}}(\lambda_2 - 1)\right] \tag{43}$$

$$CTRD-II : \Phi_X(t) = \sum_{i=0}^{\infty} \frac{it^i}{i!} \frac{\sigma^i}{3^{\frac{i}{2}}} \Gamma\left(\frac{i+2}{2}\right)\left[\left(2^{\frac{i}{2}} - \lambda\right)3^{\frac{i}{2}} + \lambda 2^{\frac{i}{2}}\right] \tag{44}$$

$$CTRD-III : \Phi_X(t) = \sum_{i=0}^{\infty} \frac{it^i}{i!} \frac{\sigma^i}{3^{\frac{i}{2}}} \Gamma\left(\frac{i+2}{2}\right)\left[\left((\lambda_2+\lambda_1-1)2^{\frac{i}{2}} - 2\lambda_2 - \lambda_1\right)3^{\frac{i}{2}} + 2^{\frac{i}{2}}\lambda_2\right] \tag{45}$$

$$CTRD-IV : \Phi_X(t) = \sum_{i=0}^{\infty} \frac{it^i}{i!} \frac{\sigma^i}{3^{\frac{i}{2}}} \Gamma\left(\frac{i+2}{2}\right)\left[-\left((\lambda-1)2^{\frac{i}{2}} - 3\lambda\right)3^{\frac{i}{2}} - \lambda 2^{\frac{i+2}{2}}\right] \tag{46}$$

3.8 Simulation

The statement that the random number from different types of CTRD is achieved by solving the following equation.

$$F(x) = p,$$

Where,

$p \sim U(0,1)$

Using the above equation, we may generate a random sample of CTRD distributions.

3.9 Shannon Entropy

Entropy measures the degree of uncertainty associated with a random variable X. The probability of a random variable X in the CTRD-III is provided as per Shannon's definition of entropy.

$$H = -E[\log(f(x))]$$

$$H = -E\left[\log\left(\frac{-x}{\sigma^2}\left[(\lambda_1+\lambda_2-1)e^{\frac{-x^2}{2\sigma^2}} - (4\lambda_2+2\lambda_1)e^{\frac{-x^2}{\sigma^2}} + 3\lambda_2 e^{\frac{-3x^2}{2\sigma^2}}\right]\right)\right] \tag{47}$$

$$H = -(I_1 + I_2)$$

Where,

$$I_1 = E\left(\log\left\{-\frac{x}{\sigma^2}\right\}\right)$$

And

$$I_2 = E\left[\log\left(-\left[(\lambda_1+\lambda_2-1)e^{\frac{-x^2}{2\sigma^2}} - (4\lambda_2+2\lambda_1)e^{\frac{-x^2}{\sigma^2}} + 3\lambda_2 e^{\frac{-3x^2}{2\sigma^2}}\right]\right)\right]$$

On simplification and by using expressions I_1 *and* I_2, the Shannon entropy is obtained.

4. PARAMETER ESTIMATION

4.1 MLE for CTRD-I

The *pdf* of the CTRD-I is

$$g(x) = \frac{-x}{\sigma^2} e^{\frac{-3x^2}{2\sigma^2}}\left[(\lambda_2+\lambda_1-3)e^{\frac{x^2}{\sigma^2}} + (-4\lambda_2-2\lambda_1+6)e^{\frac{x^2}{2\sigma^2}} + 3\lambda_2 - 3\right]$$

The likelihood function under this model is

$$L(x_i,\sigma,\lambda_1,\lambda_2) = \prod_{i=1}^{n} g(x_i,\sigma,\lambda_1,\lambda_2)$$

$$L(x_i,\sigma,\lambda_1,\lambda_2) = \prod_{i=1}^{n}\left[\frac{-x_i}{\sigma^2} e^{\frac{-3x_i^2}{2\sigma^2}}\left[(\lambda_2+\lambda_1-3)e^{\frac{x_i^2}{\sigma^2}} + (-4\lambda_2-2\lambda_1+6)e^{\frac{x_i^2}{2\sigma^2}} + 3\lambda_2 - 3\right]\right]$$

The log-likelihood function is

$$l(x_i) = \log L(x_i,\sigma,\lambda_1,\lambda_2)$$

$$l(x_i) = \log\left[\prod_{i=1}^{n}\left[\frac{-x_i}{\sigma^2} e^{\frac{-3x_i^2}{2\sigma^2}}\left[(\lambda_2+\lambda_1-3)e^{\frac{x_i^2}{\sigma^2}} + (-4\lambda_2-2\lambda_1+6)e^{\frac{x_i^2}{2\sigma^2}} + 3\lambda_2 - 3\right]\right]\right]$$

$$l(x_i) = -\sum_{i=1}^{n} log(x_i) + n\,log\,\sigma^2 -$$

$$\frac{3}{2\sigma^2}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} log\left[(\lambda_2+\lambda_1-3)e^{\frac{x_i^2}{\sigma^2}} + (-4\lambda_2-2\lambda_1+6)e^{\frac{x_i^2}{2\sigma^2}} + 3\lambda_2 - 3\right]$$

Now setting $l_{\lambda_1} = 0,\ l_{\lambda_2} = 0$ and $l_\sigma = 0$,

We have

$$l_{\lambda_1} = \frac{dl(x_i)}{d\lambda_1} = \sum \frac{e^{\frac{x_i^2}{\sigma^2}} - 2e^{\frac{x_i^2}{2\sigma^2}}}{(\lambda_2+\lambda_1-3)e^{\frac{x_i^2}{\sigma^2}} + (-4\lambda_2-2\lambda_1+6)e^{\frac{x_i^2}{2\sigma^2}} + 3\lambda_2 - 3} = 0$$

$$l_{\lambda_2} = \frac{dl(x_i)}{d\lambda_2} = \sum \frac{e^{\frac{x_i^2}{\sigma^2}} - 4e^{\frac{x_i^2}{2\sigma^2}} + 3}{(\lambda_2+\lambda_1-3)e^{\frac{x_i^2}{\sigma^2}} + (-4\lambda_2-2\lambda_1+6)e^{\frac{x_i^2}{2\sigma^2}} + 3\lambda_2 - 3} = 0$$

And

$$l_\sigma = \frac{dl(x_i)}{d\sigma} = \frac{2n}{\sigma} + \frac{3}{\sigma^3}\sum_{i=1}^{n} x_i^2 + \sum \frac{\frac{-2x_i^2}{\sigma^3}(\lambda_2+\lambda_1-3)e^{\frac{x_i^2}{\sigma^2}} - \frac{x_i^2}{\sigma^3}(-4\lambda_2-2\lambda_1+6)e^{\frac{x_i^2}{2\sigma^2}}}{(\lambda_2+\lambda_1-3)e^{\frac{x_i^2}{\sigma^2}} + (-4\lambda_2-2\lambda_1+6)e^{\frac{x_i^2}{2\sigma^2}} + 3\lambda_2 - 3} = 0$$

This nonlinear system of equations is solved to yield the MLEs $\hat{\theta} = (\hat{\sigma}, \hat{\lambda}_1, \hat{\lambda}_2)$ of $\theta = (\sigma, \lambda_1, \lambda_2)$. Nonlinear optimization procedures are frequently more convenient for numerically optimizing the sample likelihood function. We can solve these equations numerically using statistical tools like R programming.

To construct Fisher's information matrix, we must first determine the second partial derivatives of the model parameters. The information matrix that has been seen will be $\sqrt{n}(\hat{\theta} - \theta) \sim N_3(0, V^{-1})$

Where V is the expected information matrix.

The information matrix is

$$V^{-1} = -E\begin{bmatrix} \frac{\partial^2 l}{\partial \sigma^2} & \frac{\partial^2 l}{\partial\sigma\partial\lambda_1} & \frac{\partial^2 l}{\partial\sigma\partial\lambda_2} \\ \frac{\partial^2 l}{\partial\lambda_1\partial\sigma} & \frac{\partial^2 l}{\partial\lambda_1{}^2} & \frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} \\ \frac{\partial^2 l}{\partial\lambda_2\partial\sigma} & \frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & \frac{\partial^2 l}{\partial\lambda_2^2} \end{bmatrix}$$

The asymptotic variance and covariance of the MLEs $\hat{\sigma}, \hat{\lambda}_1,$ and $\hat{\lambda}_2$ are given by solving the inverse matrix of the observed information matrix. The approximate $100(1-\alpha)\%$ asymptotic confidence intervals $\sigma, \lambda_1,$ and λ_2 are given by

$$\hat{\sigma} \pm Z_{\frac{\alpha}{2}}\sqrt{\mathrm{var}(\hat{\sigma})}, \quad \hat{\lambda}_1 \pm Z_{\frac{\alpha}{2}}\sqrt{\mathrm{var}(\hat{\lambda}_1)}, \quad \hat{\lambda}_2 \pm Z_{\frac{\alpha}{2}}\sqrt{\mathrm{var}(\hat{\lambda}_2)}$$

We know that $Z_{\frac{\alpha}{2}}$ is the α^{th} percentile of the standard normal distribution.

4.2 MLE for CTRD-II

The *pdf* of the CTRD-II is

$$g(x) = \frac{x}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}\left[1 - 2\lambda e^{\frac{-x^2}{2\sigma^2}} + 3\lambda e^{\frac{-x^2}{\sigma^2}}\right]$$

The likelihood function under this model is

$$l(x_i) = -\sum_{i=1}^{n}\log(x_i) - n\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2 + \sum_{i=1}^{n}\log\left[1 - 2\lambda e^{\frac{-x_i^2}{2\sigma^2}} + 3\lambda e^{\frac{-x_i^2}{\sigma^2}}\right]$$

Now setting $l_\lambda = 0$ and $l_\sigma = 0,$

We have

$$l_\lambda = \frac{dl(x_i)}{d\lambda} = \sum \frac{-2e^{-\frac{x_i^2}{2\sigma^2}} + 3e^{\frac{-x_i^2}{\sigma^2}}}{\left[1-2\lambda e^{\frac{-x_i^2}{2\sigma^2}} + 3\lambda e^{\frac{-x_i^2}{\sigma^2}}\right]} = 0$$

$$l_\sigma = \frac{dl(x_i)}{d\sigma} = \frac{-2n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n} x_i^2 + \sum \frac{\left(\frac{x_i^2}{\sigma^3}\right)\left(2\lambda e^{-\frac{x_i^2}{2\sigma^2}} + 6\lambda e^{\frac{-x_i^2}{\sigma^2}}\right)}{\left[1-2\lambda e^{\frac{-x_i^2}{2\sigma^2}} + 3\lambda e^{\frac{-x_i^2}{\sigma^2}}\right]} = 0$$

To acquire the MLEs $\hat{\theta} = (\hat{\sigma}, \hat{\lambda})$ of $\theta = (\sigma, \lambda)$, we must first solve the nonlinear system of equations. Statistical tools such as the *maxLik* package in R can be used to solve the following equations statistically.

We employ second partial derivatives of the log-likelihood function to compute Fisher's information matrix. It is used to calculate interval estimates and hypothesis testing for model parameters. The information matrix observed will be

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\lambda} \end{pmatrix} \sim N\left[\begin{pmatrix} \sigma \\ \lambda \end{pmatrix}\begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} \\ \hat{V}_{21} & \hat{V}_{22} \end{pmatrix}\right]$$

Where, $\hat{V}_{ij} = V_{ij}\big|_{\theta=\hat{\theta}}$ and

The information matrix is

$$V^{-1} = I_x(\sigma, \lambda) = \begin{bmatrix} -E\left(\frac{\partial^2 l}{\partial \sigma^2}\right) & -E\left(\frac{\partial^2 l}{\partial \sigma \partial \lambda}\right) \\ -E\left(\frac{\partial^2 l}{\partial \lambda \partial \sigma}\right) & -E\left(\frac{\partial^2 l}{\partial \lambda^2}\right) \end{bmatrix}$$

The asymptotic variance and covariance of these ML estimators will be produced by solving the above-mentioned information matrix. The approximate $100(1-\alpha)\%$ confidence intervals σ and λ can be determined as

$$\hat{\sigma} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}, \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}}$$

Where $Z_{\frac{\alpha}{2}}$ is denote the upper α^{th} percentile of the standard normal distribution.

4.3 MLE for CTRD-III

The *pdf* of the CTRD-III is

$$g(x) = \frac{-x}{\sigma^2}\left[(\lambda_1 + \lambda_2 - 1)e^{\frac{-x^2}{2\sigma^2}} - (4\lambda_2 + 2\lambda_1)e^{\frac{-x^2}{\sigma^2}} + 3\lambda_2 e^{\frac{-3x^2}{2\sigma^2}}\right]$$

The likelihood function under this model is

$$L(x_i,\sigma,\lambda_1,\lambda_2)=\prod_{i=1}^{n}\left[\frac{-x_i}{\sigma^2}\left[(\lambda_1+\lambda_2-1)e^{\frac{-x_i^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-x_i^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3x_i^2}{2\sigma^2}}\right]\right]$$

The log-likelihood function

$$l(x_i)=-\sum_{i=1}^{n}\log(x_i)+n\log\sigma^2+\sum_{i=1}^{n}\log\left[\left[(\lambda_1+\lambda_2-1)e^{\frac{-x_i^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-x_i^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3x_i^2}{2\sigma^2}}\right]\right]$$

Now setting $l_{\lambda_1}=0,\ l_{\lambda_2}=0$ and $l_\sigma=0,$ we have

$$l_{\lambda_1}=\frac{dl(x_i)}{d\lambda_1}=\sum\frac{e^{\frac{x_i^2}{2\sigma^2}}-2e^{\frac{x_i^2}{\sigma^2}}}{\left[(\lambda_1+\lambda_2-1)e^{\frac{-x_i^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-x_i^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3x_i^2}{2\sigma^2}}\right]}=0$$

$$l_{\lambda_2}=\frac{dl(x_i)}{d\lambda_2}=\sum\frac{e^{\frac{x_i^2}{2\sigma^2}}-4e^{\frac{-x_i^2}{\sigma^2}}+3e^{\frac{-3x_i^2}{2\sigma^2}}}{\left[(\lambda_1+\lambda_2-1)e^{\frac{-x_i^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-x_i^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3x_i^2}{2\sigma^2}}\right]}=0$$

And

$$l_\sigma=\frac{dl(x_i)}{d\sigma}=\frac{2n}{\sigma}+\sum\frac{\frac{x_i^2}{\sigma^3}(\lambda_1+\lambda_2-1)e^{\frac{-x_i^2}{2\sigma^2}}+\frac{2x_i^2}{\sigma^3}(-4\lambda_2-2\lambda_1)e^{\frac{-x_i^2}{\sigma^2}}+\frac{9x_i^2}{\sigma^3}\lambda_2 e^{\frac{-3x_i^2}{2\sigma^2}}}{(\lambda_1+\lambda_2-1)e^{\frac{-x_i^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-x_i^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3x_i^2}{2\sigma^2}}}=0$$

The MLEs $\hat{\theta}=(\hat{\sigma},\hat{\lambda}_1,\hat{\lambda}_2)$ of $\theta=(\sigma,\lambda_1,\lambda_2)$ is produced by solving this nonlinear system of equations. We can use statistical software like R programming to solve these equations numerically. *MaxLik* package is most preferably using packages.

4.4 MLE for CTRD-IV

The *pdf* of the CTRD-IV is

$$g(x)=\frac{x}{\sigma^2}\left[(1-\lambda)e^{\frac{-x^2}{2\sigma^2}}+6\lambda e^{\frac{-x^2}{\sigma^2}}-6\lambda e^{\frac{-3x^2}{2\sigma^2}}\right]$$

The log-likelihood function under this model is

$$l(x_i)=\sum_{i=1}^{n}\log(x_i)-n\log\sigma^2+\sum_{i=1}^{n}\log\left[(1-\lambda)e^{\frac{-x_i^2}{2\sigma^2}}+6\lambda e^{\frac{-x_i^2}{\sigma^2}}-6\lambda e^{\frac{-3x_i^2}{2\sigma^2}}\right]$$

Now setting $l_\lambda=0$ and $l_\sigma=0,$

We have

$$l_\lambda=\frac{dl(x_i)}{d\lambda}=\sum\frac{-e^{-\frac{x_i^2}{2\sigma^2}}+6e^{\frac{-x_i^2}{\sigma^2}}-6e^{\frac{-3x_i^2}{2\sigma^2}}}{\left[(1-\lambda)e^{\frac{-x_i^2}{2\sigma^2}}+6\lambda e^{\frac{-x_i^2}{\sigma^2}}-6\lambda e^{\frac{-3x_i^2}{2\sigma^2}}\right]}=0$$

and

$$l_\sigma=\frac{dl(x_i)}{d\sigma}=\frac{-2n}{\sigma}+\sum\frac{\left(\frac{x_i^2}{\sigma^3}\right)\left[(1-\lambda)e^{\frac{-x_i^2}{2\sigma^2}}+6\lambda e^{\frac{-x_i^2}{\sigma^2}}-18\lambda e^{\frac{-3x_i^2}{2\sigma^2}}\right]}{\left[(1-\lambda)e^{\frac{-x_i^2}{2\sigma^2}}+6\lambda e^{\frac{-x_i^2}{\sigma^2}}-6\lambda e^{\frac{-3x_i^2}{2\sigma^2}}\right]}=0$$

Solving the sets of equations will give the MLEs $\hat{\theta}=\left(\hat{\sigma},\hat{\lambda}\right)$ of $\theta=(\sigma,\lambda)$. To numerically maximize the sample likelihood function, it is typically more advantageous to utilize an iterative process such as the Newton-Raphson method or the quasi-newton maximization algorithm. The above-mentioned equations can be quantitatively solved using statistical software such as the *maxLik* package in R.

5. RELIABILITY FUNCTION

The reliability function $R(t)$, which is the chance of an item breaking before time *t*, is

$$R(t)=1-F(t)$$

The hazard rate function is commonly provided by

$$h(t)=\frac{f(t)}{1-F(t)}$$

The reliability and hazard function of various forms of CTRD, as well as their forms, are given below.

5.1 CTRD-I

$$R(t)=e^{\frac{-t^2}{2\sigma^2}}\left[1-\left(1-e^{\frac{-t^2}{2\sigma^2}}\right)\left(\lambda_2+\lambda_1-2+e^{\frac{-t^2}{2\sigma^2}}-\lambda_2 e^{\frac{-t^2}{2\sigma^2}}\right)\right] \tag{48}$$

$$h(t)=\frac{\frac{-t}{\sigma^2}e^{\frac{-3t^2}{2\sigma^2}}\left[(\lambda_2+\lambda_1-3)e^{\frac{t^2}{\sigma^2}}+(-4\lambda_2-2\lambda_1+6)e^{\frac{t^2}{2\sigma^2}}+3\lambda_2-3\right]}{e^{\frac{-t^2}{2\sigma^2}}\left[1-\left(1-e^{\frac{-t^2}{2\sigma^2}}\right)\left(\lambda_2+\lambda_1-2+e^{\frac{-t^2}{2\sigma^2}}-\lambda_2 e^{\frac{-t^2}{2\sigma^2}}\right)\right]} \tag{49}$$

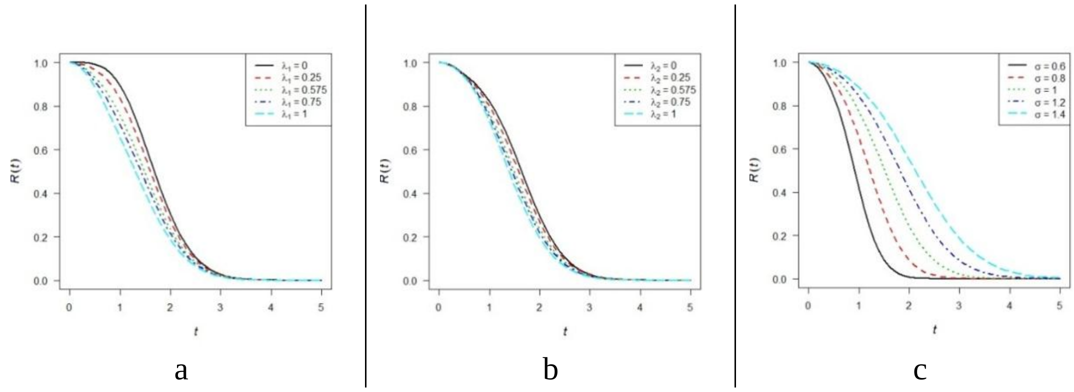


FIGURE 9. Different shapes of the survival function of CTRD-I for varied parameter values.

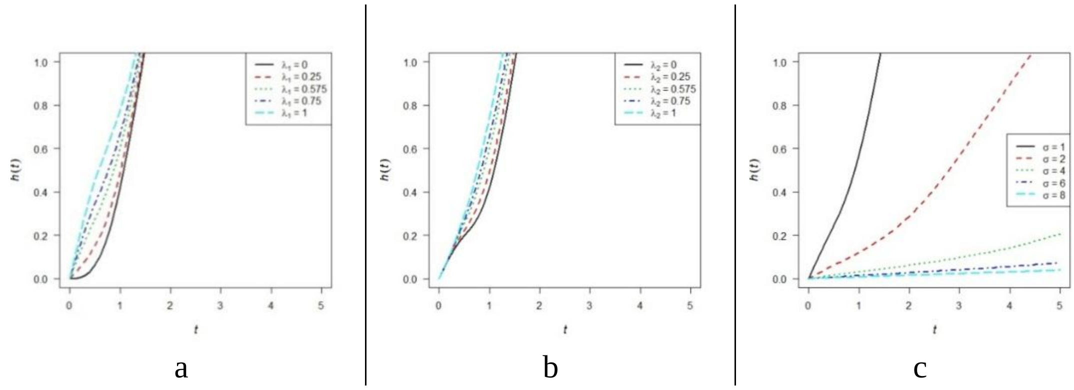


FIGURE 10. Various shapes of hazard function of CTRD-I for varying parameter values.

5.2 CTRD-II

$$R(t) = e^{\frac{-t^2}{2\sigma^2}} - \lambda e^{\frac{-t^2}{\sigma^2}} - \lambda e^{\frac{-3t^2}{2\sigma^2}} \tag{50}$$

$$h(t) = \frac{\frac{t}{\sigma^2} e^{\frac{-t^2}{2\sigma^2}} \left[1 - 2\lambda e^{\frac{-t^2}{2\sigma^2}} + 3\lambda e^{\frac{-t^2}{\sigma^2}} \right]}{e^{\frac{-t^2}{2\sigma^2}} - \lambda e^{\frac{-t^2}{\sigma^2}} - \lambda e^{\frac{-3t^2}{2\sigma^2}}} \tag{51}$$

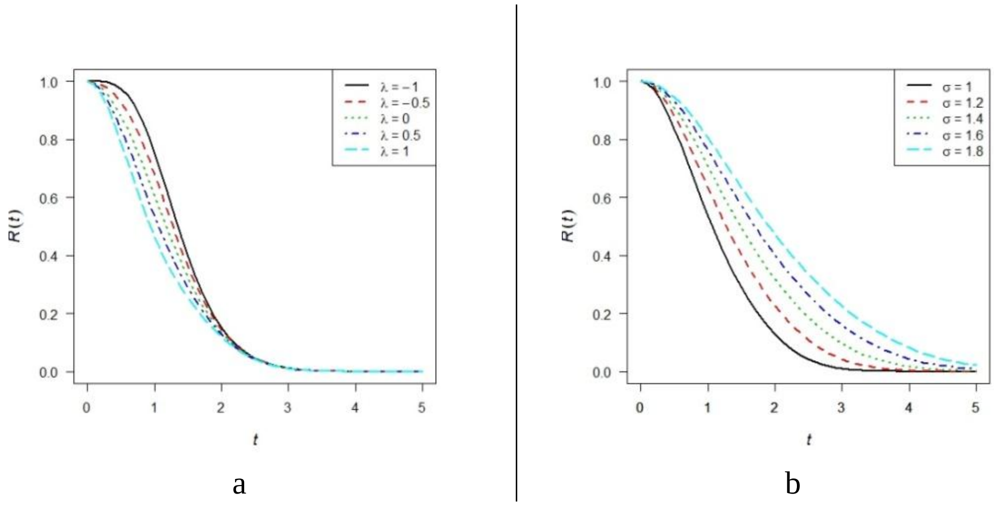


FIGURE 11. Different shapes of CTRD-II survival function for varied parameter values

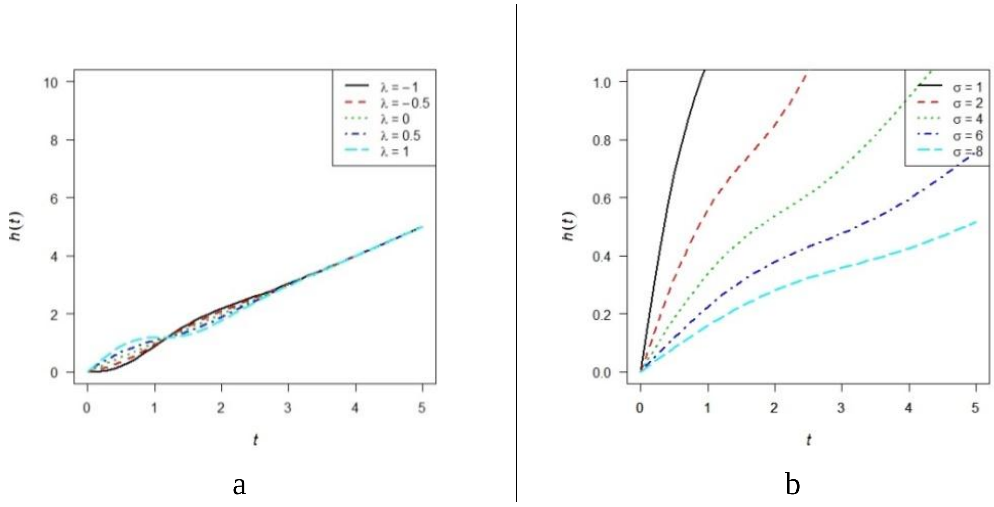


FIGURE 12. Various shapes of hazard function of CTRD-II for varying parameter values**.**

5.3 CTRD-III

$$R(t) = e^{\frac{-t^2}{2\sigma^2}} - \left(1 - e^{\frac{-t^2}{2\sigma^2}}\right)\left[(\lambda_1 + \lambda_2)e^{\frac{-t^2}{2\sigma^2}} - \lambda_2 e^{\frac{-t^2}{\sigma^2}}\right] \tag{52}$$

$$h(t)=\frac{\dfrac{-t}{\sigma^2}\left[(\lambda_1+\lambda_2-1)e^{\frac{-t^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-t^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3t^2}{2\sigma^2}}\right]}{e^{\frac{-t^2}{2\sigma^2}}-\left(1-e^{\frac{-t^2}{2\sigma^2}}\right)\left[(\lambda_1+\lambda_2)e^{\frac{-t^2}{2\sigma^2}}-\lambda_2 e^{\frac{-t^2}{\sigma^2}}\right]} \tag{53}$$

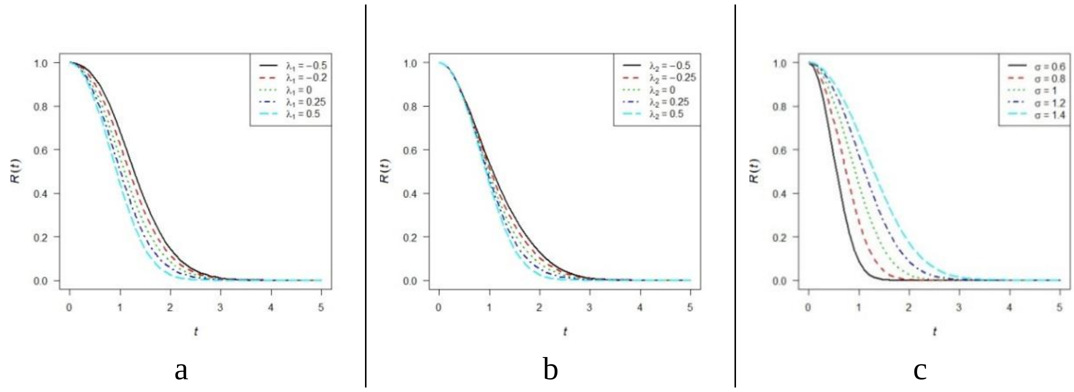


FIGURE 13. Different shapes of the survival function of CTRD-III for varied parameter values

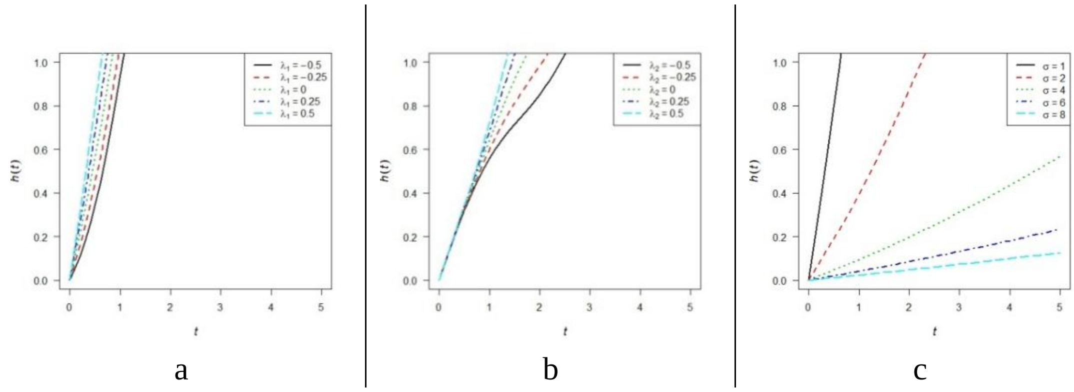
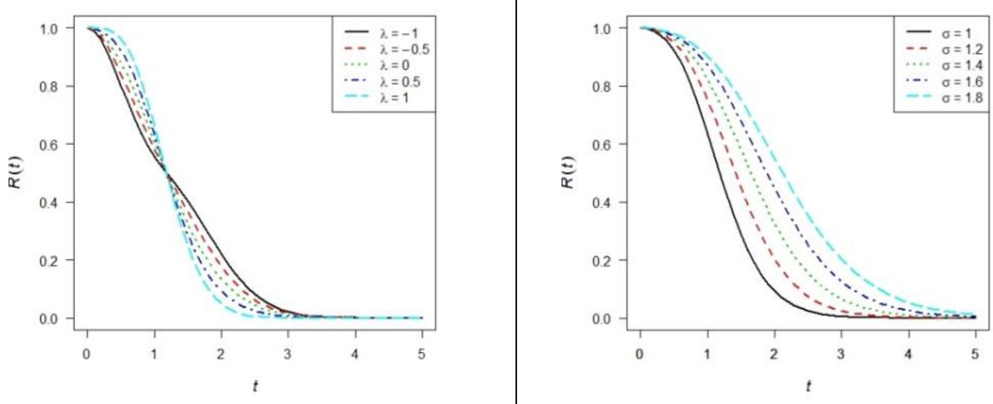


FIGURE 14. Various shapes of hazard function of CTRD-III for varying parameter values**.**

5.4 CTRD-IV

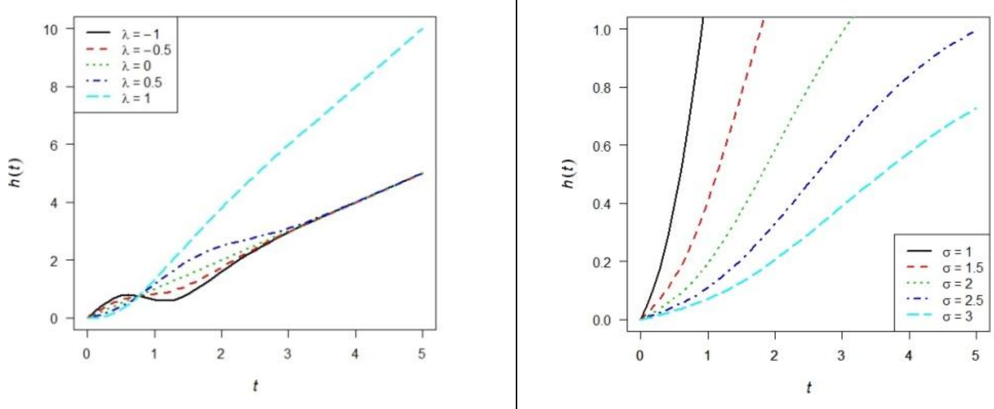
$$R(t)=e^{\frac{-t^2}{2\sigma^2}}+3\lambda e^{\frac{-t^2}{\sigma^2}}-2\lambda e^{\frac{-3t^2}{2\sigma^2}}-\lambda e^{\frac{-t^2}{2\sigma^2}} \tag{54}$$

$$h(t)=\frac{\frac{t}{\sigma^2}\left[(1-\lambda)e^{\frac{-t^2}{2\sigma^2}}+6\lambda e^{\frac{-t^2}{\sigma^2}}-6\lambda e^{\frac{-3t^2}{2\sigma^2}}\right]}{e^{\frac{-t^2}{2\sigma^2}}+3\lambda e^{\frac{-t^2}{\sigma^2}}-2\lambda e^{\frac{-3t^2}{2\sigma^2}}-\lambda e^{\frac{-t^2}{2\sigma^2}}} \tag{55}$$



a b

FIGURE 15. Different shapes of the survival function of CTRD-IV for various parameter values



a b

FIGURE 16. Various shapes of hazard function of CTRD-IV for varying parameter values**.**

6. ORDER STATISTICS

The n^{th} order statistic is defined as

$X_{(n)} = \max\left(X_1, X_2, \ \ldots \ , X_n\right)$

then, the order statistics function is

$range\left(X_1, X_2, \ \ldots \ , X_n\right) = X_{(n)} - X_{(1)}$

If $X_{(1)} \le X_{(2)} \le \ \ldots \ \le X_{(n)}$ represents the order statistic of a random sample from *cdf* $F_X(x)$ and *pdf* $f_X(x)$ then the *pdf* $X_{(r)}$ is given by

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)\left[F_X(x)\right]^{(r-1)}\left[1 - F_X(x)\right]^{(n-r)} \tag{56}$$

For, $r = 1, 2, \ \ldots \ , n$. The *pdf* of the r^{th} order statistic for various types of CTRD is computed, and the *pdf*s of the largest order statistic $X_{(n)}$ and smallest order statistic $X_{(1)}$ are shown below.

6.1 CTRD-I

$$f_{X_{(n)}}(x) = n\left(\frac{-x}{\sigma^2} e^{\frac{-3x^2}{2\sigma^2}}\left[(\lambda_2 + \lambda_1 - 3)e^{\frac{x^2}{\sigma^2}} + (-4\lambda_2 - 2\lambda_1 + 6)e^{\frac{x^2}{2\sigma^2}} + 3\lambda_2 - 3\right]\right)$$
$$\left[\left(1 - e^{\frac{-x}{2\sigma^2}}\right)\left[1 + e^{\frac{-x^2}{2\sigma^2}}\left(\lambda_2 + \lambda_1 - 2 + e^{\frac{-x^2}{2\sigma^2}} - \lambda_2 e^{\frac{-x^2}{2\sigma^2}}\right)\right]\right]^{(n-1)} \tag{57}$$

$$f_{X_{(1)}}(x) = n\left(\frac{-x}{\sigma^2} e^{\frac{-3x^2}{2\sigma^2}}\left[(\lambda_2 + \lambda_1 - 3)e^{\frac{x^2}{\sigma^2}} + (-4\lambda_2 - 2\lambda_1 + 6)e^{\frac{x^2}{2\sigma^2}} + 3\lambda_2 - 3\right]\right)$$
$$\left[\left(e^{\frac{-x}{2\sigma^2}}\right)\left[1 - \left(1 - e^{\frac{-x^2}{2\sigma^2}}\right)\left(\lambda_2 + \lambda_1 - 2 + e^{\frac{-x^2}{2\sigma^2}} - \lambda_2 e^{\frac{-x^2}{2\sigma^2}}\right)\right]\right]^{(n-1)} \tag{58}$$

6.2 CTRD-II

$$f_{X_{(n)}}(x) = n\left(\frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}\left[1 - 2\lambda e^{\frac{-x^2}{2\sigma^2}} + 3\lambda e^{\frac{-x^2}{\sigma^2}}\right]\right)\left[1 - e^{\frac{-x^2}{2\sigma^2}} + \lambda e^{\frac{-x^2}{\sigma^2}} - \lambda e^{\frac{-3x^2}{2\sigma^2}}\right]^{(n-1)} \tag{59}$$

$$f_{X_{(1)}}(x) = n\left(\frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}\left[1 - 2\lambda e^{\frac{-x^2}{2\sigma^2}} + 3\lambda e^{\frac{-x^2}{\sigma^2}}\right]\right)\left[e^{\frac{-x^2}{2\sigma^2}} - \lambda e^{\frac{-x^2}{\sigma^2}} + \lambda e^{\frac{-3x^2}{2\sigma^2}}\right]^{(n-1)} \tag{60}$$

6.3 CTRD-III

$$f_{X_{(n)}}(x) = n\left(\frac{-x}{\sigma^2}\left[\left(\lambda_1 + \lambda_2 - 1\right)e^{\frac{-x^2}{2\sigma^2}} - \left(4\lambda_2 + 2\lambda_1\right)e^{\frac{-x^2}{\sigma^2}} + 3\lambda_2 e^{\frac{-3x^2}{2\sigma^2}}\right]\right)$$
$$\left[\left(1 - e^{\frac{-x^2}{2\sigma^2}}\right)\left[1 + \left(\lambda_1 + \lambda_2\right)e^{\frac{-x^2}{2\sigma^2}} - \lambda_2 e^{\frac{-x^2}{\sigma^2}}\right]\right]^{(n-1)} \tag{61}$$

$$f_{X_{(1)}}(x)=n\left(\frac{-x}{\sigma^2}\left[(\lambda_1+\lambda_2-1)e^{\frac{-x^2}{2\sigma^2}}-(4\lambda_2+2\lambda_1)e^{\frac{-x^2}{\sigma^2}}+3\lambda_2 e^{\frac{-3x^2}{2\sigma^2}}\right]\right)$$
$$\left[e^{\frac{-x^2}{2\sigma^2}}-\left(1-e^{\frac{-x^2}{2\sigma^2}}\right)\left[1+(\lambda_1+\lambda_2)e^{\frac{-x^2}{2\sigma^2}}-\lambda_2 e^{\frac{-x^2}{\sigma^2}}\right]\right]^{(n-1)} \quad (62)$$

6.4 CTRD-IV

$$f_{X_{(n)}}(x)=n\left(\frac{x}{\sigma^2}\left[(1-\lambda)e^{\frac{-x^2}{2\sigma^2}}+6\lambda e^{\frac{-x^2}{\sigma^2}}-6\lambda e^{\frac{-3x^2}{2\sigma^2}}\right]\right)$$
$$\left[1-e^{\frac{-x^2}{2\sigma^2}}-3\lambda e^{\frac{-x^2}{\sigma^2}}+2\lambda e^{\frac{-3x^2}{2\sigma^2}}+\lambda e^{\frac{-x^2}{2\sigma^2}}\right]^{(n-1)} \quad (63)$$

$$f_{X_{(1)}}(x)=n\left(\frac{x}{\sigma^2}\left[(1-\lambda)e^{\frac{-x^2}{2\sigma^2}}+6\lambda e^{\frac{-x^2}{\sigma^2}}-6\lambda e^{\frac{-3x^2}{2\sigma^2}}\right]\right)$$
$$\left[e^{\frac{-x^2}{2\sigma^2}}+3\lambda e^{\frac{-x^2}{\sigma^2}}-2\lambda e^{\frac{-3x^2}{2\sigma^2}}-\lambda e^{\frac{-x^2}{2\sigma^2}}\right]^{(n-1)} \quad (64)$$

7. APPLICATION

We employ three real-time data sets in this part. CTRD-I is based on survival time data from guinea pigs treated with varying amounts of tubercle bacilli. In addition, the CTRD-I is compared to fundamental distributions such as the Rayleigh, gamma, and Weibull distributions. We use the breaking strength of carbon fiber data for CTRD-II and CTRD-III. Further, we compare the distributions (CTRD-II, CTRD-III) with the Rayleigh distribution. Finally, we compared CTRD-IV to Rayleigh distribution and transmuted Rayleigh distributions using nicotine readings taken from numerous brands of cigarettes in the 1998 data.

The AIC, BIC, K-S test, Anderson Darling's test (AD), and CVM test were used to evaluate the goodness of fit for these models.

7.1 CTRD-I

Sankudey et al. (2017) compared the suggested distribution to the Rayleigh, Weibull, and Gamma distributions using the data set containing 72 observations of the survival periods (in days) of guinea pigs given various dosages of tubercle bacilli. We also compared the CTRD-I distribution with the Rayleigh, Weibull, and Gamma distributions for this data set.

The goodness-of-fit metrics for the four probability distributions of the data stated above are shown in Table 1. It shows that the log-likelihood, AIC, and BIC values obtained by the CTRD-I were comparatively better. Figure 17 demonstrates that the

CTRD-I distribution fits the guinea pig data better than other traditional distributions.

TABLE 1: Using distributional fitting, determine the model that best fits the data from Guinea pigs.

Distribution	Log-Lik	AIC	BIC	K-S (P-value)	AD (P-value)	CVM (P-value)
Rayleigh	91.48	-178.96	-174.41	0.25(0.00)	6.15(0.00)	1.28(0.001)
Weibull	99.83	-195.66	-191.10	0.15(0.88)	2.36(0.59)	0.41(0.067)
Gamma	102.83	-201.66	-197.11	0.99(0.00)	781.6(0.00)	23.99(0.00)
CTRD-I	109.88	-213.74	-206.92	0.082(0.71)	0.62(0.63)	0.116(0.51)

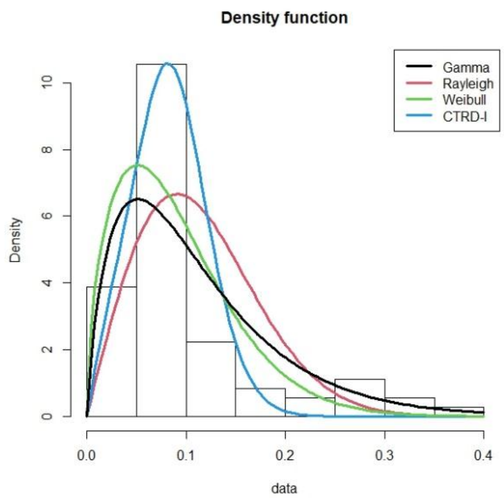


FIGURE 17: The comparison of four distributions fits.

7.2 CTRD-II and CTRD-III

The second data set was obtained from Cordeiro, Ortega, and Popovic (2014). It is also utilized in Nadarajah, Cardeiro, and Ortega (2013), as well as Sankudey, Enayetur Raheem, and Saikat Mukherjee (2017).

TABLE 3: A brief statistical analysis of carbon fibers' breaking strength data

N	Min	Q_1	Median	Mean	Q_3	Max
100	0.390	1.847	2.700	2.622	3.223	5.560

The variance-covariance matrix of the MLEs under CTRD-II and CTRD-III are computed as

$$I(\hat{\theta})^{-1} = \begin{pmatrix} 0.006298 & 0.003071 \\ 0.003071 & 0.019920 \end{pmatrix}$$

$$I(\hat{\theta})^{-1} = \begin{pmatrix} 0.01128 & -0.00309 & 0.04831 \\ -0.00309 & 0.02385 & -0.05099 \\ 0.04831 & -0.05099 & 0.34767 \end{pmatrix}$$

The variances of the MLE of the parameters of CTRD-II of σ and λ are $\text{var}(\hat{\sigma}) = 0.006298$ and $\text{var}(\hat{\lambda}) = 0.019920$. And 95% confidence intervals of σ and λ are $[1.708, 2.019]$ and $[-1.215, -0.662]$ respectively.

The variances of the MLE of the parameters of CTRD-III σ, λ_1, and λ_2 are $\text{var}(\hat{\sigma}) = 0.01128$, $\text{var}(\hat{\lambda}_1) = 0.02385$ and $\text{var}(\hat{\lambda}_2) = 0.34767$. And 95% confidence intervals of σ, λ_1, and λ_2 are $[1.3992, 1.8154]$, $[-1.1901, -0.5848]$ and $[-1.4120, 0.8993]$ respectively.

TABLE 4: Estimated parameter values for the Rayleigh, CTRD-II, and CTRD-III distributions

Model	Parameter Estimate	Standard Error	Log-Lik
CTRD-III	$\hat{\sigma} = 1.6073$ $\hat{\lambda}_1 = -0.8874$ $\hat{\lambda}_2 = -0.2564$	0.1062 0.1544 0.5896	-141.3554
CTRD-II	$\hat{\sigma} = 1.8635$ $\hat{\lambda} = -0.9385$	0.0794 0.1411	-142.1786
Rayleigh	$\hat{\sigma} = 1.9867$	0.0993	-149.5185

TABLE 5: Criteria for comparison

Distribution	-2 *l*	AIC	AICC	BIC	CVM (P-value)	AD (P-value)	K-S (P-value)
Rayleigh	299.037	301.037	301.078	303.642	0.6355 (0.0181)	3.5538 (0.0145)	0.3856 (0.000)
CTRD-II	284.3572	288.357	288.481	293.568	0.1255 (0.4743)	0.6297 (0.6199)	0.0858 (0.4527)
CTRD-III	282.711	288.711	288.961	296.526	0.0745 (0.7253)	0.4126 (0.8358)	0.0696 (0.7176)

In general, the better distribution is one whose values of KS,-*2l,* AIC, AICC, and BIC are minimum. It suggests that the newly suggested CTRD-II and CTRD-III distributions provide a better match than the Rayleigh distribution.

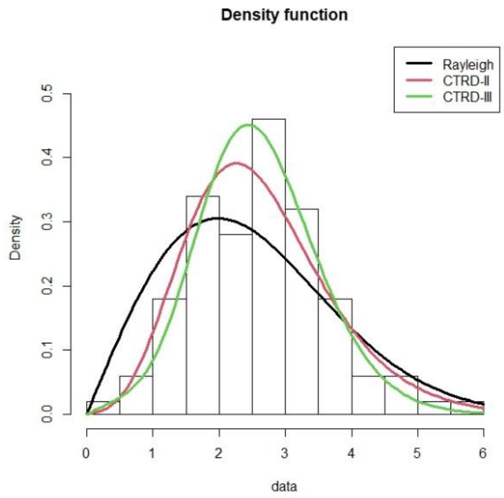


FIGURE 18: Comparison of model fit of these distributions.

7.3 CTRD-IV

In this subsection, we utilize real-time data to demonstrate how the suggested distribution matched the data better than Rayleigh and Transmuted Rayleigh distributions. Faton Merovci is already making use of the data (2013). The data comprises information such as NIC, TAR, CO, Brand Name, LEN, FLTR, Pack, Strength, Style, and Pack kind. However, we only utilize nicotine values taken from multiple brands of cigarettes in 1998. This information was formally gathered by the Federal Trade Commission and is primarily used to safeguard American consumers. It is an autonomous US government agency.

The data set has 345 observations. We examine nicotine data summary statistics. The distribution comparison criteria were computed and tabulated below.

TABLE 6 Criteria for comparison

Distribution	-2 *l*	K-S	AIC	AICC	BIC
Rayleigh	284.714	0.184	286.714	285.714	296.407
Transmuted Rayleigh	242.448	0.124	246.448	243.445	265.833
CTRD-IV	228.424	0.122	232.424	232.460	240.115

It implies that compared to the Rayleigh distribution and Transmuted Rayleigh distribution, the new CTRD-IV provides a better fit.

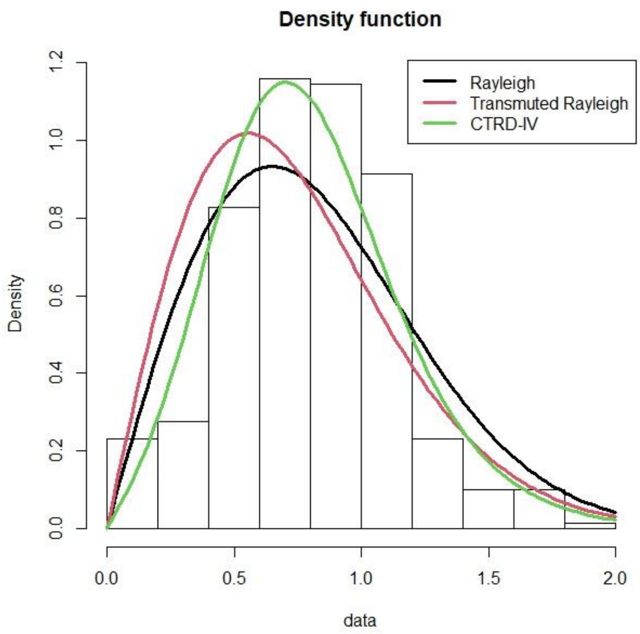


FIGURE 19: Comparison of model fit of these distributions.

8. CONCLUSION

We explained and explored four types of CTRD in this study. We derived expansions of statistical measures such as mean, variance, moments, moment-generating function, and others. Further, the parameters of various forms of CTRD are estimated using the maximum likelihood estimation approach along with the Fisher information matrix is provided. The properties of various types of CTRD are investigated including hazard function and reliability function, etc. The real-time data sets were employed to show the efficiency and appropriateness of the proposed models.

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