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The Riemann Hypothesis and the C-Transformation of the Riemann Zeta Function

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Abstract

Criteria relevant to the Riemann hypothesis are derived using Caceres' C-transformation of the Riemann zeta function. Fourier transforms are used to analyze results.

Keywords: Riemann zeta function, C-transformation, Riemann Hypothesis, Fourier Transform

1. INTRODUCTION

Caceres [1] defined the C-transformation as

$$C_n\{f\} = \sum_{k=1}^{n} f(k) - \int f(n)dn \tag{1}$$

and derived the following function

$$Y(z,n) = n^{(1-\alpha)}\frac{1}{[(1-\alpha)^2+\beta^2]}[((1-\alpha)\cdot\cos(\beta\ln(n))+\beta\cdot\sin(\beta\ln(n)))+ \tag{2}$$

$$i(\beta\cdot\cos(\beta\ln(n))-(1-\alpha)\cdot\sin(\beta\ln(n)))]. \tag{3}$$

Caceres' equation [24] is

$$Re(C_n\{f\}) = \sum_{k=1}^{n} k^{-\alpha}(\cos(\beta\cdot\ln(k))+ \tag{4}$$

$$\frac{1}{[(1-\alpha)^2+\beta^2]}(n^{(1-\alpha)}[(1-\alpha)\cdot\cos(\beta\cdot\ln(n))+\beta\cdot\sin(\beta\cdot\ln(n))]) \tag{5}$$

where $f = \frac{1}{n^z}$. Caceres' equation [25] is

$$Im(C_n\{f\}) = -\sum_{k=1}^{n} k^{-\alpha}(\sin(\beta \cdot \ln(k))+ \tag{6}$$

$$\frac{1}{[(1-\alpha)^2+\beta^2]}(n^{(1-\alpha)}[\beta \cdot \cos(\beta \cdot \ln(n)) - (1-\alpha) \cdot \sin(\beta \cdot \ln(n))]) \tag{7}$$

2. PROPERTIES OF A VARIANT ZETA FUNCTION AND ITS C-TRANSFORMATION

Let $\zeta'(z,n)$ denote $k \cdot n^{-z}$, $k = 1, 2, 3, \ldots, n$. This function's C-transformation (denoted by $C_n\{f\}$) is derived from the above by substituting $\sum_{k=1}^{n} n^{-\alpha}(\cos(\beta \cdot \ln(n))$ for $\sum_{k=1}^{n} k^{-\alpha}(\cos(\beta \cdot \ln(k))$ and by substituting $\sum_{k=1}^{n} n^{-\alpha}(\sin(\beta \cdot \ln(n))$ for $\sum_{k=1}^{n} k^{-\alpha}(\sin(\beta \cdot \ln(k))$ in the above formula.

By definition the real and imaginary components of $\zeta'(z,n)$ are linear, but different lines are generated depending on the last k value. The real and imaginary components of $C_n\{f\} - \zeta'(z,n)$ are also linear. For $z = (0.5, 1.0)$ and $n \leq 10000$, the last value of $C_n\{f\} - \zeta'(z,n)$ is (22.0600145777, 44.1200291554). In general, the ratio between the last values is 2β. A plot of $C_n\{f\} - \zeta'(z,n)$ for the first non-trivial zeta function zero and $n \leq 10000$ is

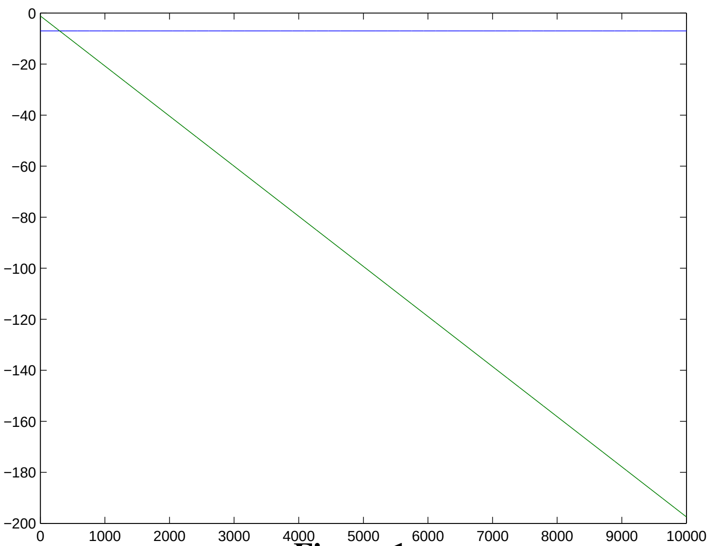

Figure 1

The horizontal line is the real component of $C_n\{f\} - \zeta'(z,n)$. Its value is -6.9854739188.

By definition, $Re(C_n\{f\} - \zeta'(z,n)) = Re(C_n\{\frac{1}{n^z}\} - \zeta(z,n))$. A plot of the imaginary components of the two curves is

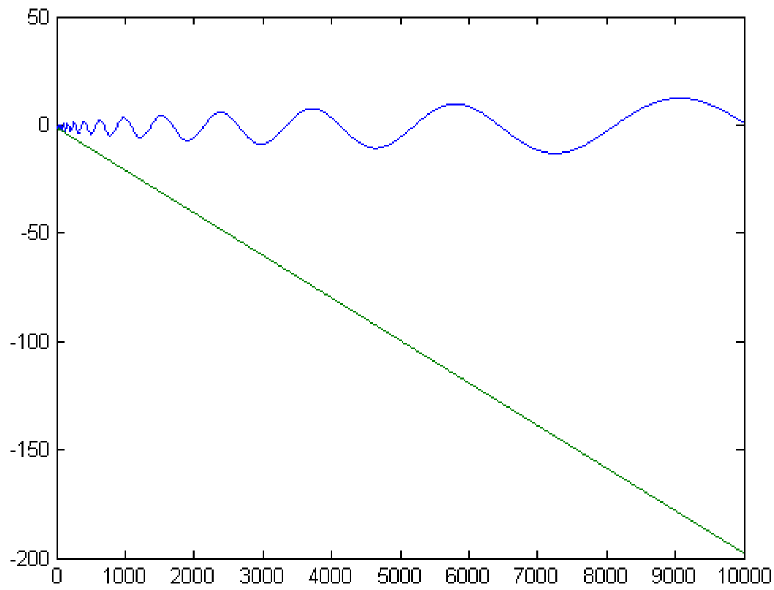

Figure 2

3. PROPERTIES OF $Y(z,n)$

Let m denote the last value in a finite sequence of n values starting with 1, 2, 3,.... A variant of $Y(z,n)$ is

$$Y'(z,n,m) = n^{(1-\alpha)} \frac{1}{[(1-\alpha)^2 + \beta^2]} [((1-\alpha) \cdot \cos(\beta \ln(m)) + \beta \cdot \sin(\beta \ln(m))) + \quad (8)$$

$$i(\beta \cdot \cos(\beta \ln(m)) - (1-\alpha) \cdot \sin(\beta \ln(m)))]. \quad (9)$$

A "polynomial" form of $Y'(z,n,m)$ is

$$|Y'(z,n,m)|^2 \geq n^{2(1-a)}/[(1-\alpha)^2 + \beta^2] \quad (10)$$

A plot of $|Y'(z,n,m)|^2$ and the real and imaginary components of the right-hand side of the inequality for the first zeta function zero and $n \leq 1000$ is

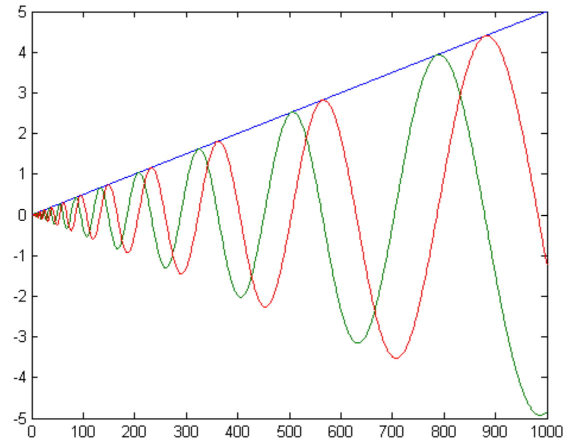

Figure 3

The slope of $|Y'(z,n,m)|^2$ is

$$d(|Y'(z,n.m)|^2)/dn = 2(1-\alpha)n^{1-2\alpha}/[(1-\alpha)^2+\beta^2] \tag{11}$$

A plot of $|Y'(z,n,m)|^2$ and the real and imaginary components of the right-hand side of the inequality for $z=(0.5,34.0)$ is

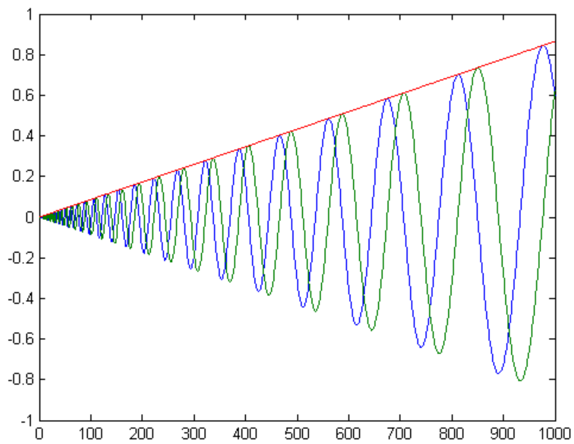

Figure 4

The slopes of the two above lines are 0.004999 and 0.0008649 (relative to n). The ratio of these two slopes is about 5.78. The ratio of $(1-0.5)^2+34.0^2$ and $(1-0.5)^2+14.13472514173470^2$ is about 5.78.

A plot of the real components of $Y'(z,n,m)$ versus $\sqrt{n}$ for the first zeta function zero and $n \leq 10000$ is

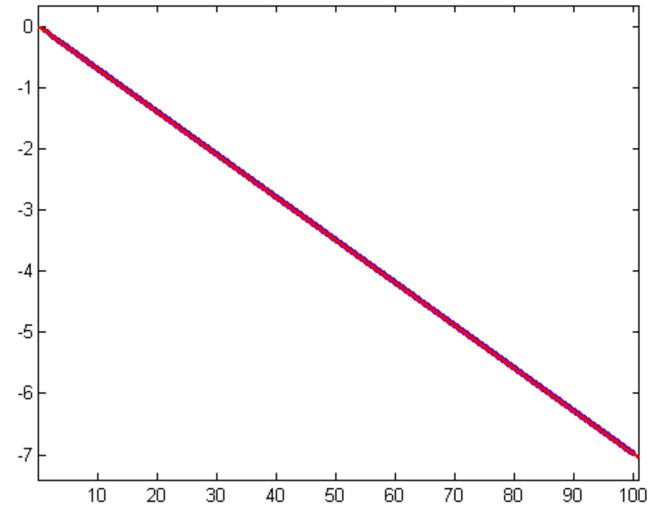

Figure 5

For a linear least-squares fit of the curve, $p_1 = -0.06985$ with a 95% confidence interval of (-0.06985, -0.06985), $p_2 = 4.734 \cdot 10^{-13}$ with a 95% confidence interval of ($-1.22 \cdot 10^{-12}$, $2.167 \cdot 10^{-12}$), SSE=$8.291 \cdot 10^{-18}$, R-squared=1, and RMSE=$2.88 \cdot 10^{-11}$. Taking into account the limitations of 64-bit floating point arithmetic, the second parameter is effectively zero.

A plot of the imaginary components of $Y'(z, n.m)$ versus $\sqrt{n}$ for the first zeta function zero and $n \leq 10000$ is

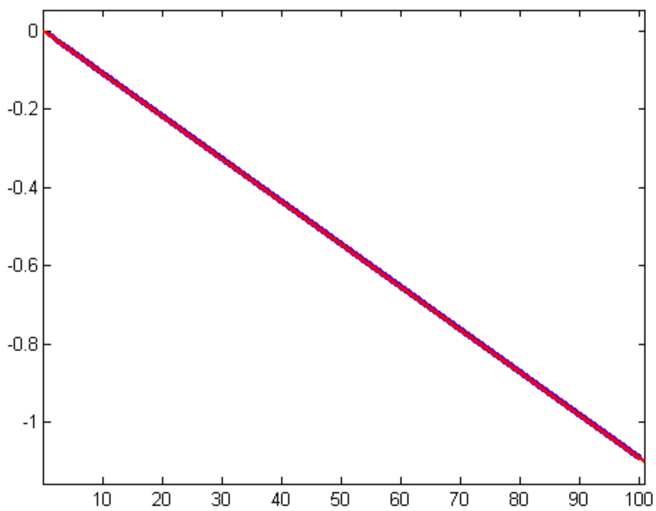

Figure 6

For a linear least-squares fit of the curve, $p_1 = -0.01092$ with a 95% confidence interval of (-0.01092, -0.01092), $p_2 = 7.064 \cdot 10^{-13}$ with a 95% confidence interval of $(-9.949 \cdot 10^{-13}$, $2.408 \cdot 10^{-12})$, SSE=$8.356 \cdot 10^{-18}$, R-squared=1, and RMSE=$2.893 \cdot 10^{-11}$.

The curves are linear only if z has a real component of 1/2.

For β=1.0, 2.0, 3.0,..., 40 the slopes of the real components are -0.02206, -0.00887, 0.1513, -0.1661, 0.1664, $-0.1.55$, 0.1411, -0.1243, 0.1059, -0.008649, 0.006691, -0.00477, 0.002939, -0.01244, -0.002774, 0.01595, -0.02686, 0.03537, -0.04142, 0.04504, -0.04632, 0.04544, -0.04262, 0.03815, -0.03235, 0.02556, -0.01814, 0.01044, -0.002815, -0.004429, 0.01101, -0.01668, 0.02127, -0.02463, 0.02671, -0.02747, 0.02696, -0.02528, 0.02255, and -0.01894. The slopes of the imaginary components are -0.8668, 0.4769, -0.2919, 0.1843, -0.1121, 0.05964, -0.02003, -0.01025, 0.03317, -0.04994, 0.061410000, -0.06825, 0.07103, -0.07029, 0.06657, -0.0604, 0.0523, -0.04281, 0.03244, -0.02168, 0.01099, -0.0007945, -0.008548, 0.01673, -0.02351, 0.02873, -0.03228, 0.03415, -0.03436, 0.03303, -0.03032, 0.02642, -0.02158, 0.01606, -0.01014, 0.004099, 0.001801, -0.00731, 0.01221, and -0.01631. (These slopes change for every upper bound of n values.) A plot of the reciprocal of the sum of the squares of the slopes of the real and imaginary components versus $\sqrt{i}$, $i = 1, 2, 3, \ldots, 40$ is

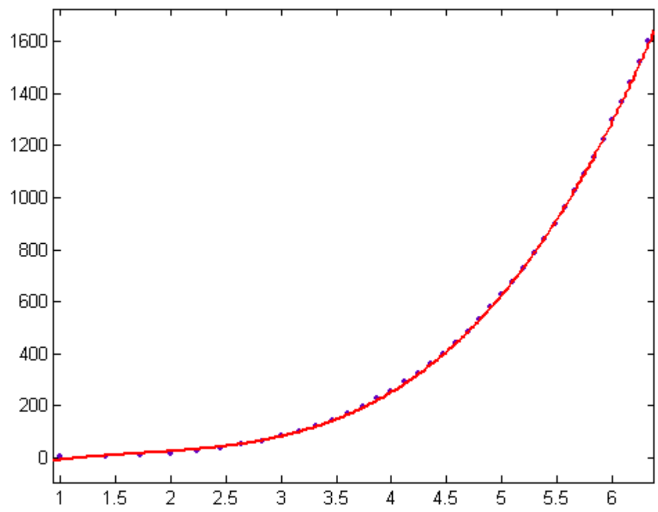

Figure 7

For a cubic least-squares fit of the curve, $p_1 = 14.97$ with a 95% confidence interval of (14.5, 15.44), $p_2 = -77.65$ with a 95% confidence interval of $(-83.03, -72.27)$, $p_3 = 161.1$ with a 95% confidence interval of (142.2, 180.1), $p_4 = -108.3$ with a 95% confidence interval of $(-128.5, -88.1)$, SSE=675.8, R-squared=0.9999, and RMSE=4.333.

A plot of the phase angles of the Fourier transform of the above slopes (where the slopes of the real components and the slopes of the imaginary components are taken to comprise complex numbers) is

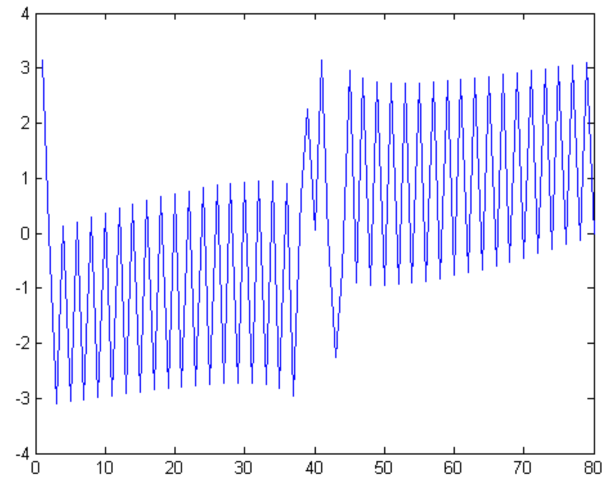

Figure 8

A plot of the complex magnitudes of the Fourier transform is

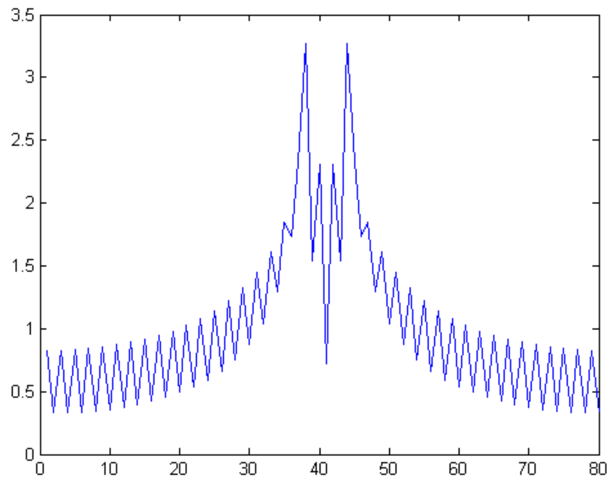

Figure 9

From this perspective, the slopes are not random. The original signal can be reconstructed from the phase angles and complex magnitudes.

4. PROPERTIES OF THE SUM OF $\zeta'(z,n)$ AND $Y'(z,n,m)$

A plot of the real components of $\zeta'(z,n) + Y'(z,n.m)$ for the first zeta function zero versus $\sqrt{n}$ for $n \leq 100000$ is

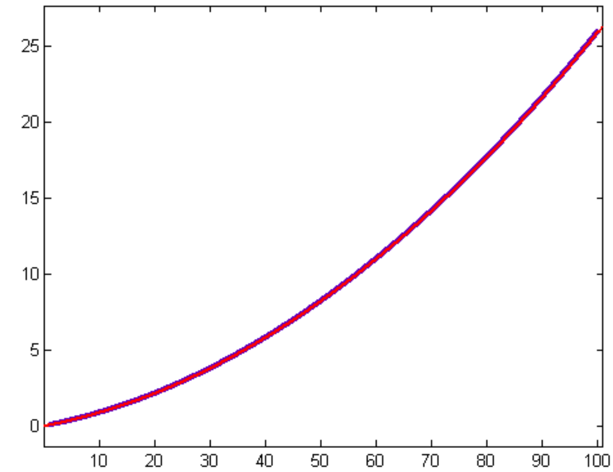

Figure 10

For a quadratic least-squares fit of the curve, $p_1 = -0.001893$ with a 95% confidence interval of (-0.001893, -0.001893), $p_2 = -0.06985$ with a 95% confidence interval of (-0.06985, -0.06985), $p_3 = 3.435 \cdot 10^{-12}$ with a 95% confidence interval of $(-1.378 \cdot^{-12}$, $8.248 \cdot 10^{-12})$, SSE=$1.671 \cdot 10^{-17}$, R-squared=1, and RMSE=$4.088 \cdot 10^{-11}$. Taking into account the limitations of 64-bit floating point arithmetic, the third parameter is effectively zero.

A plot of the imaginary components of $\zeta'(z,n) + Y'(z,n,m)$ for the first zeta function zero versus $\sqrt{n}$ for $n \leq 100000$ is

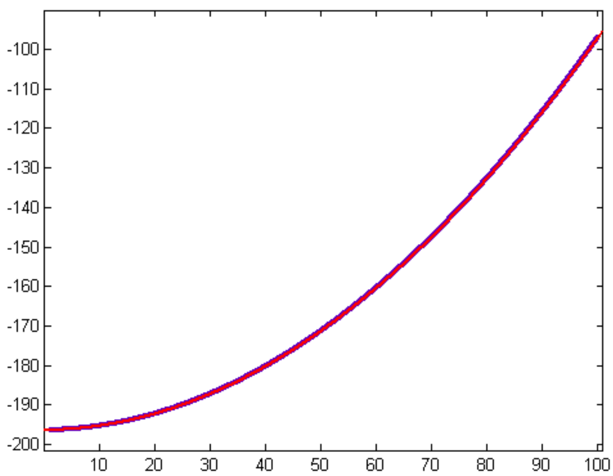

Figure 11

For a quadratic least-squares fit of the curve, $p_1 = 0.009819$ with a 95% confidence interval of (0.009819, 0.009819), $p_2 = -0.01092$ with a 95% confidence interval of (-0.01092, -0.01092), $p_3 = 7.201 \cdot 10^{-12}$ with a 95% confidence interval of ($2.42 \cdot 10^{-12}$, $1.198 \cdot 10^{-11}$), SSE=$1.649 \cdot 10^{-17}$, R-squared=1, and RMSE=$4.062 \cdot 10^{-11}$.

The fits are exactly quadratic (and the p_3 parameters effectively 0) only when the real component of z is 1/2.

Denote the p_1 and p_2 parameters of the real components by a_i and b_i and the p_1 and p_2 parameters of the imaginary components by c_i and d_i. A plot of $1/(a_i^2 + b_i^2 + c_i^2 + d_i^2)$ for the quadratic least-squares fits of the above curves for $\alpha = 0.5$ and $\beta = 1.0, 2.0, 3.0, ..., 20$ and i, $i = 1, 2, 3, ..., 20$ is

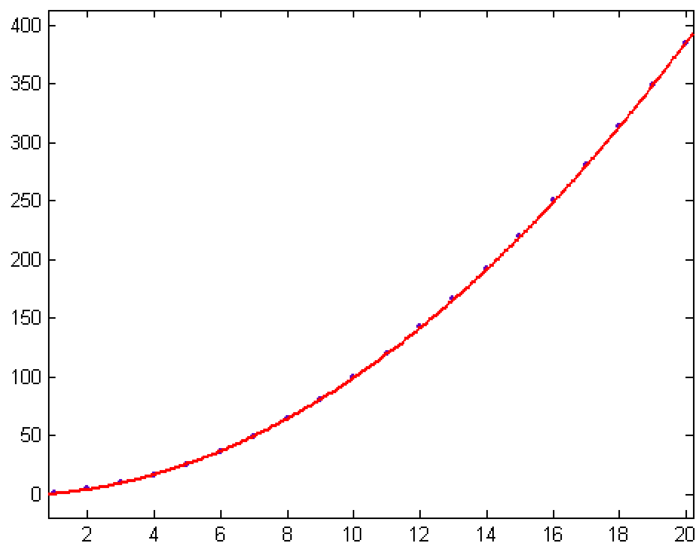

Figure 12

For a quadratic least-squares fit of the curve, $p_1 = 0.9286$ with a 95% confidence interval of (0.9185, 0.9388), $p_2 = 0.8084$ with a 95% confidence interval of (0.5886, 1.028), $p_3 = -1.509$ with a 95% confidence interval of (-2.511, -0.5068), SSE=6.927, R-squared=1, and RMSE=0.6383.

A plot of the phase angles of the Fourier transform of the concatenated a_i, b_i, c_i, and d_i values is

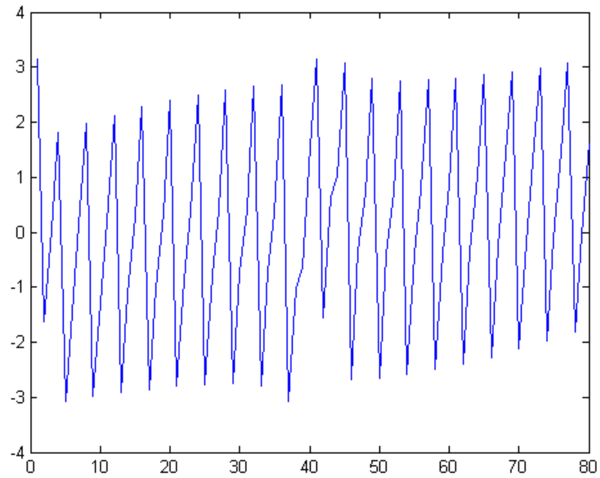

Figure 13

A plot of the complex magnitudes of the Fourier transform is

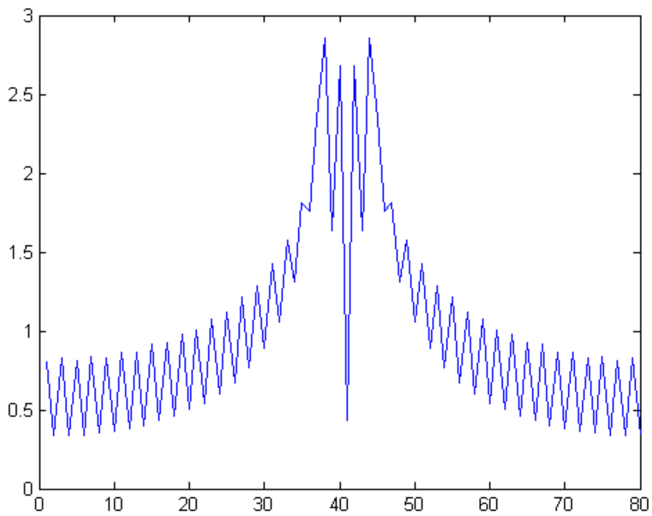

Figure 14

The complex magnitudes are not quite as large as those for $Y'(z, n, m)$ but are otherwise similar.

5. PROPERTIES OF $(C_n\{f\} - \zeta'(z, n)) + Y(z, n)$

Let m denote the upper bound of n values as above. A plot of the real components of

$(C_n\{f\} - \zeta'(z,n)) + Y(z,m)$ versus $\sqrt{n}$ for $z = (0.5, 1.0)$ and $n \leq 10000$ is

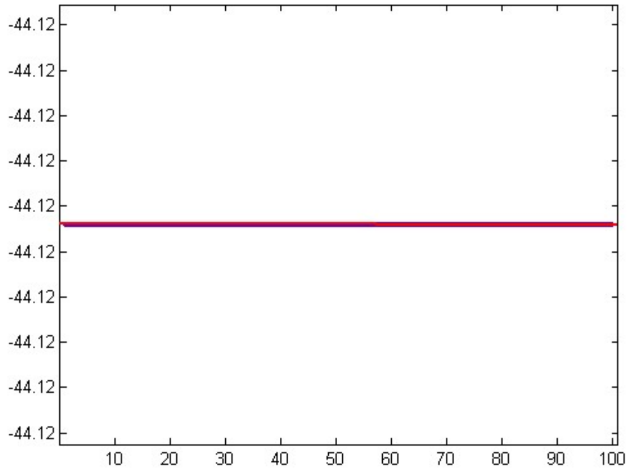

Figure 15

For a linear least-squares fit of the curve, $p_1 = -1.672 \cdot 10^{-15}$ with a 95% confidence interval of $(-1.706 \cdot 10^{-15}, -1.638 \cdot 10^{-15})$, $p_2 = -44.12$ with a 95% confidence interval of $(-44.12, -44.12)$, SSE=$1.695 \cdot 10^{-23}$, R-squared=0.9996, and RMSE=$4.118 \cdot 10^{-14}$. The first parameter is effectively zero.

A plot of the imaginary components is

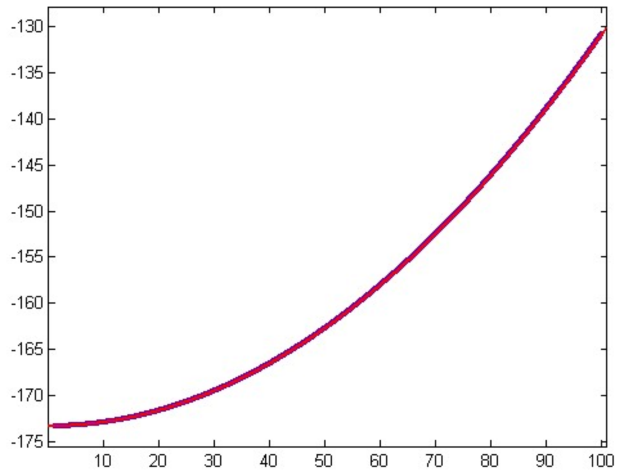

Figure 16

For a quadratic least-squares fit of the curve, $p_1 = 0.004256$ with a 95% confidence interval of (0.004256, 0.004256), $p_2 = -2.122 \cdot 10^{-14}$ with a 95% confidence interval of $(-1.475 \cdot 10^{-13}, 9.905 \cdot 10^{-14})$, $p_3 = -173.4$ with a 95% confidence interval of $(-173.4, -173.4)$, SSE=$8.351 \cdot 10^{-18}$, R-squared=1, and RMSE=$2.89 \cdot 10^{-11}$. The second parameter is effectively zero.

Let a_i, b_i, and c_i denote the non-zero parameters. A plot of $1.0/(a_i^2 + b_i^2 + c_i^2)$ for z with a real component of 1/2 and imaginary components of 1.0, 2.0, 3.0,..., 20 is

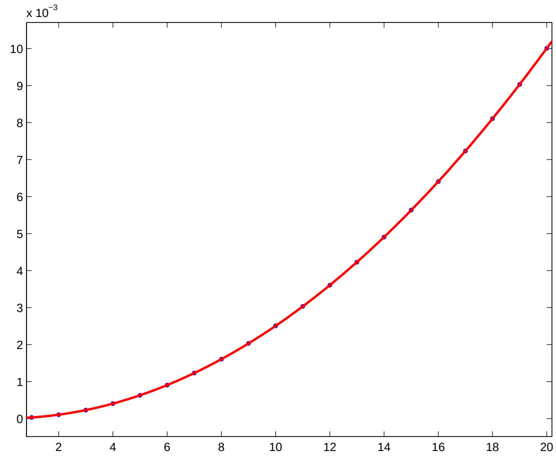

Figure 17

For a quadratic least-squares fit of the curve, $p_1 = 2.5 \cdot 10^{-5}$ with a 95% confidence interval of $(2.498 \cdot 10^{-5}, 2.502 \cdot 10^{-5})$, $p_2 = -3.262 \cdot 10^{-8}$ with a 95% confidence interval of $(-4.459 \cdot 10^{-7}, -3.807 \cdot 10^{-7})$, $p_3 = 6.58 \cdot 10^{-6}$ with a 95% confidence interval of $(4.374 \cdot 10^{-6}, 8.143 \cdot 10^{-6})$, SSE=$2.45 \cdot 10^{-11}$, R-squared=1, and RMSE=$1.201 \cdot 10^{-6}$.

A plot of the phase angles of the Fourier transform of the concatenated a_i, b_i, and c_i values is

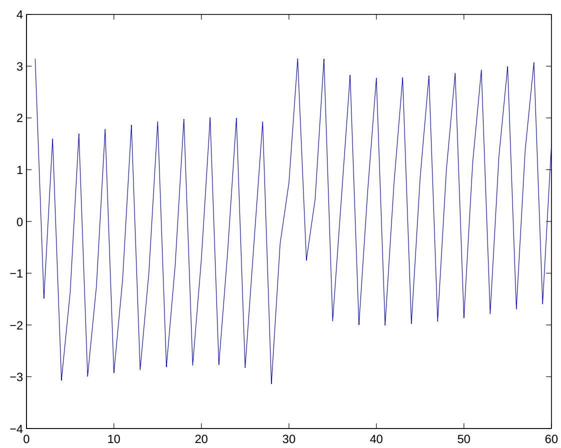

Figure 18

A plot of the complex magnitudes of the Fourier transform is

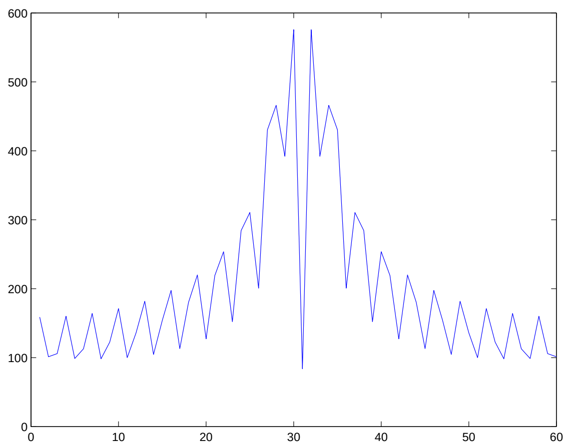

Figure 19

Let d_i, e_i, and f_i denote the corresponding parameters for the first twenty zeta function zeros. A plot of $1.0/(a_i^2 + b_i^2 + c_i^2 + d_i^2 + e_i^2 + f_i^2)$ for $i = 1$, 2, 3, ..., 20 is

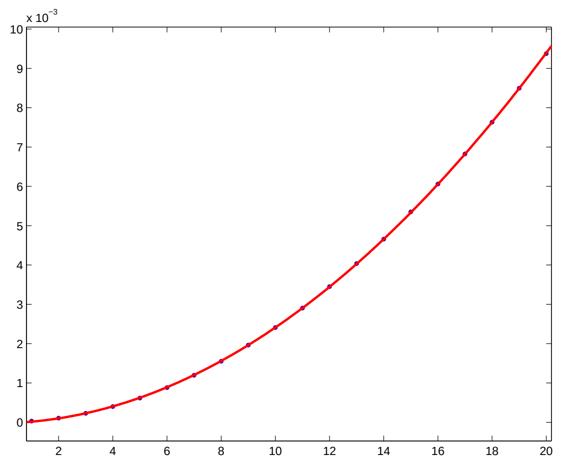

Figure 20

For a quadratic least-squares fit of the curve, $p_1 = 2.274 \cdot 10^{-5}$ with a 95% confidence interval of $(2.26 \cdot 10^{-5}, 2.289 \cdot 10^{-5})$, $p_2 = 1.593 \cdot 10^{-5}$ with a 95% confidence interval of $(1.286 \cdot 10^{-5}, 1.9 \cdot 10^{-5})$, $p_3 = -2.257 \cdot 10^{-5}$ with a 95% confidence interval of $(-3.657 \cdot 10^{-5}, -8.577 \cdot 10^{-6})$, SSE=$1.352 \cdot 10^{-9}$, R-squared=1, and RMSE=$8.918 \cdot 10^{-6}$.

6. SQUARES OF ABSOLUTE VALUES

A plot of $|\zeta(z,n)|^2 - |Y'(z,n,m)|^2 - |C_n\{f\} - \zeta'(z,n)|^2$ versus n for the first zero function zero and $n \leq 10000$ is

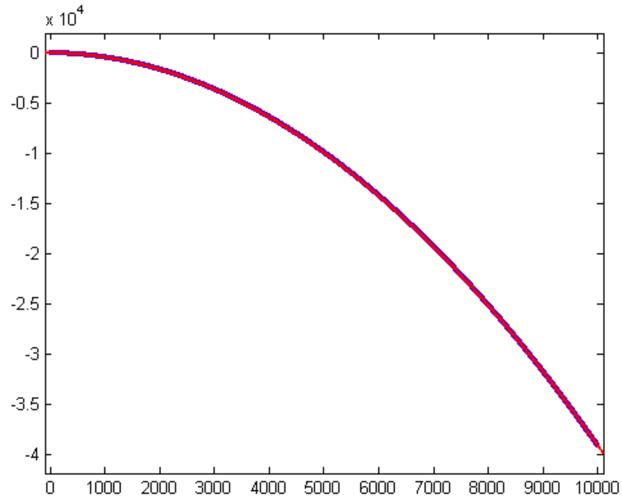

Figure 21

For a quadratic least-squares fit of the curve, $p_1 = -0.0003857$ with a 95% confidence interval of $(-0.0003857, -0.0003857)$, $p_2 = -0.0429$ with a 95% confidence interval of $(-0.0429, -0.0429)$, $p_3 = -49.99$ with a 95% confidence interval of $(-49.99, -49.99)$, SSE=1.018, R-squared=1, and RMSE=0.01099. This is not a perfect fit.

For the first twenty zeta function zeros, the p_1 parameters are -0.0003857, -0.0003358, -0.0002912, -0.0001357, -0.0003873, -0.0001299, -0.000005339, -0.000002235, -0.0002141, -0.00002234, -0.0002562, -0.0003992, -0.0000003754, -0.0003104, -0.00004336, -0.0003064, -0.0000442, -0.0002404, -0.00001113, and -0.0001023. The p_2 parameters are -0.0429, 0.0736, -0.06881, -0.0615, 0.02311, -0.04937, 0.01123, -0.006878, 0.04202, 0.0185, -0.03578, -0.002521, 0.002062, -0.02699, 0.01914, 0.02558, 0.01807, -0.02695, 0.008696, and -0.02254. The p_3 parameters are -49.99, -22.61, -15.98, -10.8, -9.212, -7.071, -5.967, -5.319, -4.332, -4.031, -3.554, -3.129, -2.833, -2.693, -2.347, -2.215, -2.058, -1.91, -1.735, and -1.673.

Let a_i, b_i, and c_i denote the parameters. The c_i values for the first forty zeta function zeros are -49.99, -22.61, -15.98, -10.8, -9.212, -7.071, -5.967, -5.319, -4.332, -4.031, -3.554, -3.129, -2.833, -2.693, -2.347, -2.215, -2.058, -1.91, -1.735, -1.673, -1.382, -1.299, -1.253, -1.149, -1.108, -1.077, -1.002, -0.9558, -0.9178, -0.8885, -0.8466, -0.7937, -0.7869, -0.7508, -0.7199,

-0.681, -0.6637, -0.6505, -0.6272, and -0.5822. A plot of these values versus $\log(i)$, $i = 1$, 2, 3, ..., 40 is

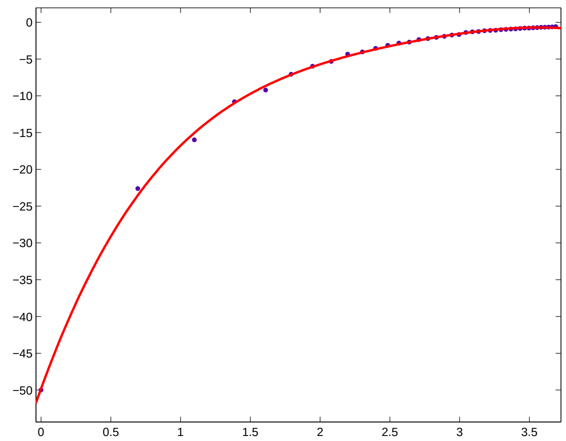

Figure 22

For a quartic least-squares fit of the curve, $p_1 = -0.5172$ with a 95% confidence interval of $(-0.6167, -0.4156)$, $p_2 = 5.617$ with a 95% confidence interval of (4.825, 6.41), $p_3 = -24.24$ with a 95% confidence interval of $(-26.29, -22.19)$, $p_4 = 52.2$ with a 95% confidence interval of (50.31, 54.04), $p_5 = -49.82$ with a 95% confidence interval of $(-50.35, -49.29)$, SSE=2.432, R-squared=0.9992, and RMSE=0.2636.

A plot of the Fourier transform of the concatenated a_i, b_i, and c_i values for the first forty zeta function zeros is

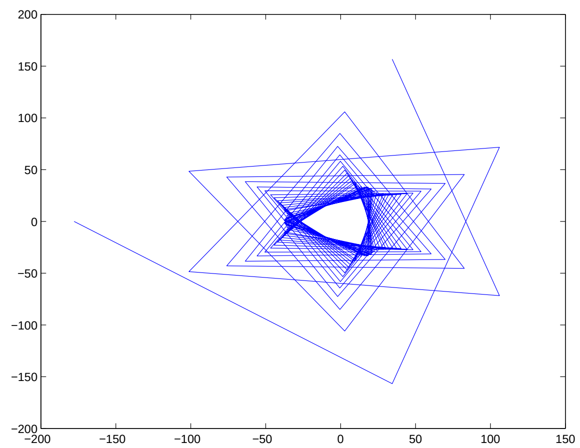

Figure 23

Like the logarithmic spirals of the zeta function zeros, the Fourier transform is centered on (0, 0). A plot of the phase angles of the Fourier transform is

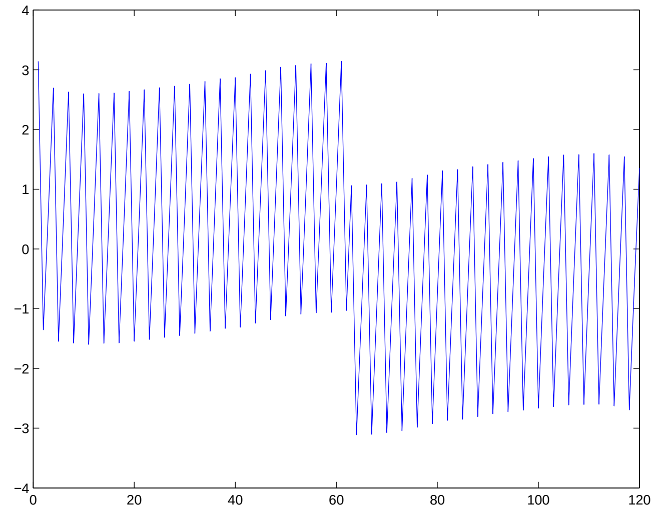

Figure 24

A plot of the complex magnitudes of the Fourier transform is

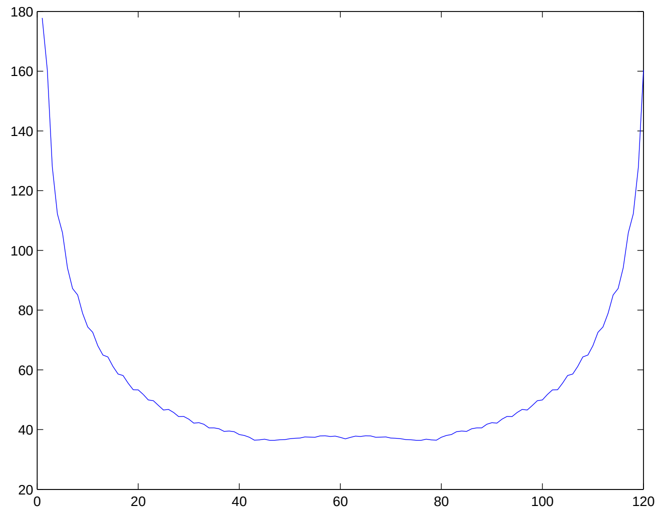

Figure 25

A plot of $|\zeta(z,n)|^2 - |Y'(z,n,m)|^2 - |C_n\{f\} - \zeta'(z,n)|^2$ for $z = (0.5, 10.0)$ versus n is

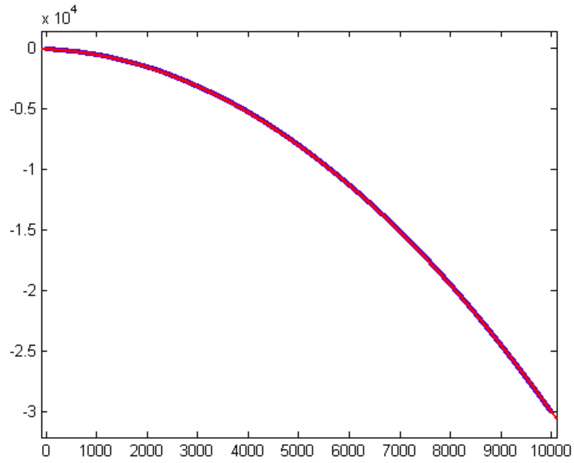

Figure 26

For a quadratic least-squares fit of the curve, $p_1 = -0.0002823$ with a 95% confidence interval of $(-0.0002824, -0.0002823)$, $p_2 = -0.1662$ with a 95%confidence interval of $(-0.1666, -0.1658)$, $p_3 = -100.1$ with a 95% confidence interval of $(-101, -99.26)$, SSE=$2.142 \cdot 10^6$, R-squared=1, and RMSE=14.64. This is a poor fit compared to that of the zeta function zeros.

7. SIMILAR PROPERTIES OF THE USUAL RIEMANN ZETA FUNCTION

A plot of the real components of $\zeta(z,n) - Y'(z,n,m) - C_n\{\frac{1}{n^z}\}$ versus $\sqrt{n}$ for the first zeta function zero and $n \le 10000$ is

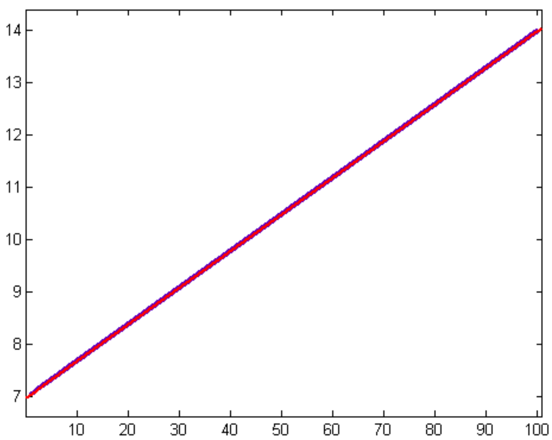

Figure 27

For a linear least-squares fit of the curve, $p_1 = 0.06985$ with a 95% confidence interval of (0.06985, 0.06985), $p_2 = 6.985$ with a 95% confidence interval of (6.985, 6.985), SSE=$1.049 \cdot 10^{-17}$, R-squared=1, and RMSE=$3.239 \cdot 10^{-11}$ (effectively a perfect fit). The p_2 parameter is the negative of the p_1 parameter of the linear least-squares fit of the real components of $C_n\{\frac{1}{n^z}\} - \zeta(z, n)$ versus n. The p_1 parameter is $p_2/\sqrt{n}$.

A plot of the imaginary components is

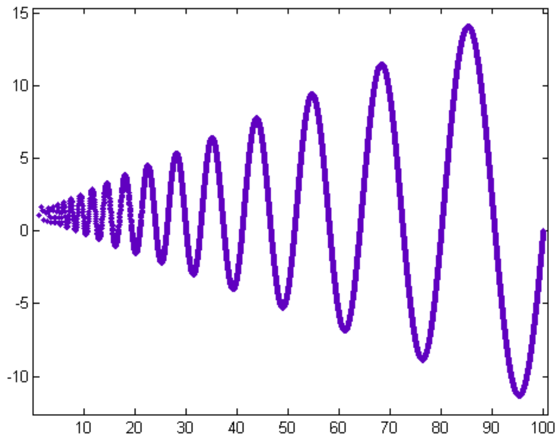

Figure 28

A plot of the imaginary components of $\zeta(z, n)$ versus $\sqrt{n}$ for the first zeta function zero and $n \leq 10000$ is

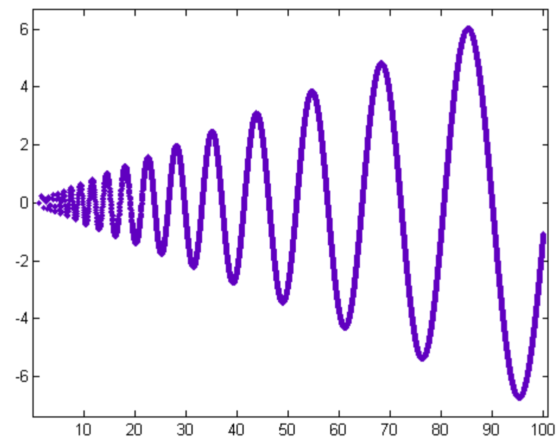

Figure 29

This is similar to the above plot except for scaling of the amplitude and translation along the y-axis.

A plot of the real components of $\zeta(z,n)$ and the imaginary components of $\zeta(z,n) - Y'(z,n,m) - C_n\{\frac{1}{n^z}\}$ for $n \leq 50000$ is

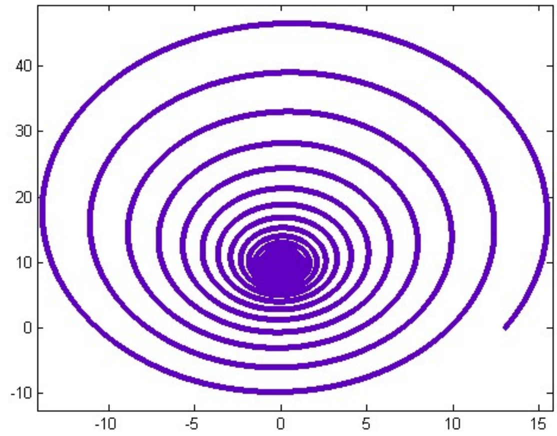

Figure 30

This is a 3-dimensional logarithmic spiral. Such spirals end with a point that has an imaginary component of approximately 0.0. For this m value, the logarithmic spiral is approximately centered on (0.0, 9.0). The imaginary components are decremented by 9.0 so that the inflection points can be computed. The inflection points (where the curve crosses the x-axis from above) are at $n = 4$, 7, 10, 16, 25, 40, 62, 97, 151, 236, 367, 573, 894, 1395, 2176, 3395, 5295, 8260, 12884, 20096, 31346, and 48893. A plot of the logarithms of the n values of the inflection points is

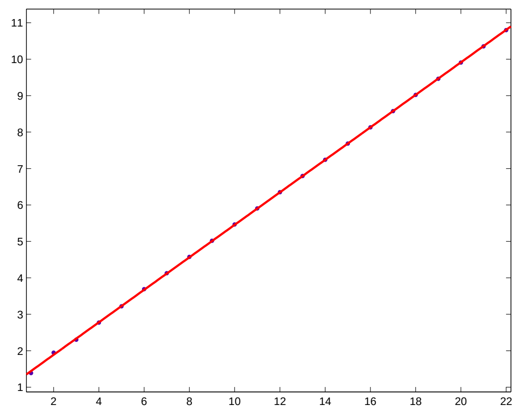

Figure 31

For a linear least-squares fit of the curve, $p_1 = 0.4458$ with a 95% confidence interval of (0.4444, 0.4473), $p_2 = 0.9969$ with a 95% confidence interval of (0.978, 1.016), SSE=0.008437, R-squared=1.0, and RMSE=0.02054.

For zeta function zeros, the logarithmic spirals are already centered on (0, 0). A plot of $\zeta(z, n)$ for the first zeta function zero and $n \leq 50000$ is

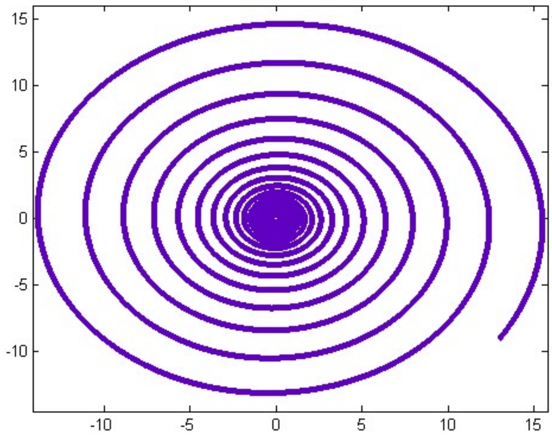

Figure 32

The inflection points (where the curve approaches the x-axis from above and then increases) are at $n = 4$, 6, 9, 14, 22, 35, 54, 85, 133, 207, 323, 503, 785, 1224, 1909, 2978, 4644, 7244, 11299, 17623, 27487, and 42873. A plot of the above n values of inflection points versus these n values of inflection points is

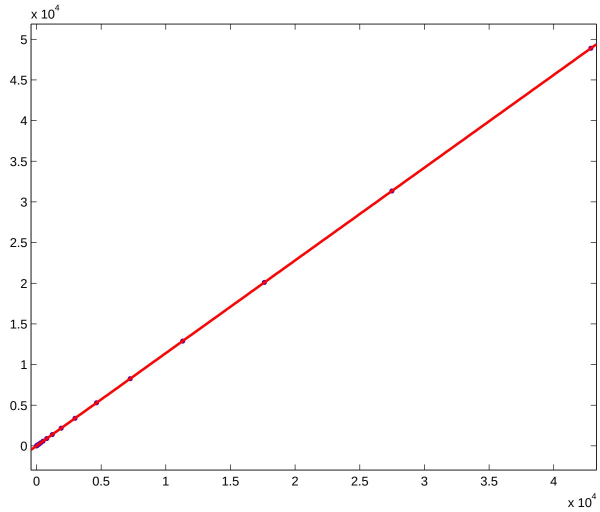

Figure 33

For a linear least-squares fit of the curve, $p_1 = 1.14$ with a 95% confidence interval of (1.14, 1.14), $p_2 = -0.5703$ with a 95% confidence interval of $(-0.8815, -0.2592)$, SSE=7.755, R-squared=1.0, and RMSE=0.6227.

A plot of the real components of $\zeta(z,n)$ and the imaginary components of $\zeta(z,n) - Y'(z,n,m) - C_n\{\frac{1}{n^z}\}$ for the tenth zeta function zero and $n \leq 50000$ is

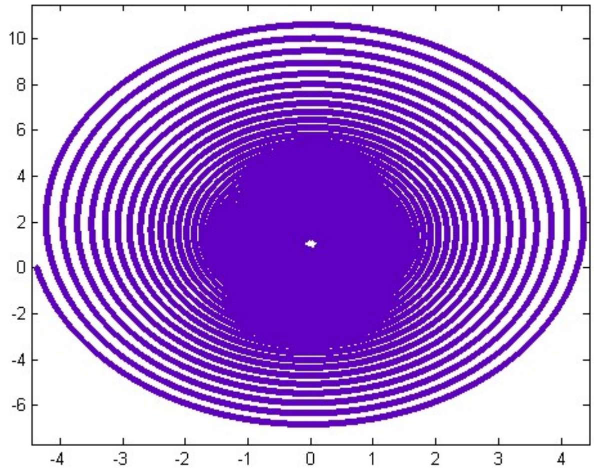

Figure 34

The curve is approximately centered on (0, 0.95). The inflection points are determined as above. A plot of the logarithms of the n values of the inflection points is

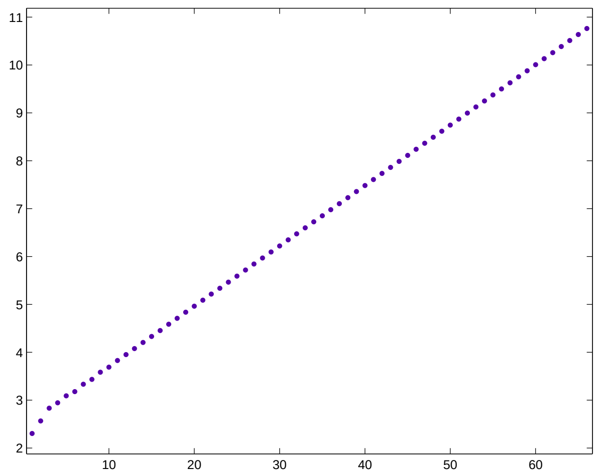

Figure 35

For a linear least-squares fit of the curve (disregarding the first two points), $p_1 = 0.1261$ with a 95% confidence interval of (0.126, 0.1261), $p_2 = 2.44$ with a 95% confidence interval of (2.437, 2.442), SSE=0.001605, R-squared=1, and RMSE=0.005088.

A plot of the n values of these inflection points versus the n values of the inflection points of $\zeta(z, n)$ for the tenth zeta function zero and $n \leq 50000$ is

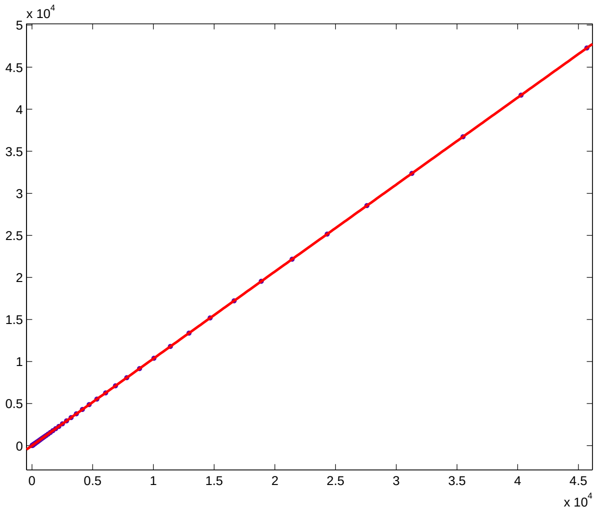

Figure 36

For a linear least-squares fit of the curve, $p_1 = 1.035$ with a 95% confidence interval of (1.035, 1.035), $p_2 = 0.8611$ with a 95% confidence interval of (0.6338, 1.088), SSE=41.53, R-squared=1, and RMSE=0.8056.
A plot of the real components of $\zeta(z, n) - Y'(z, n, m) - C_n\{\frac{1}{n^z}\}$ versus $\sqrt{n}$ for $z = (0.40, 14.13472514173470)$ and $n \leq 10000$ is

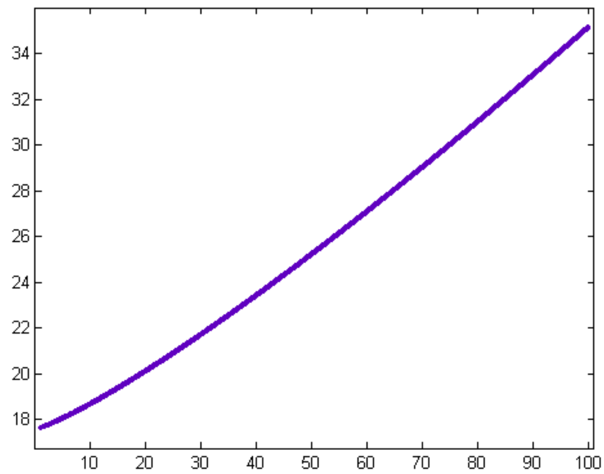

Figure 37

The curve is linear only when z has a real part of 1/2.

A plot of the imaginary components is

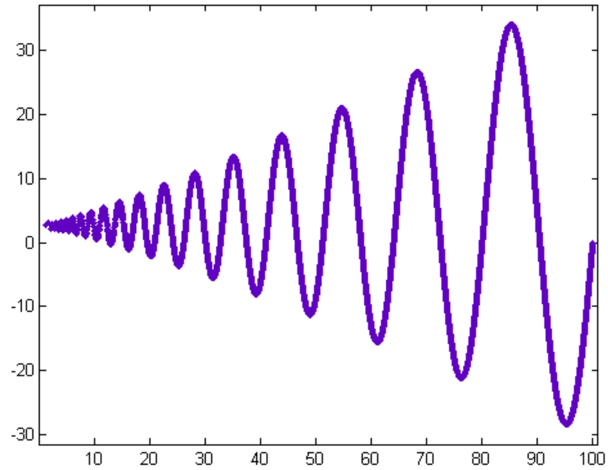

Figure 38

The amplitude of the peaks does not increase linearly as above.

8. PROPERTIES OF THE SUM OF $\zeta(z,n)$ AND $Y'(z,n,m)$

A plot of $\zeta(z,n) + Y'(z,n,m)$ for the first zeta function zero $(z = (0.50, 14.13472514173470))$ and $n \leq 10000$ is

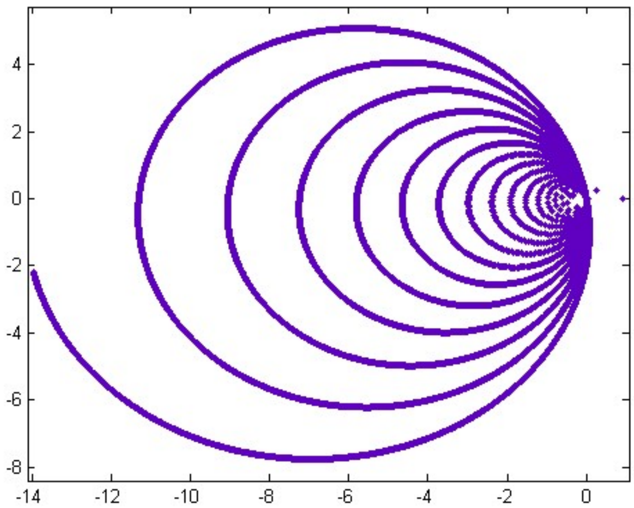

Figure 39

The logarithmic spiral is centered on (0.0). A plot of the logarithms of the n values of the inflection points (where the curve crosses the x-axis from above) is

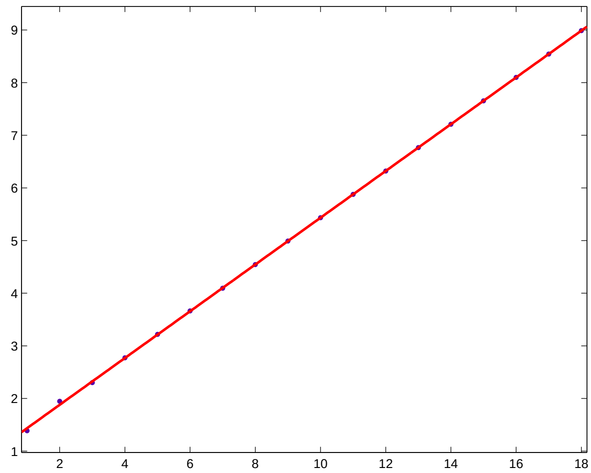

Figure 40

For a linear least-squares fit of the curve (disregarding the first four points), $p_1 = 0.4448$ with a 95% confidence interval of (0.444, 0.4456), $p_2 = 1.427$ with a 95% confidence interval of (1.418, 1.436), SSE=0.0003557, R-squared=1. and RMSE=0.005444.
A plot of $\zeta(z, n) + Y'(z, n, m)$ for $z = (0.5, 100)$ and $n \leq 10000$ is

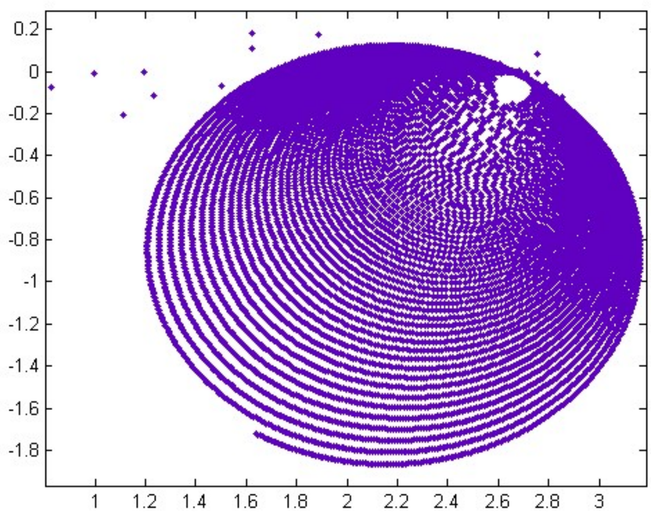

Figure 41

The logarithmic spiral is no longer centered on (0, 0).

A plot of $\zeta(z,n) + Y'(z,n,m)$ for $z = (0.10, 14.13472514173470)$ and $n \leq 10000$ is

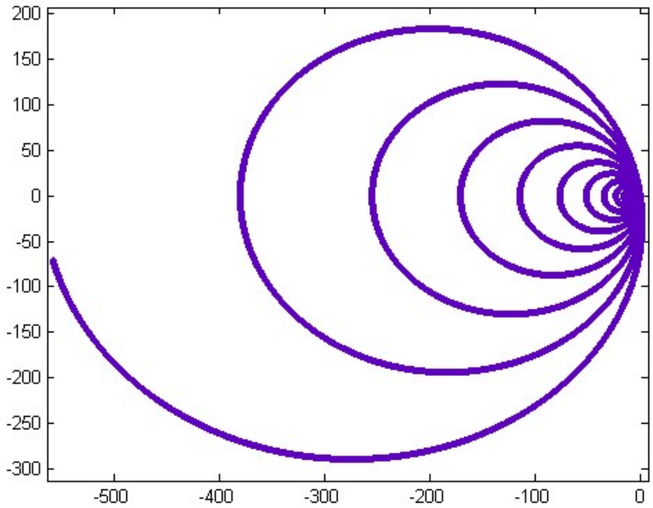

Figure 42

The logarithmic spiral is centered on (0, 0). A plot of the logarithms of the n values of the inflection points is

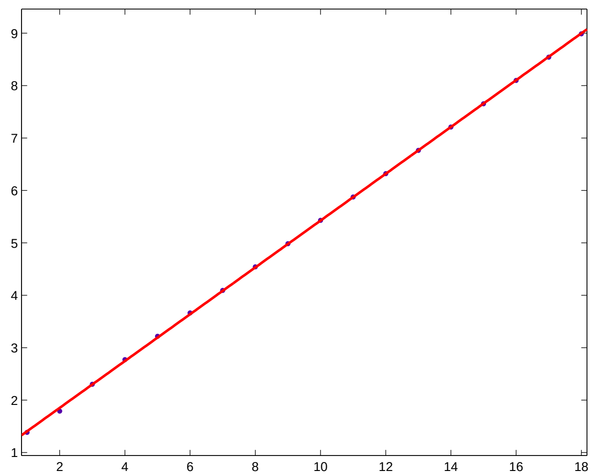

Figure 43

For a linear least-squares fit of the curve, $p_1 = 0.4466$ with a 95% confidence interval of (0.4447, 0.4486), $p_2 = 0.9596$ with a 95% confidence interval of (0.9384, 0.9807), SSE=0.006598, R-squared=0.9999, and RMSE=0.02031.

A plot of $\zeta(z,n)+Y'(z,n,m)$ for $z=(0.90,14.13472514173470)$ and $n\leq 10000$ is

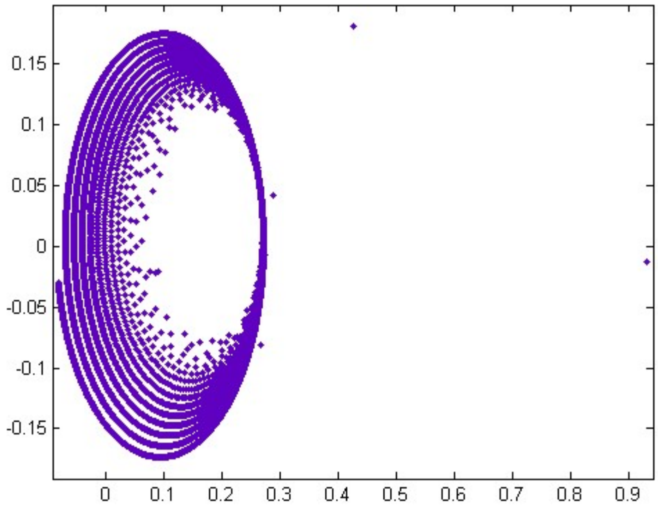

Figure 44

A plot of the logarithms of the n values of the inflection points is

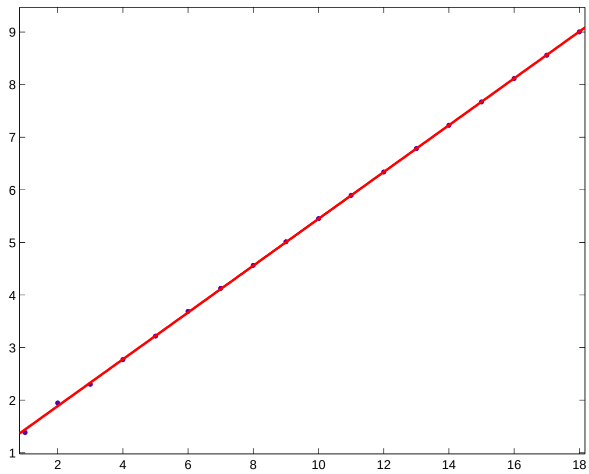

Figure 45

For a linear least-squares fit of the curve, $p_1=0.4451$ with a 95% confidence interval of (0.4429, 0.4473), $p_2=0.9969$ with a 95% confidence interval of (0.9731, 1.021), SSE=0.008357, R-squared=0.9999, and RMSE=0.02285.

The slopes and intercepts are approximately equal even though the real part of z is not 1/2.

9. A VARIANT OF CACERES' $X(z,n)$ FUNCTION

Caceres defined the companion function of $Y(z,n)$ as

$$X(z,n) = \sum_{k=1}^{n} k^{-\alpha}(\cos(\beta \cdot \ln(k)) + \frac{1}{2}n^{-\alpha}\cos(\beta \ln(n)) + \tag{12}$$

$$i(\sum_{k=1}^{n} k^{-\alpha}(\sin(\beta \cdot \ln(k)) + \frac{1}{2}n^{-\alpha}\sin(\beta \ln(n))). \tag{13}$$

A variant definition (where m is the last n value as above) is

$$X'(z,n,m) = \sum_{k=1}^{n} k^{-\alpha}(\cos(\beta \cdot \ln(k)) + \frac{1}{2}m^{-\alpha}\cos(\beta \ln(m)) + \tag{14}$$

$$i(\sum_{k=1}^{n} k^{-\alpha}(\sin(\beta \cdot \ln(k)) + \frac{1}{2}m^{-\alpha}\sin(\beta \ln(m))). \tag{15}$$

The corresponding $Y(z,n)$ definition is

$$Y''(z,n,m) = n^{(1-\alpha)}\frac{1}{[(1-\alpha)^2+\beta^2]}[((1-\alpha)\cdot\cos(\beta\ln(n))+\beta\cdot\sin(\beta\ln(m)))+ \tag{16}$$

$$i(\beta \cdot \cos(\beta \ln(m)) - (1-\alpha)\cdot \sin(\beta \ln(n)))]. \tag{17}$$

A plot of $|Y''(z,n,m)|^2$ for the tenth zeta function zero and $n \leq 10000$ is

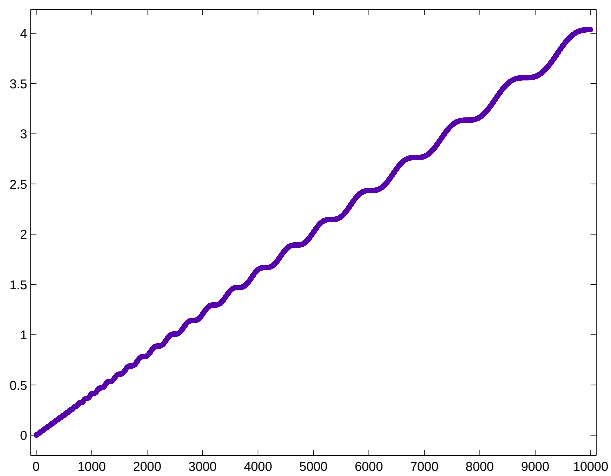

Figure 46

The square of the absolute value is no longer linear.

A plot of $X'(z, n, m) - Y''(z, n, m)$ for the tenth zeta function zero and $n \leq 10000$ is

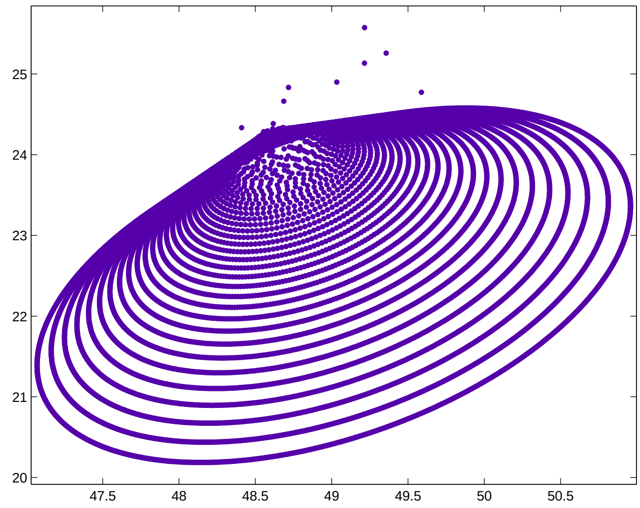

Figure 47

This is a three-dimensional logarithmic spiral. A plot of the logarithms of the n values of the inflection points is

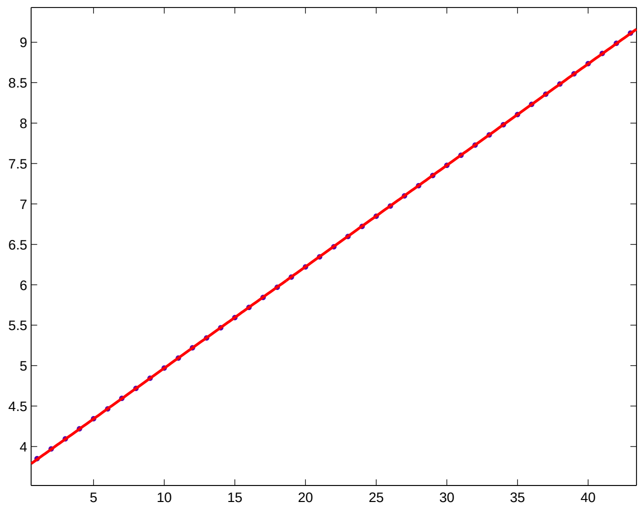

Figure 48

For a linear least-squares fit of the curve, $p_1 = 0.1254$ with a 95% confidence interval of (0.1254, 0.1255), $p_2 = 3.715$ with a 95% confidence interval of (3.713, 3.717), SSE=0.0003602, R-squared=1, and RMSE=0.002964.

There are 43 points in the curve.
A plot of $\zeta(z, n)$ for the tenth zeta function zero and $n \leq 10000$ is

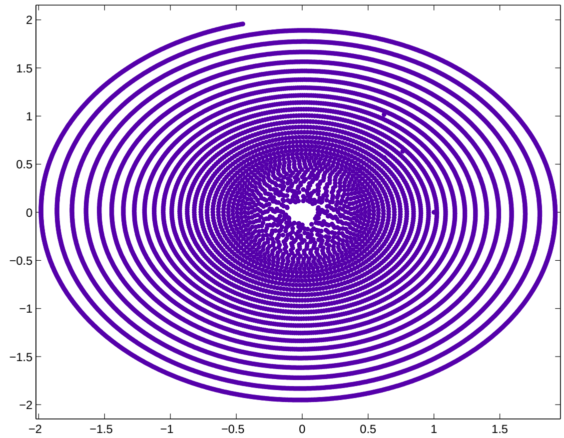

Figure 49

A plot of the logarithms of the n values of the inflection points is

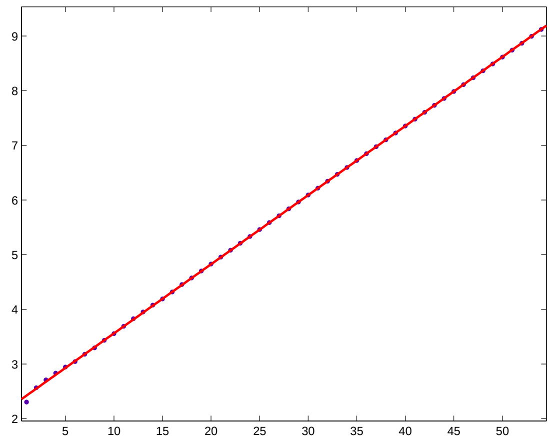

Figure 50

For a linear least-squares fit of the curve, $p_1 = 0.1264$ with a 95% confidence interval of (0.1261, 0.1268), $p_2 = 2.296$ with a 95% confidence interval of (2.286, 2.306), SSE=0.01802, R-squared=0.9999, and RMSE=0.01861. There are 54 points in the curve.
In general, $X'(z, n, m) - Y''(z, n, m)$ appears to equal a three-dimensional version of $\zeta(z, n)$.

10. A SIMILARITY BETWEEN $X'(z,n,m)$ AND $\zeta(z,n)$

A plot of the real components of $X'(z,n,m) - Y'(z,n,m) - C_n\{\frac{1}{n^z}\}$ versus $\sqrt{n}$ for the first zeta function zero and $n \leq 10000$ is

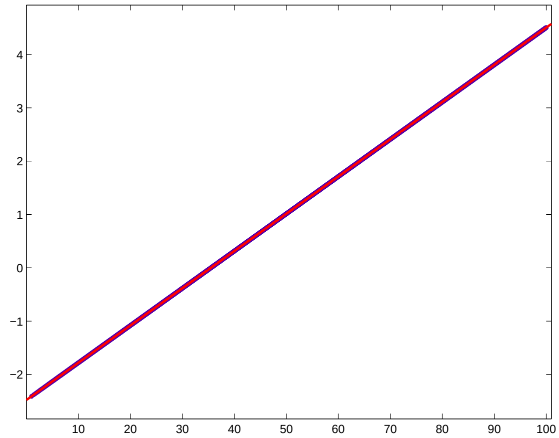

Figure 51

For a linear least-squares fit of the curve, $p_1 = 0.06985$ with a 95% confidence interval of (0.06985, 0.06985), $p_2 = -2.48$ with a 95% confidence interval of (-2.48, -2.48), SSE=$3.234 \cdot 10^{-17}$, R-squared=1.0, and RMSE=$8.687 \cdot 10^{-11}$. The fit is effectively perfect.

A plot of the imaginary components is

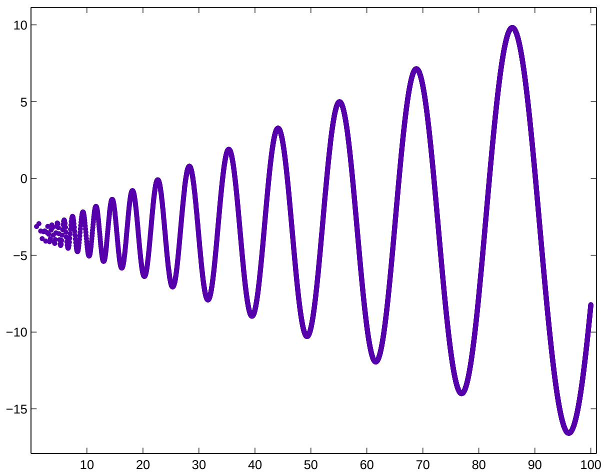

Figure 52

A plot of the real components of $X'(z,n,m)$ and the imaginary components of $X'(z,n,m) - Y'(z,n,m) - C_n\{\frac{1}{n^z}\}$ is

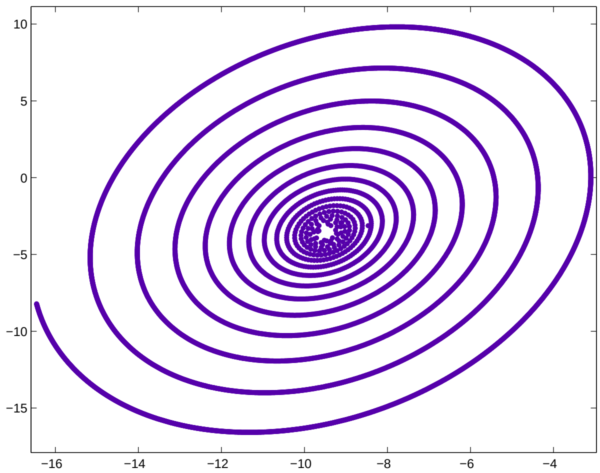

Figure 53

The inflection points are at $n = 3$, 24, 38, 60, 94, 148, 232, 363, 568, 888, 1387, 2168, 3386, 5286, and 8252. A plot of the logarithms of the n values of the inflection points is

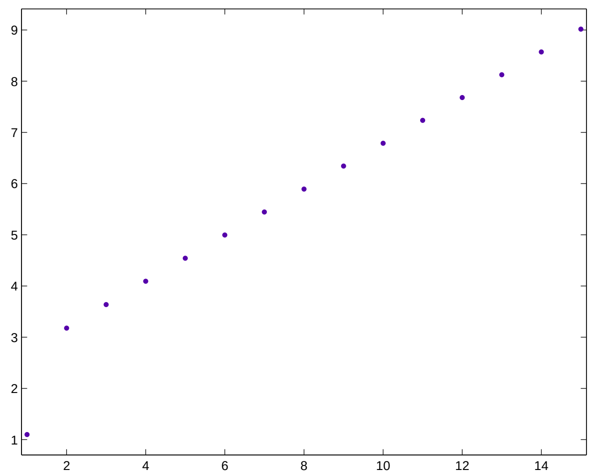

Figure 54

For a linear least-squares fit of the curve (disregarding the first point), $p_1 = 0.4487$ with a 95% confidence interval of (0.4475, 0.4498), $p_2 = 2.298$ with a 95% confidence

interval of (2.287, 2.308), SSE=0.0007187, R-squared=1, and RMSE=0.007739.
A plot of $X'(z,n,m)$ for the first zeta function zero and $n \leq 10000$ is

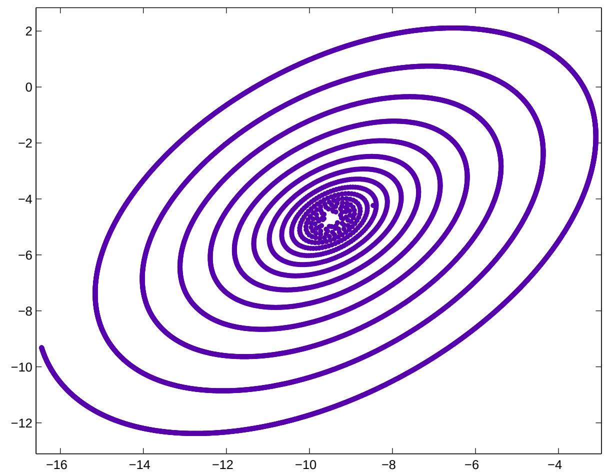

Figure 55

The inflection points are at n = 18, 27, 42, 65, 101, 156, 243, 377, 587, 913, 1422, 2216, 3453, 5381, and 8387. While these values are not equal to the ones above, there is an approximately linear relationship between them. A plot of these n values of inflection points versus the above group of n values of inflection points is

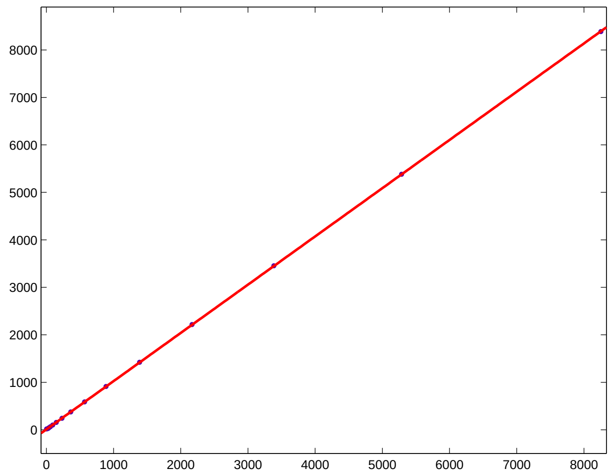

Figure 56

For a linear least-squares fit of the curve, $p_1 = 1.016$ with a 95% confidence interval of (1.015, 1.017), $p_2 = 8.289$ with a 95% confidence interval of (5.905, 11.17), SSE=242.5, R-squared=1, and RMSE=4.319.

A plot of the real component of $X'(z,n,m) - Y'(z,n,m) - C_n\{\frac{1}{n^z}\}$ for $z = (0.40, 14.13472514173470)$ and $n \leq 10000$ is

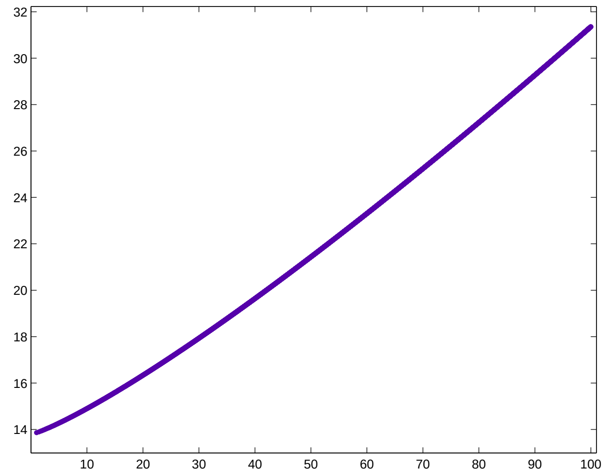

Figure 57

The curve is linear only if the real part of z equals 1/2.

11. A TRANSFORMATION OF $\zeta(z,n)$

A plot of $\zeta(z,n) - Y''(z,n,m)$ for the twentieth zeta function zero and $n \leq 10000$ is

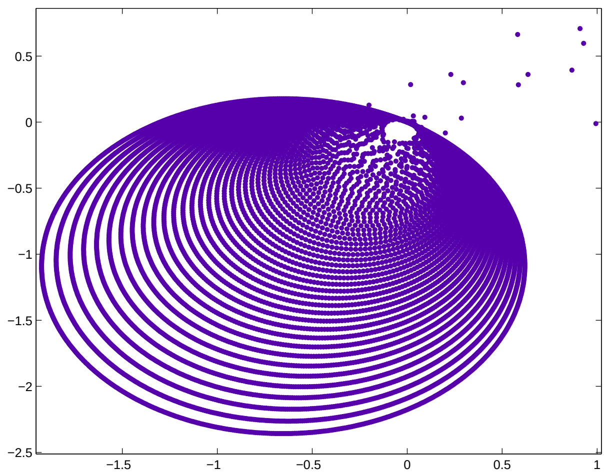

Figure 58

A plot of the logarithms of the n values of the inflection points is

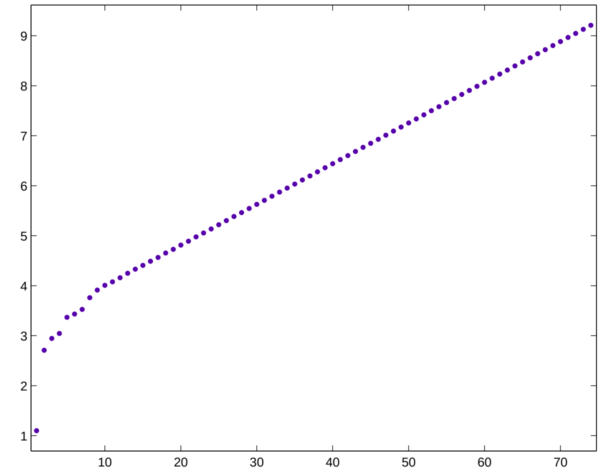

Figure 59

For a linear least-squares fit of the curve (disregarding the first eight points), $p_1 = 0.08143$ with a 95% confidence interval of (0.0814, 0.08145), $p_2 = 3.184$ with a 95% confidence interval of (3.183, 3.184), SSE=0.000282, R-squared=1, and RMSE=0.002099.

A plot of $\zeta(z, n)$ for the twentieth zeta function zero and $n \leq 10000$ is

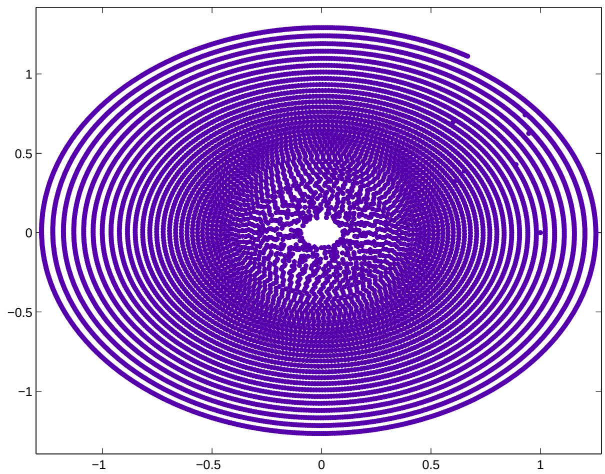

Figure 60

A plot of the logarithms of the n values of the inflection points is

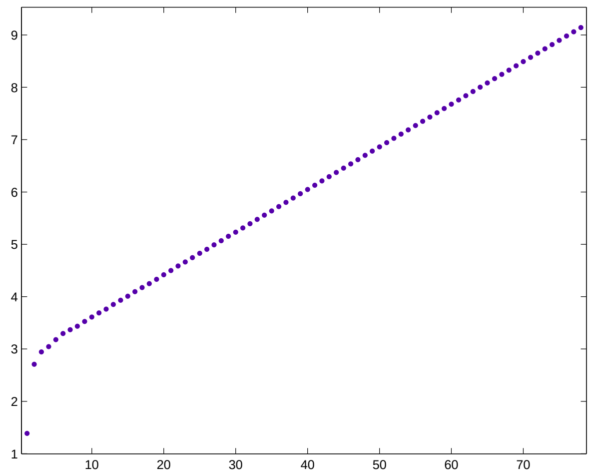

Figure 61

For a linear least-squares fit of the curve (neglecting the first six points), $p_1 = 0.08144$ with a 95% confidence interval of (0.08142, 0.08147), $p_2 = 2.79$ with a 95% confidence interval of (2.788, 2.791), SSE=0.0003846, R-squared=1, and RMSE=0.002344.

Subtracting $Y''(z, n, m)$ from $\zeta(z, n)$ converts it to a three-dimensional logarithmic spiral.

12. CONCLUSION

Several necessary conditions for the Riemann hypothesis to be true are empirically derived.

13. METHODS

The C code for computing $\zeta(z, n)$, $Y(z, n)$ and $C_n\{\frac{1}{n^z}\}$ is as follows.

```
#include <math.h>
#include <stdio.h>
//
// zeta function (when noinc=0)
// X(z,n) (when noinc=1)
//
unsigned int max=1000;
double s=0.50;
```

```
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int xmin=0; // usually set to 0
unsigned int noinc=0; // if set, don't add additional term
unsigned int out=1; // set to 1 when not finding inflection points
void main() {
unsigned int x;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt;
FILE *Outfp;
Outfp = fopen("spiral1.dat","w");
sumr=0.0;
sumi=0.0;
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(x)));
    b=temp1*(sin(t*log(x)));
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    if (s>=0.0)
        temp1=pow((double)max,s);
    else {
        temp1=pow((double)max,-s);
```

```
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(max)));
    tempt=temp1*(sin(t*log(max)));
    if (noinc==0) {
        temps=sumr+temps/2.0;
        tempt=sumi+temps/2.0;
        }
    else {
        temps=sumr;
        tempt=sumi;
        }
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
            fprintf(Outfp,"                                          if
((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
            fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
        olds=temps;
        oldt=tempt;
        }
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
//
// Y'(z,n,m)
//
```

```
unsigned int maxn=1000;
unsigned int out=1;
double s=0.50;
//double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
void main() {
double temp1,temps,tempt,x,olds,oldt;
unsigned int n,max;
FILE *Outfp;
Outfp = fopen("spiral2.dat","w");
olds=0.0;
oldt=0.0;
for (n=1; n<=maxn; n++) {
    x=1.0-s;
    if (x>=0.0)
        temp1=pow((double)n,x);
    else {
        temp1=pow((double)n,-x);
        temp1=1.0/temp1;
        }
    temps=temp1*(x*cos(t*log(max))+t*sin(t*log(max)));
    temps=temps/(x*x+t*t);
    tempt=temp1*(t*cos(t*log(max))-x*sin(t*log(max)));
    tempt=tempt/(x*x+t*t);
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(n>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",n,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(n>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",n,temps,tempt);
```

```
    if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(n>1)))
        fprintf(Outfp,” %d %.10llf %.10llf \n”,n,temps,tempt);
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(n>1)))
        fprintf(Outfp,” %d %.10llf %.10llf \n”,n,temps,tempt);
    olds=temps;
    oldt=tempt;
    }
printf(” %.10llf %.10llf \n”,temps,tempt);
fclose(Outfp);
return;
}

#include <math.h>
#include <stdio.h>
//
// C-transformation
//
unsigned int max=1000;
double s=0.50;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
unsigned int out=1; // use out=3 otherwise
void main() {
unsigned int x,z;
double temp1,temps,tempt,sumr,sumi,a,b,olds,oldt,y;
FILE *Outfp;
Outfp = fopen(”ctrans.dat”,”w”);
sumr=0.0;
sumi=0.0;
```

```
olds=0.0;
oldt=0.0;
for (x=1; x<=max; x++) {
    if (s>=0.0)
        temp1=pow((double)x,s);
    else {
        temp1=pow((double)x,-s);
        temp1=1.0/temp1;
        }
    a=temp1*(cos(t*log(x)));
    b=temp1*(sin(t*log(x)));
    temp1=a*a+b*b;
    sumr=sumr+a/temp1;
    sumi=sumi-b/temp1;
    y=1-s;
    if (y>=0.0)
        temp1=pow((double)max,y);
    else {
        temp1=pow((double)max,-y);
        temp1=1.0/temp1;
        }
    temps=temp1*(y*cos(t*log(max))+t*sin(t*log(max)));
    tempt=temp1*(t*cos(t*log(max))-y*sin(t*log(max)));
    temps=temps/(y*y+t*t);
    tempt=tempt/(y*y+t*t);
    temps=sumr+temps;
    tempt=tempt-sumi;
    if (out==1)
        fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
    if ((out==2)&&((olds>0.0)&&(temps<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==3)&&((oldt>0.0)&&(tempt<0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==4)&&((olds<0.0)&&(temps>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    if ((out==5)&&((oldt<0.0)&&(tempt>0.0)&&(x>1)))
        fprintf(Outfp," %d %.10llf %.10llf \n",x,temps,tempt);
    olds=temps;
```

```
    oldt=tempt;
    }
printf(" %.10llf %.10llf \n",temps,tempt);
fclose(Outfp);
return; }
```

REFERENCES

[1] Caceres, Pedro."$\zeta(z) = X(z) - Y(z)$ A decomposition of the Riemann Zeta Function for $Re(z) > 0$, $z \neq 1$", 2020. VIXRA:https://vixra.org/2003.0189, accessed March 2020.