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Comparative Estimation of the General Connectivity Indices and Its Chemical Applicability

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Abstract

For any non-zero real number α. The general sum-connectivity and product-connectivity (or general Randic) indices of a non-trivial, connected graph $G = (V, E)$ are defined as $S^{\alpha}(G) = \sum_{uv \in E(G)} \left[d_G(u) + d_G(v)\right]^{\alpha}$ and $P^{\alpha}(G) = \sum_{uv \in E(G)} \left[d_G(u) d_G(v)\right]^{\alpha}$, respectively. In this paper, we compute the general sum and product connectivity indices of some standard class of graphs and line graphs. Also, we obtained the comparative estimation (i.e., some bounds and characterization) of general sum and product connectivity indices of graphs and line graphs. Finally, we explore chemical applicability of the Zigzag line coronoid fused with starphene nanotubes.

Keywords : Graph; Topological indices; Sum connectivity index; Product connectivity index.

2020 Mathematics Subject Classification: 05C05, 05C07, 05C35

1. INTRODUCTION

All graphs considered in this paper are finite, connected, undirected without loops and multiple edges. Let $G = (V, E)$ be a connected graph with $\mid V \mid= p$ and $\mid E \mid= q$. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v. The edge connecting the vertices u and v will be denoted by uv. Let $\delta(G)$ and $\Delta(G)$ be the minimum and

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maximum degree of the vertex. An r-regular graph is a graph where each vertex has the same degree r. i.e., $d_G(u) = d_G(v) = r$. Any undefined term in this paper may be found in Harary [17].

A molecular graph is a graph such that its vertices correspond to the atoms and the edges to the bonds. In Chemical Science, the physico- chemical properties of chemical compounds are often modeled by means of molecular graph based structure descriptors, which are also referred to as topological indices, see [16].

The degree–based graph invariants $M_1(G)$ and $M_2(G)$, called Zagreb indices. For the their history, applications, and mathematical properties, see [7, 8, 12, 16] and the references cited therein.

The first and second Zagreb indices take into account the contributions of pairs of adjacent vertices and are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \qquad \text{or} \qquad M_1(G) = \sum_{uv \in E(G)} \big[d_G(u) + d_G(v)\big].$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)\, d_G(v).$$

One of the best known and widely used topological index is the product-connectivity index (or Randic index or connectivity index) by Randic [20], who has shown this index to reflect molecular branching and is defined as

$$P(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}}.$$

Later, the general Randic index was proposed by Gutman and Pavlovic [14] and studied by Bollobas and Erdos [6] and is defined as

$$P^{\alpha}(G) = \sum_{uv \in E(G)} [d_G(u) d_G(v)]^{\alpha}.$$

Motivated by Randic's definition of the product connectivity index, the sum connectivity index of a graph was initiated by Zhou and Trinajstic [22] and is defined by

$$S(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) + d_G(v)}}.$$

Anologously, Zhou and Trinajstic [23] are initiate the study of general sum-connectivity index of a graph as

$$S^{\alpha}(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^{\alpha}.$$

2. MAIN RESULTS

Theorem 2.1. *Let G be an r- regular graph with $p \geq 2$ vertices. Then*

(i) $S^{\alpha}(G) = 2^{\alpha} q r^{\alpha} = p r^{\alpha+1} 2^{\alpha-1}$.

(ii) $P^{\alpha}(G) = q r^{2\alpha} = \frac{p r^{\alpha+1}}{2}$.

Proof. Let G be an r-regular graph with $p \geq 2$ vertices. Since every vertex is of same degree r, we have $2q = pr$.

(i) The general sum-connectivity index is

$$\begin{aligned} S^{\alpha}(G) &= \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^{\alpha} \\ &= \sum_{uv \in E(G)} [r + r]^{\alpha} = 2^{\alpha} q r^{\alpha} = p r^{\alpha+1} 2^{\alpha-1}. \end{aligned}$$

(ii) The general product-connectivity index is

$$\begin{aligned} P^{\alpha}(G) &= \sum_{uv \in E(G)} [d_G(u).d_G(v)]^{\alpha} \\ &= \sum_{uv \in E(G)} [r.r]^{\alpha} = q r^{2\alpha} = \frac{p r^{\alpha+1}}{2}. \end{aligned}$$

Thus the result follows. □

Corollary 2.1. *If cycle C_p with $p \geq 3$ is a 2-regular graph, then*

$$S^{\alpha}(C_p) = P^{\alpha}(C_p) = 2^{2\alpha} q = 2^{2\alpha} p.$$

Corollary 2.2. *If complete graph K_p with $p \geq 2$ is a $(p-1)$-regular graph, then*

(i) $S^{\alpha}(K_p) = 2^{\alpha} q (p-1)^{\alpha} = 2^{\alpha}(p)(p-1)^{\alpha+1}$.

(ii) $P^{\alpha}(K_p) = q(p-1)^{2\alpha} = \frac{p(p-1)^{2\alpha+1}}{2}$.

Corollary 2.3. *If complete regular bipartite graph $K_{m,m}$ with $m \geq 1$ is a m-regular graph, then*

(i) $S^{\alpha}(K_{m,m}) = 2^{\alpha} q m^{\alpha} = p m^{\alpha+1} 2^{\alpha-1}$.

(ii) $P^{\alpha}(K_{m,m}) = q m^{2\alpha} = \dfrac{p m^{2\alpha+1}}{2}$.

2.1. Line graph

The Line graph $L(G)$ is the graph with vertex set $V(L) = E(G)$ and whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common. For more details, we refer to [5, 11].

Theorem 2.2. *Let G be a r- regular graph with $p \geq 2$ vertices. Then*

$$P^{\alpha}(L(G)) = (r-1)^{\alpha}\, S^{\alpha}(L(G)).$$

Proof. Let G be a r- regular graph with $p \geq 2$ vertices. By algebraic method, we have $|V(L(G))| = \frac{pr}{2}$ and $|E(L(G))| = \frac{pr}{2}(r-1)$. Since line graph of a r - regular graph is $(2r-2)$ - regular. Hence, we have the following cases:
Case 1. The general sum-connectivity index of a line graph $L(G)$ of a r-regular graph G is

$$\begin{aligned} S^{\alpha}(L(G)) &= \sum_{uv \in E(L(G))} \left[d_{L(G)}(u) + d_{L(G)}(v)\right]^{\alpha} \\ &= \frac{pr}{2}(r-1)\left[(2r-2) + (2r-2))\right]^{\alpha} \\ &= \frac{pr}{2}(r-1)\, 4^{\alpha}\, (r-1)^{\alpha} \\ &= 2^{(2\alpha-1)}\, pr(r-1)^{\alpha+1}. \end{aligned}$$

Case 2. The general product-connectivity index of a line graph $L(G)$ of a r-regular graph G is

$$\begin{aligned} P^{\alpha}(L(G)) &= \sum_{uv \in E(L(G))} \left[d_{L(G)}(u)\, d_{L(G)}(v)\right]^{\alpha} \\ &= \frac{pr}{2}(r-1)\left[(2r-2)^{2}\right]^{\alpha} \\ &= \frac{pr}{2}(r-1)\, 2^{2\alpha}\, (r-1)^{2\alpha} \\ &= 2^{(2\alpha-1)}\, pr\, (r-1)^{2\alpha+1}. \end{aligned}$$

Thus the result follows. □

Corollary 2.4. *If $L(C_p)$ is a line graph of a cycle C_p with $p \geq 3$, then*

(i) $S^{\alpha}(L(C_p)) = S^{\alpha}(C_p) = 2^{2\alpha}q = 2^{2\alpha}p$.

(ii) $P^{\alpha}(L(C_p)) = P^{\alpha}(C_p) = 2^{2\alpha}q = 2^{2\alpha}p$.

Corollary 2.5. *If $L(K_p)$ is a line graph of a complete graph K_p with $p \geq 2$, then*

(i) $S^{\alpha}(L(K_p)) = p(p-1)(p-2)^{\alpha+1}2^{2\alpha-1}$.

(ii) $P^{\alpha}(L(K_p)) = p(p-1)(p-2)^{2\alpha+1}2^{2\alpha-1}$.

Corollary 2.6. *If $L(K_{m,m})$ is a line graph of a complete regular bipartite graph $K_{m,m}$ with $m \geq 1$, then*

(i) $S^{\alpha}(L(K_{m,m})) = 2^{2\alpha-1}pm(m-1)^{\alpha+1}$.

(ii) $P^{\alpha}(L(K_{m,m})) = 2^{2\alpha-1}pm(m-1)^{2\alpha+1}$.

In order to prove our next results of r- regular graph with $n \geq 2$ vertices in terms of the sum-connectivity and Atom-bond sum connectivity indices, and the product-connectivity and Atom-bond connectivity indices we make use of the following definitions:
The atom-bond sum connectivity index of a graph G is defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u) + d_G(v)}}.$$

This index was introduced by Ali et al., in [2].
The atom-bond connectivity index of a graph G is defined as

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u)d_G(v)}}.$$

This index was proposed by Estrada et al., in [10].

Theorem 2.3. *Let G be a r- regular graph with $p \geq 2$ vertices. Then*

(i) $ABS(G) = \sqrt{2(r-1)}\, S^{-\frac{1}{2}}(G)$.

(ii) $ABC(G) = \sqrt{2(r-1)}\, P^{-\frac{1}{2}}(G)$.

Proof. Let G be a r- regular graph with $n \geq 2$ vertices. Then

(i) the atom-bond sum connectivity index of G is

$$\begin{aligned} ABS(G) &= \sum_{e=uv \in E(G)} \sqrt{\frac{d_G(e)}{d_G(u)+d_G(v)}} \\ &= \frac{pr}{2}\sqrt{\frac{2r-2}{2r}} \\ &= \frac{p}{2}\big(\sqrt{r(r-1)}\big). \end{aligned}$$

By Theorem 2.1(i), if $\alpha = -\frac{1}{2}$, then we have the sum-connectivity index of G is given by

$$S^{-\frac{1}{2}}(G) = \frac{p\sqrt{r}}{2\sqrt{2}}.$$

Thus the result (i) follows.

(i) the atom-bond connectivity index of G is

$$\begin{aligned} ABS(G) &= \sum_{e=uv \in E(G)} \sqrt{\frac{d_G(e)}{d_G(u)d_G(v)}} \\ &= \frac{pr}{2}\sqrt{\frac{2r-2}{r^2}} \\ &= \frac{p}{2}\big(\sqrt{2(r-1)}\big). \end{aligned}$$

By Theorem 2.1(ii), if $\alpha = -\frac{1}{2}$, then we have the product-connectivity index of G is given by

$$P^{-\frac{1}{2}}(G) = \frac{p}{2}.$$

Thus the result (ii) follows.

Hence, the proof. □

Theorem 2.4. *Let G be a r- regular graph with $p \geq 2$ vertices. Then*

(i) $ABS(L(G)) = \big(\sqrt{4r-6}\big)\, S^{-\frac{1}{2}}(L(G))$.

(ii) $ABC(L(G)) = \big(\sqrt{4r-6}\big)\, P^{-\frac{1}{2}}(L(G))$.

Proof. Let G be a r- regular graph with $p \geq 2$ vertices. Then

(i) the atom-bond sum connectivity index of a line graph of a r-regular graph G is

$$\begin{aligned} ABS(L(G)) &= \sum_{e=uv \in E(L(G))} \sqrt{\frac{d_{L(G)}(e)}{d_{L(G)}(u) + d_{L(G)}(v)}} \\ &= \frac{pr}{2}(r-1)\sqrt{\frac{2(2r-2)-2}{2(2r-2)}} \\ &= \frac{pr}{4}\big(\sqrt{r-1}\big)\big(\sqrt{4r-6}\big). \end{aligned}$$

By Theorem 2.2, if $\alpha = -\frac{1}{2}$, then we have the sum-connectivity index of line graph of G is given by

$$S^{-\frac{1}{2}}(L(G)) = \frac{pr}{4}\big(\sqrt{r-1}\big).$$

Thus the result (i) follows.

(ii) the atom-bond connectivity index of a line graph of a r-regular graph G is

$$\begin{aligned} ABC(L(G)) &= \sum_{e=uv \in E(L(G))} \sqrt{\frac{d_{L(G)}(e)}{d_{L(G)}(u)\, d_{L(G)}(v)}} \\ &= \frac{pr}{2}(r-1)\sqrt{\frac{2(2r-2)-2}{(2r-2)^2}} \\ &= \frac{pr}{4}\big(\sqrt{4r-6}\big). \end{aligned}$$

By Theorem 2.2, if $\alpha = -\frac{1}{2}$, then we have the product-connectivity index of line graph of G is given by

$$P^{-\frac{1}{2}}(L(G)) = \frac{pr}{4}.$$

Thus the result (ii) follows.

Hence, the proof. □

Theorem 2.5. *Let G be a nontrivial graph with $\alpha > 0$. Then*

$$\frac{2^\alpha}{\Delta(G)^\alpha} P^\alpha(G) \leq S^\alpha(G) \leq \frac{2^\alpha}{\delta(G)^\alpha} P^\alpha(G).$$

Further, both left and right inequalities are holds if and only if G is regular.

Proof. Let G be a nontrivial with $\alpha > 0$. Then

$$[d_G(u) + d_G(v)]^\alpha = d_G(u)^\alpha d_G(v)^\alpha \left[\frac{1}{d_G(v)} + \frac{1}{d_G(u)}\right]^\alpha$$

$$d_G(u)^\alpha d_G(v)^\alpha \left[\frac{2}{\Delta(G)}\right]^\alpha \leq [d_G(u) + d_G(v)]^\alpha \leq d_G(u)^\alpha d_G(v)^\alpha \left[\frac{2}{\delta(G)}\right]^\alpha.$$

The above inequality satisfies for each $uv = e \in E(G)$ and the sum of all these inequalities arise

$$\Rightarrow \sum_{uv\in E(G)} d_G(u)^\alpha d_G(v)^\alpha \left[\frac{2}{\Delta(G)}\right]^\alpha \leq \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^\alpha$$

$$\leq \sum_{uv\in E(G)} d_G(u)^\alpha d_G(v)^\alpha \left[\frac{2}{\delta(G)}\right]^\alpha$$

$$\Rightarrow \left[\frac{2}{\Delta(G)}\right]^\alpha \sum_{uv\in E(G)} d_G(u)^\alpha d_G(v)^\alpha \leq \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^\alpha$$

$$\leq \left[\frac{2}{\delta(G)}\right]^\alpha \sum_{uv\in E(G)} d_G(u)^\alpha d_G(v)^\alpha$$

$$\frac{2^\alpha}{\Delta(G)^\alpha} P^\alpha(G) \leq S^\alpha(G) \leq \frac{2^\alpha}{\delta(G)^\alpha} P^\alpha(G).$$

By Theorem 2.1, both left and right inequalities are holds if and only if G is regular. Thus the result follows. □

Theorem 2.6. *For any nontrivial graph G with $\alpha > 0$,*

$$2^\alpha P^{\frac{\alpha}{2}}(G) \leq S^\alpha(G) \leq 2^\alpha P^\alpha(G).$$

Proof. Let G be a nontrivial graph with $\alpha 0$. Since $d_G(u), d_G(v) \geq 0$ for all $e = uv \in E(G)$. By the relationship between Arithmetic and Geometric mean inequality (i.e., $AM \geq GM$) in terms of $d_G(u)$ and $d_G(v)$, we have

$$\sqrt{d_G(u)d_G(v)} \leq \frac{d_G(u) + d_G(v)}{2}.$$

For any $\alpha > 0$, we have $2^\alpha \left(d_G(u)d_G(v)\right)^{\frac{\alpha}{2}} \leq \left(d_G(u) + d_G(v)\right)^\alpha$. This inequality holds good for each edge $e = uv \in E(G)$. Adding all those inequalities, we have

$$\begin{aligned}\sum_{uv\in E(G)} 2^\alpha \left(d_G(u)d_G(v)\right)^{\frac{\alpha}{2}} &\leq \sum_{uv\in E(G)} \left(d_G(u)+d_G(v)\right)^\alpha \\ 2^\alpha P^{\frac{\alpha}{2}}(G) &\leq S^\alpha(G).\end{aligned}$$

Now we prove the second part.

Clearly, $d_G(u) \leq d_G(u)d_G(v)$ and $d_G(v) \leq d_G(u)d_G(v)$ for all $e = uv \in E(G)$. Adding above inequalities, we have $d_G(u) + d_G(v) \leq 2d_G(u)d_G(v)$. For any $\alpha > 0$, $(d_G(u) + d_G(v))^\alpha \leq (2d_G(u)d_G(v))^\alpha$.
Therefore

$$\begin{aligned}\sum_{uv\in E(G)} \left(d_G(u)+d_G(v)\right)^\alpha &\leq \sum_{uv\in E(G)} \left(2d_G(u)d_G(v)\right)^\alpha \\ S_\alpha(G) &\leq 2^\alpha P^\alpha(G).\end{aligned}$$

Thus the result follows. □

Theorem 2.7. *For any connected graph G with $\alpha > 0$,*

$$2^\alpha(11q - 12p + P^{\frac{\alpha}{2}}(G)) \leq S_\alpha(G) \leq 2^b(2q^2 - \frac{1}{2}P^\alpha(G)).$$

Proof. By Theorems 2.5 and 2.6, we have desired bounds of $S^\alpha(G)$ in terms of order, size and $P^\alpha(G)$. □

3. CHEMICAL APPLICABILITY

We consider the system which is a composite benzenoid obtained by a zigzag-line coronoid $ZC(k, l, m)$ with a starphene $Zt(k, l, m)$. This system is called a zigzag line coronoid fused with starphene nanotubes denoted by $ZCS(k, l, m)$, see Figure 1(a). The subdivision graph of $ZCS(k, l, m)$ is depicted in Figure 1(b) and the line graph of the subdivision graph of $ZCS(k, l, m)$ is depicted in Figure 1(c). For more details, we refer to [1] and [18].

We compute the general sum and product connectivity indices of the line graph of the subdivision graph of a zigzag-edge coronoid fused with starphene nanotubes.

Theorem 3.1. *Let G^* be the line graph of the subdivision graph of a zigzag-line coronoid fused with starphene nanotubes $ZCS(k, l, m)$ for every $k = l = m \geq 4$. Then*

(i) $S^\alpha(G^*) = (6(k + l + m) - 30)(4^\alpha) + (12(k + l + m) - 84)(5^\alpha) + (21(k + l +$

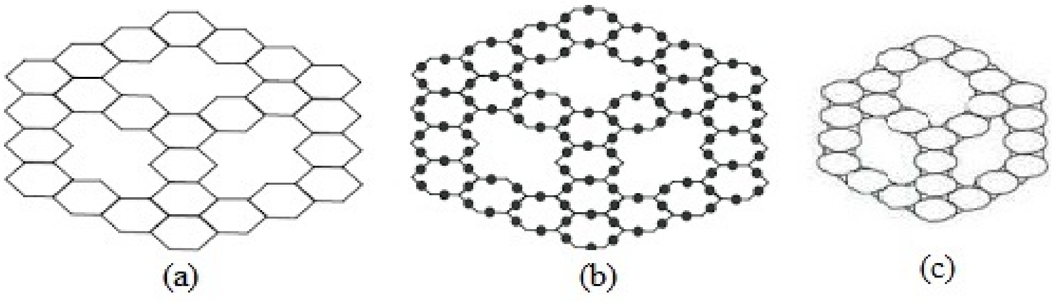

Figure 1: Zigzag line coronoid fused with starphene nanotubes.

$m) - 39)(6^\alpha)$.

(ii) $P^\alpha(G^*) = (6(k+l+m) - 30)(4^\alpha) + (12(k+l+m) - 84)(6^\alpha) + (21(k+l+m) - 39)(9^\alpha)$.

where α is a real number.

Proof. Let G^* be the line graph of the subdivision graph of a zigzag-line coronoid fused with starphene nanotubes $ZCS(k,l,m)$ for every $k = l = m \geq 4$. By algebraic method, we obtain
$|E(G^*)| = 39(k+l+m) - 153$, see Figure 1(c). Also by algebraic method, we obtain three edge partitions of G^* based on the sum of degrees of the end-vertices of each edge as follows.
$E_1 = \{uv \in E(G^*) : d_{G^*}(u) = d_{G^*}(v) = 2\}; |E_1| = 6(k+l+m) - 30$,
$E_2 = \{uv \in E(G^*) : d_{G^*}(u) = 2, d_{G^*}(v) = 3\}; |E_2| = 12(k+l+m) - 84, and E_3 = \{uv \in E(G^*) : d_{G^*}(u) = d_{G^*}(v) = 3\}; |E_3| = 21(k+l+m) - 39$. The vertex-edge degree partition of G^* is given in Table 1.

$\big(d_{G^*}(u), d_{G^*}(v)\big) : uv \in E(G^*)$	Number of edges
$(2,2)$	$6(k+l+m) - 30$
$(2,3)$	$12(k+l+m) - 84$
$(3,3)$	$21(k+l+m) - 39$

Table 1: The vertex-edge degree partition of G^*.

By the definitions of $S^\alpha(G)$ and $P^\alpha(G)$ with the cardinalities of the vertex-line degree partitions of the molecular graph of zigzag-edge coronoid fused with starphene nanotubes $G^* \cong ZCS(k,l,m)$ for every $k = l = m \geq 4$ and $E(G^*) = E_1(G^*) \cup E_2(G^*) \cup E_3(G^*)$ (or, simply $E = E_1 \cup E_2 \cup E_3$).

Therefore, the general sum connectivity index of G^* is given by

$$\begin{aligned}
S^\alpha(G^*) &= \sum_{uv\in E(G^*)} [d_{G^*}(u)+d_{G^*}(v)]^\alpha \\
&= \sum_{e\in E_1}[d_{G^*}(u)+d_{G^*}(v)]^\alpha + \sum_{e\in E_2}[d_{G^*}(u)+d_{G^*}(v)]^\alpha \\
&+ \sum_{e\in E_3}[d_{G^*}(u)+d_{G^*}(v)]^\alpha \\
&= (6(k+l+m)-30)(4^\alpha)+(12(k+l+m)-84)(5^\alpha) \\
&+ (21(k+l+m)-39)(6^\alpha).
\end{aligned}$$

Also, the general product connectivity index of G^* is given by

$$\begin{aligned}
P^\alpha(G^*) &= \sum_{uv\in E(G^*)} [d_{G^*}(u)\times d_G(v)]^\alpha \\
&= \sum_{e\in E_1}[d_G(u)\times d_{G^*}(v)]^\alpha + \sum_{e\in E_2}[d_{G^*}(u)\times d_{G^*}(v)]^\alpha \\
&+ \sum_{e\in E_3}[d_{G^*}(u)\times d_{G^*}(v)]^\alpha \\
&= (6(k+l+m)-30)(4^\alpha)+(12(k+l+m)-84)(6^\alpha) \\
&+ (21(k+l+m)-39)(9^\alpha).
\end{aligned}$$

Hence, the proof. □

3.1. Comparative Estimation

The comparative estimation of general sum and product connectivity indices of the molecular graph of zigzag-line coronoid fused with starphene nanotubes $G^* \cong ZCS(k,l,m)$ with $k=l=m=4$, we have

(i) $S^\alpha(G^*) = (42\times 4^\alpha)+(60\times 5^\alpha)+(213\times 6^\alpha)$.

(ii) $P^\alpha(G^*) = (42\times 4^\alpha)+(60\times 6^\alpha)+(213\times 9^\alpha)$.

For different growth of non-zero real number α, we consider the computed values of $1\le\alpha\le 4$ as in Table 2.

In view of the Table 2 and Figure 2, the comparative Estimation of $S^\alpha(G^*)$ and $P^\alpha(G^*)$ with negative growth (i.e., $\alpha=-2,-1$ and $-\frac{1}{2}$) and positive growth (i.e., $\alpha=\frac{1}{2},1$ and 2). In general, $S^\alpha(G^*)\le P^\alpha(G^*)$ for positive growth of α and $S^\alpha(G^*)\ge P^\alpha(G^*)$

Indices	Negative Growth			Positive Growth		
	$\alpha=-2$	$\alpha=-1$	$\alpha=\frac{-1}{2}$	$\alpha=\frac{1}{2}$	$\alpha=1$	$\alpha=2$
$S^{\alpha}(G^{*})$	10.941	58	134.789	739.905	1746	9840
$P^{\alpha}(G^{*})$	6.921	44.1166	116.494	869.969	2445	20085

Table 2: Different growth of non-zero real number α.

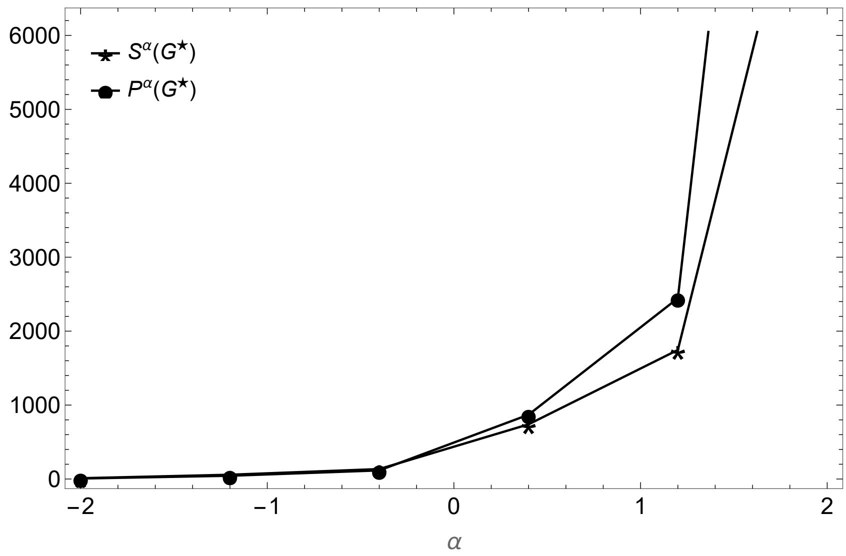

Figure 2: Comparative Estimation of $S^{\alpha}(G^{*})$ and $P^{\alpha}(G^{*})$ with respect to α.

for negative growth of α. However, the difference between $S^{\alpha}(G^{*})$ and $P^{\alpha}(G^{*})$ is arbitrarily large for $\alpha \geq 2$.

4. CONCLUDING REMARKS

In this paper, we compute the general sum and product connectivity indices of some standard class of graphs and line graphs for highlighting their intrinsic connection to established atom-bond connectivity related graphical indices under specific variable assignments of non-zero real number α. The computation of these indices for the molecular graph of a zigzag-line coronoid fused with starphene nanotubes $ZCS(k,l,m)$ for every $k=l=m \geq 4$, which offer insights into their structural characteristics. This comprehensive investigation underscores the significance of these indices, contributing to their broader understanding and application within the realm of chemical graph theory. For this reason, we pose the following open problems.

Open Problem 1. Obtain some bounds and characterization of $S^{\alpha}(G)$ and $P^{\alpha}(G)$ in terms of other generalized graphical indices such as generalized zagreb indices generalized geometric-arthimatic index, generalized harmonic index and so on.

Open Problem 2. Obtain some results on generalized sum and product connectivity indices of some graph operations, product graphs, tranformation graphs, signed graphs and derived graphs.

Open Problem 3. Obtain some results towards QSPR/QSAR/QSTR model on generalized sum and product connectivity indices of some chemical graphs.

Open Problem 4. Prove or disprove for any non-trivial connected graph $S^\alpha(G) = P^\alpha(G)$ if and only if $G \cong C_p; p \geq 3$. Analogously, this result can be extended to some class of derived graphs such as complementary graph, line graph, subdivision graph, total graph, middle graph, Jump graph, Parity graph and so on.

5. CONFLICTS OF INTEREST

The authors have disclosed that they have no conflicts of interest.

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