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# Certain Subclass of Bi-univalent Functions Involving Fractional Calculus Operators via Chebyshev Polynomials

**Jae Ho Choi**

*Department of Mathematics Education*
*Daegu National University of Education*
*219 Jungangdaero, Namgu, Daegu 42411, Korea.*
*E-mail ID: choijh@dnue.ac.kr*

## Abstract

The object of this paper is to study certain subclass $\mathcal{B}_{\Sigma}^{\alpha,\beta}(\gamma, t, s, p, b)$ of analytic and bi-univalent functions involving fractional calculus operators in the open unit disk. For the defined new subclass, we find initial estimates on Taylor-Maclaurin coefficients and investigate the upper bounds for Feketo-Szegö inequality. Moreover, it is remarked that the given bounds improve and generalize some of the pervious results.

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## 1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in the unit disk $\mathbb{U}$.

For analytic functions $f$ and $g$ with $f(0) = g(0)$, $f$ is said to be subordinate to $g$ if there exists an analytic function $\omega$ on $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$. The subordination will be denoted by

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \qquad \text{in } \mathbb{U}.$$

Note that $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ when $g$ is univalent in $\mathbb{U}$.

The well known Koebe one-quarter theorem [1] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius 1/4. Hence every function $f \in \mathcal{S}$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z (z \in U)$ and

$$f^{-1}(f(w)) = w \qquad (|w| < r_0(f); r_0(f) \geq 1/4),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-uninalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of the class $\Sigma$, see [2].

In 1967, Lewin [3] investigated the class $\Sigma$ of bi-univalent functions and showed that $|a_2| < 1.51$. Subsequently, Netanyahu [4] showed that $\max_{f \in \Sigma} |a_2| = 4/3$ and Suffridge [5] has given an example of $f \in \Sigma$ for which $|a_2| = 4/3$. Later, Brannan and Clunie [6] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. A brief summery of functions in the family $\Sigma$ can be found in the study of Srivastava *et al.* [2], which is a basic research on the bi-univalent function family $\Sigma$ (also, see the references cited therein). In a number of sequels to [2], bounds for the first two coefficients $|a_2|$ and $|a_3|$ of different subclasses of bi-univalent functions were given, for example, see [5, 7, 8, 9, 10]. But the coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} = \{1, 2, 3, \cdots\})$ is still an open problem. In recent years, Srivastava *et al.*'s pioneering research on the subject [2] has successfully revitalized the study of bi-univalent functions to have produced numerous bi-univalent function papers. There are also several papers dealing with bi-univalent functions defined by subordination, for example, see [11, 12, 13].

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. The Chebyshev polynomials of the first and second kinds are well known (see [14, 15, 16]). Recently, Kizilateş *et al.* [17] defined $(p, b)$-Chebyshev polynomials of the first and second kinds and derived explicit formulas, generating functions and some interesting properties of these polynomials.

For any integer $n \geq 2$ and $0 < b < p \leq 1$, the $(p, b)$-Chebyshev polynomials of the second kind is defined by the following recurrence relation:

$$U_n(t, s, p, b) = (p^n + b^n)tU_{n-1}(t, s, p, b) + (pb)^{n-1}sU_{n-2}(t, s, p, b) \qquad (1.3)$$

with the initial values $U_0(t, s, p, b) = 1$, $U_1(t, s, p, b) = (p + b)t$ and $s$ is a real variable. Also, it follows readily from (1.3) that

$$\begin{aligned} U_2(t, s, p, b) &= t^2(p + b)(p^2 + r^2) + pbs, & (1.4) \\ U_3(t, s, p, b) &= t^3(p + b)(p^2 + b^2)(p^3 + b^3) + pbst(p^3 + b^3) + (pb)^2st(p + b), \cdots . \end{aligned}$$

By assuming various values of $t,\ s,\ p$ and $b$, we get some interesting polynomials as follows (see [17] and [18]):

(i) When $t = t/2$, $s = s$, $p = p$ and $b = q$, the $(p, b)$-Chebyshev polynomials of the second kind becomes $(p, q)$-Fibonacci polynomials.

(ii) When $t = t$, $s = -1$, $p = 1$ and $b = q$, the $(p, b)$-Chebyshev polynomials of the second kind becomes $q$-Chebyshev polynomials of the second kind.

(iii) When $t = t$, $s = 1$, $p = 1$ and $b = 1$, the $(p, b)$-Chebyshev polynomials of the second kind becomes Pell polynomials.

(iv) When $t = 1/2$, $s = 2y$, $p = 1$ and $b = 1$, the $(p, b)$-Chebyshev polynomials of the second kind becomes Jacobi polynomials.

The generating function of the $(p, b)$-Chebyshev polynomials of the second kind is as follows:

$$\begin{aligned} G_{p,b}(z) &= \frac{1}{1 - tpz\eta_p - tbz\eta_b - spbz^2\eta_{p,b}} \\ &= \sum_{n=0}^{\infty} U_n(t, s, p, b)z^n \qquad (z \in \mathbb{U}), \end{aligned} \tag{1.5}$$

where the Fibonacci operator $\eta_b$ was introduced by Mason and Handscomb [9], by $\eta_b f(z) = f(bz)$. Similarly, we define another operator $\eta_{p,b} f(z) = f(pbz)$.

Various definitions of operators of fractional calculus (that is, fractional integral and fractional derivative) are available in the literature (*cf.*, *e.g.*, [19, 20, 21, 22]). We state the following definitions due to Owa [23] which have been used rather frequently in the theory of analytic functions (see also [24, 25, 26]).

**Definition 1.1.** The fractional integral of order $\lambda$ $(\lambda > 0)$ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \tag{1.6}$$

and the fractional derivative of order $\lambda$ $(0 \leq \lambda < 1)$ by

$$\mathcal{D}_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \tag{1.7}$$

where $f(z)$ is an analytic function in a simply-connected region of the $z$-plane containing the origin, and the multiplicity of $(z - \zeta)^{\lambda-1}$ involved in (1.5) (and that of $(z - \zeta)^{-\lambda}$ in (1.6)) is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

**Definition 1.2.** Under the hypotheses of Definition 1.1, the fractional derivative of order $n+\lambda$ $(0 \leq \lambda < 1; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ is defined by

$$\mathcal{D}_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} \mathcal{D}_z^{\lambda} f(z). \tag{1.8}$$

With the aid of the above definitions, Owa and Srivastava [26] defined the fractional calculus operator $\Omega^{\lambda}$ $(\lambda \in \mathbb{R}; \lambda \neq 2, 3, 4, \cdots)$ by

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} \mathcal{D}_z^{\lambda} f(z) \tag{1.9}$$

for functions (1.1) belonging to the class $\mathcal{A}$ (see also [27, 28]).

Now we define the following subclass of function class $\Sigma$.

**Definition 1.3.** Let $\alpha < 2$, $\beta < 2$, $0 < b < p \leq 1$, $\gamma \in \mathbb{C} \setminus \{0\}$ and $\frac{1}{2} < t < 1$. A function $f \in \Sigma$ is said to be in the subclass $\mathcal{B}_{\Sigma}^{\alpha,\beta}(\gamma, t, s, p, b)$, if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{\Omega^{\alpha} f(z)}{\Omega^{\beta} f(z)} - 1\right) \prec G_{p,b}(z) \quad (z \in \mathbb{U})$$

and

$$1 + \frac{1}{\gamma}\left(\frac{\Omega^{\alpha} g(w)}{\Omega^{\beta} g(w)} - 1\right) \prec G_{p,b}(w) \quad (w \in \mathbb{U}),$$

where $g = f^{-1}$ and $G_{p,b}$ is given by (1.5).

The object of the present paper is to study the Chebyshev polynomial expansions to provide estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclass $\mathcal{B}_{\Sigma}^{\alpha,\beta}(\gamma, t, s, p, b)$. Also, we investigate the Feketo-Szegö inequalities for the class $\mathcal{B}_{\Sigma}^{\alpha,\beta}(\gamma, t, s, p, b)$.

## 2. Main Results

In order to establish our results, we shall need the following lemma.

**Lemma 2.1.** ([1]) *Let $\mathcal{P}$ be the class of all functions $h$ analytic in $\mathbb{U}$ of the form*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

*which satisfy* $\operatorname{Re}(h(z)) > 0$ *for all $z \in \mathbb{U}$. Then if $h \in \mathcal{P}$, then $|c_n| \leq 2$ $(n \in \mathbb{N})$.*

We begin by proving the following result.

**Theorem 2.2.** *Let* $\beta < \alpha < 2$, $0 < r < p \leq 1$, $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $\frac{1}{2} < t < 1$. *If the function* $f(z)$ *given by* $(1.1)$ *belongs to* $\mathcal{B}_{\Sigma}^{\alpha,\beta}(\gamma, t, s, p, b)$, *then*

$$|a_2| \leq \frac{|\gamma|(p+b)^{\frac{3}{2}}\, t\sqrt{t}}{\sqrt{|\Theta_{\beta}^{\alpha}(\gamma, t, s, p, b)|}} \tag{2.1}$$

*and*

$$|a_3| \leq \frac{|\gamma|(2-\alpha)(2-\beta)(p+b)t}{2(\alpha-\beta)} \left( \frac{(2-\alpha)(2-\beta)|\gamma|(p+b)t}{2(\alpha-\beta)} + \frac{(3-\alpha)(3-\beta)}{3(5-\alpha-\beta)} \right), \tag{2.2}$$

*where*

$$\begin{aligned} \Theta_{\beta}^{\alpha}(\gamma, t, s, p, b) \;=\; & \frac{2(\alpha-\beta)}{(2-\alpha)(2-\beta)^2} \left[ (p+b)t^2 \left( \frac{\gamma(p+b)(3\beta^2 + 5\alpha\beta - 12\alpha - 27\beta + 48)}{(3-\alpha)(3-\beta)} \right.\right. \\ & \left.\left. - \frac{2(\alpha-\beta)}{2-\alpha}(p^2+b^2) \right) + \frac{2(\alpha-\beta)}{2-\alpha}\left((p+b)t - pbs\right) \right]. \end{aligned} \tag{2.3}$$

*Proof.* Using the definition of fractional calculus with (1.1), we have

$$\Omega^{\lambda} f(z) = z + \sum_{n=2}^{\infty} \varphi_n(\lambda) a_n z^n \qquad (\lambda < 2), \tag{2.4}$$

where

$$\varphi_n(\lambda) = \frac{n!\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} \qquad (n \geq 2). \tag{2.5}$$

Assume that $f \in \mathcal{B}_{\Sigma}^{\alpha,\beta}(\gamma, t, s, p, b)$ and $g$ be the analytic function of $f^{-1}$ to $\mathbb{U}$. Then there exist two functions $\phi$ and $\psi$, analytic in $\mathbb{U}$ with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$ and $|\psi(w)| < 1$ $(z, w \in \mathbb{U})$ such that

$$1 + \frac{1}{\gamma}\left( \frac{\Omega^{\alpha} f(z)}{\Omega^{\beta} f(z)} - 1 \right) = G_{p,b}(\phi(z)) \quad (z \in \mathbb{U}) \tag{2.6}$$

and

$$1 + \frac{1}{\alpha}\left( \frac{\Omega^{\alpha} g(w)}{\Omega^{\beta} g(w)} - 1 \right) = G_{p,b}(\psi(w)) \quad (w \in \mathbb{U}). \tag{2.7}$$

Next, we define the function $p, q \in \mathcal{P}$ by

$$p(z) = \frac{1+\phi(z)}{1-\phi(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(w) = \frac{1+\psi(w)}{1-\psi(w)} = 1 + q_1 w + q_2 w^2 + \cdots$$

or equivalently,

$$\phi(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\cdots \tag{2.8}$$

and

$$\psi(w)=\frac{q(w)-1}{q(w)+1}=\frac{1}{2}q_1w+\frac{1}{2}\left(q_2-\frac{1}{2}q_1^2\right)w^2+\cdots . \tag{2.9}$$

Using (2.8) and (2.9) together with (1.5), it follows that

$$\begin{aligned} G_{p,b}(\phi(z)) &= 1+\frac{U_1(t,s,p,b)}{2}p_1z \\ &\quad +\left(\frac{U_1(t,s,p,b)}{2}(p_2-\frac{1}{2}p_1^2)+\frac{U_2(t,s,p,b)}{4}p_1^2\right)z^2+\cdots \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} G_{p,b}(\psi(w)) &= 1+\frac{U_1(t,s,p,b)}{2}q_1w \\ &\quad +\left(\frac{U_1(t,s,p,b)}{2}(q_2-\frac{1}{2}q_1^2)+\frac{U_2(t,s,p,b)}{4}q_1^2\right)w^2+\cdots . \end{aligned} \tag{2.11}$$

By equating the coefficients from (2.6), (2.7), (2.10) and (2.11), we obtain

$$\frac{\varphi_2(\alpha)-\varphi_2(\beta)}{\gamma}a_2=\frac{U_1(t,s,p,b)}{2}p_1, \tag{2.12}$$

$$\begin{aligned} &\frac{1}{\gamma}\Big[(\varphi_3(\alpha)-\varphi_3(\beta))a_3+\varphi_2(\beta)(\varphi_2(\alpha)-\varphi_2(\beta))a_2^2\Big] \\ &\qquad =\frac{U_1(t,s,p,b)}{2}\left(p_2-\frac{p_1^2}{2}\right)+\frac{U_2(t,s,p,b)}{4}p_1^2, \end{aligned} \tag{2.13}$$

$$-\frac{\varphi_2(\alpha)-\varphi_2(\beta)}{\gamma}a_2=\frac{U_1(t,s,p,b)}{2}q_1, \tag{2.14}$$

$$\begin{aligned} &\frac{1}{\gamma}\Big[(\varphi_3(\alpha)-\varphi_3(\beta))(2a_2^2-a_3)+\varphi_2(\beta)(\varphi_2(\alpha)-\varphi_2(\beta))a_2^2\Big] \\ &\qquad =\frac{U_1(t,s,p,b)}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{U_2(t,s,p,b)}{4}q_1^2. \end{aligned} \tag{2.15}$$

From (2.12) and (2.14), we observe that

$$p_1=-q_1 \tag{2.16}$$

$$\frac{2}{\gamma^2}(\varphi_2(\alpha)-\varphi_2(\beta))^2a_2^2=\frac{U_1^2(t,s,p,b)}{4}(p_1^2+q_1^2). \tag{2.17}$$

If we add (2.13) to (2.15), then

$$\frac{2}{\gamma}\left(\varphi_2(\beta)(\varphi_2(\alpha)-\varphi_2(\beta))+(\varphi_3(\alpha)-\varphi_3(\beta)\right)a_2^2 = \frac{U_1(t,s,p,b)}{2}(p_2+q_2)+\frac{U_2(t,s,p,b)-U_1(t,s,p,b)}{4}(p_1^2+q_1^2). \tag{2.18}$$

By using (2.17) in equality (2.18), we have

$$\frac{2}{\gamma}\left[\varphi_2(\beta)(\varphi_2(\alpha)-\varphi_2(\beta))-\frac{(\varphi_2(\alpha)-\varphi_2(\beta))^2(U_2(t,s,p,b)-U_1(t,s,p,b))}{\gamma U_1^2(t,s,p,b)} +(\varphi_3(\alpha)-\varphi_3(\beta))\right]a_2^2=\frac{U_1(t,s,p,b)}{2}(p_2+q_2). \tag{2.19}$$

Then, by applying (1.3), (1.4), (2.5) and Lemma 2.1 in (2.19), we obtain the inequality (2.1).

Next, if we subtract (2.15) from (2.13), we get

$$\frac{2}{\gamma}(\varphi_3(\alpha)-\varphi_3(\beta))(a_3-a_2^2) = \frac{U_1(t,s,p,b)}{2}(p_2-q_2)+\frac{U_2(t,s,p,b)-U_1(t,s,p,b)}{4}(p_1^2-q_1^2). \tag{2.20}$$

By applying (2.16), (2.17) and (2.20), it is evident that

$$a_3=\frac{\gamma^2U_1^2(t,s,p,b)}{8(\varphi_2(\alpha)-\varphi_2(\beta))^2}(p_1^2+q_1^2)+\frac{\gamma U_1(t,s,p,b)}{4(\varphi_3(\alpha)-\varphi_3(\beta))}(p_2-q_2). \tag{2.21}$$

Hence, by using (1.3), (2.5) and Lemma 2.1 in (2.21), we get the inequality (2.2). This completes the proof of Theorem 2.2.

Taking $\alpha=\beta+1$ and $p=1$ in Theorem 2.2, we have the following consequences.

**Corollary 2.3.** *Let* $\beta<1$, $0<r<p\leq 1$, $\alpha\in\mathbb{C}\setminus\{0\}$ *and* $\frac{1}{2}<t<1$. *If the function* $f(z)$ *given by* (1.1) *belongs to* $\mathcal{B}_\Sigma^{\beta+1,\beta}(\gamma,t,s,p,b)$, *then*

$$|a_2|\leq\frac{|\gamma|(p+b)^{\frac{3}{2}}\,t\sqrt{t}}{\sqrt{|\zeta_\beta^{\beta+1}(\gamma,t,s,p,b)|}}$$

*and*

$$|a_3|\leq\frac{|\gamma|(p+b)t(1-\beta)(2-\beta)}{4}\left((1-\beta)(2-\beta)|\gamma|(p+b)t+\frac{3-\beta}{3}\right),$$

*where*

$$\zeta_\beta^{\beta+1}(\gamma,t,s,p,b) = \frac{8}{(1-\beta)(2-\beta)^2}\left[(p+b)t^2\left(\frac{\gamma(p+b)(4\beta^2-17\beta+18)}{(2-\beta)(3-\beta)} -\frac{4(p^2+b^2)}{1-\beta}\right)+\frac{4}{1-\beta}\left((p+b)t-pbs\right)\right]. \tag{2.22}$$

**Corollary 2.4.** *Let* $\beta < \alpha < 2$, $0 < r < 1$, $\gamma \in \mathbb{C} \setminus \{0\}$ *and* $\frac{1}{2} < t < 1$. *If the function* $f(z)$ *given by* (1.1) *belongs to* $\mathcal{B}_{\Sigma}^{\alpha,\beta}(\lambda, t, s, 1, b)$, *then*

$$|a_2| \leq \frac{|\gamma|(1+b)^{\frac{3}{2}}\, t\sqrt{t}}{\sqrt{|\vartheta_{\beta}^{\alpha}(\gamma, t, s, b)|}}$$

*and*

$$|a_3| \leq \frac{|\gamma|(2-\alpha)(2-\beta)(1+b)t}{2(\alpha-\beta)}\left(\frac{(2-\alpha)(2-\beta)|\gamma|(1+b)t}{2(\alpha-\beta)} + \frac{(3-\alpha)(3-\beta)}{3(5-\alpha-\beta)}\right),$$

*where*

$$\begin{aligned} \vartheta_{\beta}^{\alpha}(\gamma, t, s, b) \;&=\; \frac{2(\alpha-\beta)}{(2-\alpha)(2-\beta)^2}\left[(1+b)t^2\left(\frac{\gamma(1+b)(3\beta^2+5\alpha\beta-12\alpha-27\beta+48)}{(3-\alpha)(3-\beta)}\right.\right. \\ &\quad \left.\left. -\,\frac{2(\alpha-\beta)}{2-\alpha}(1+b^2)\right) + \frac{2(\alpha-\beta)}{2-\alpha}\left((1+b)t - bs\right)\right]. \end{aligned} \tag{2.23}$$

**Remark 2.5.** Taking $\alpha = \gamma = b = 1$, $\beta = 0$ and $s = -1$ in Corollary 2.4, we obtain a recent result due to Altinkaya and Yalçin [12, Corollary 8]. Furthermore, setting $\alpha = 1$ and $\beta = 0$, the class $\mathcal{B}_{\Sigma}^{1,0}(\gamma, t, s, p, b)$ reduces to the class $\mathcal{S}_{\Sigma}^{*}(\gamma, G_{p,b})$ consists of bi-starlike functions of complex order $\gamma$ of Ma-Minda type. The class $\mathcal{S}_{\Sigma}^{*}(\gamma, \phi)$ is introduced and studied by Deniz [29].

**Theorem 2.6.** *Let* $\beta < \alpha < 2$, $0 < b < p \leq 1$, $\gamma \in \mathbb{C} \setminus \{0\}$ *and* $\frac{1}{2} < t < 1$, *and let* $\mu \in \mathbb{R}$. *If the function* $f(z)$ *given by* (1.1) *belongs to* $\mathcal{B}_{\Sigma}^{\alpha,\beta}(\gamma, t, s, p, b)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)|\gamma|(p+b)t}{6(\alpha-\beta)(5-\alpha-\beta)}, \\[2ex] \qquad if\ |\mu - 1| \leq \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)|\Theta_{\beta}^{\alpha}(\gamma, t, s, p, b)|}{6(\alpha-\beta)(5-\alpha-\beta)(p+b)^2|\alpha|t^2}, \\[2ex] \dfrac{(p+b)^3|\gamma|^2|1-\mu|t^3}{|\Theta_{\beta}^{\alpha}(\gamma, t, s, p, b)|}, \\[2ex] \qquad if\ |\mu - 1| \geq \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)|\Theta_{\beta}^{\alpha}(\gamma, t, s, p, b)|}{6(\alpha-\beta)(5-\alpha-\beta)(p+b)^2|\alpha|t^2}, \end{cases} \tag{2.24}$$

*where* $\Theta_{\beta}^{\alpha}(\gamma, t, s, p, b)$ *is given by* (2.3).

*Proof.* From (2.19) and (2.20), we have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{\gamma\, U_1(t,s,p,b)}{4\varphi_3(\alpha,\beta)}(p_2-q_2) \\ &+\frac{(1-\mu)\gamma^2U_1^3(t,s,p,b)(p_2+q_2)}{4[\gamma\, U_1^2(t,s,p,b)\{\varphi_2(\beta)\cdot\varphi_2(\alpha,\beta)+\varphi_3(\alpha,\beta)\}-(\varphi_2(\alpha,\beta))^2U(t,s,p,b)]} \\ &=\gamma\, U_1(t,s,p,b)\left[\left(h(\mu)+\frac{1}{4\varphi_3(\alpha,\beta)}\right)p_2+\left(h(\mu)-\frac{1}{4\varphi_3(\alpha,\beta)}\right)q_2\right], \end{aligned}$$

where

$$\varphi_n(\alpha,\beta)=\varphi_n(\alpha)-\varphi_n(\beta), \qquad U(t,s,p,b)=U_2(t,s,p,b)-U_1(t,s,p,b)$$

and

$$h(\mu)=\frac{(1-\mu)\gamma\, U_1^2(t,s,p,b)}{4[\gamma\, U_1^2(t,s,p,b)\{\varphi_2(\beta)\cdot\varphi_2(\alpha,\beta)+\varphi_3(\alpha,\beta)\}-(\varphi_2(\alpha,\beta))^2U(t,s,p,b)]}.$$

Then, by applying Lemma 2.1, we conclude that

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{|\gamma|\, U_1(t,s,p,b)}{\varphi_3(\alpha)-\varphi_3(\beta)}, & 0\le|h(\mu)|\le\dfrac{1}{4(\varphi_3(\alpha)-\varphi_3(\beta))},\\[2ex] 4|\gamma|\, U_1(t,s,p,b)\,|h(\mu)|, & |h(\mu)|\ge\dfrac{1}{4(\varphi_3(\alpha)-\varphi_3(\beta))}.\end{cases} \tag{2.25}$$

Therefore, (2.24) can be easily obtained from (1.3) and (2.25). This evidently completes the proof of Theorem 2.6.

Putting $\alpha=\beta+1$ and $p=1$ in Theorem 2.6, we get the following corollaries.

**Corollary 2.7.** *Let* $\beta<1$, $0<r<p\le 1$, $\alpha\in\mathbb{C}\setminus\{0\}$ *and* $\frac{1}{2}<t<1$. *If the function* $f(z)$ *given by* (1.1) *belongs to* $\mathcal{B}_\Sigma^{\beta+1,\beta}(\gamma,t,s,p,b)$, *then*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{(1-\alpha)(2-\beta)(3-\beta)|\gamma|(p+b)t}{12},\\[1ex] \qquad if\ |\mu-1|\le\dfrac{(1-\alpha)(2-\beta)(3-\beta)|\zeta_\beta^{\beta+1}(\gamma,t,s,p,b)|}{12(p+b)^2|\alpha|t^2},\\[2ex] \dfrac{(p+b)^3|\gamma|^2|1-\mu|t^3}{|\zeta_\beta^{\beta+1}(\gamma,t,s,p,b)|},\\[1ex] \qquad if\ |\mu-1|\ge\dfrac{(1-\alpha)(2-\beta)(3-\beta)|\zeta_\beta^{\beta+1}(\gamma,t,s,p,b)|}{12(p+b)^2|\alpha|t^2},\end{cases} \tag{2.26}$$

*where* $\zeta_\beta^{\beta+1}(\gamma, t, s, p, b)$ *is given by* (2.22).

**Corollary 2.8.** *Let* $\beta < \alpha < 2$, $0 < r < 1$, $\gamma \in \mathbb{C} \setminus \{0\}$ *and* $\frac{1}{2} < t < 1$. *If the function* $f(z)$ *given by* $(1.1)$ *belongs to* $\mathcal{B}_\Sigma^{\alpha,\beta}(\lambda, t, s, 1, b)$*, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)|\gamma|(1+b)t}{6(\alpha-\beta)(5-\alpha-\beta)}, \\ \qquad if\ |\mu - 1| \leq \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)|\vartheta_\beta^\alpha(\gamma,t,s,b)|}{6(\alpha-\beta)(5-\alpha-\beta)(1+b)^2|\alpha|t^2}, \\ \dfrac{(1+b)^3|\gamma|^2|1-\mu|t^3}{|\vartheta_\beta^\alpha(\gamma,t,s,b)|}, \\ \qquad if\ |\mu - 1| \geq \dfrac{(2-\alpha)(2-\beta)(3-\alpha)(3-\beta)|\vartheta_\beta^\alpha(\gamma,t,s,b)|}{6(\alpha-\beta)(5-\alpha-\beta)(1+b)^2|\alpha|t^2}, \end{cases} \tag{2.27}$$

*where* $\vartheta_\beta^\alpha(\gamma, t, s, b)$ *is given by* (2.23).

**Remark 2.9.** Taking $\lambda = 0$, $\alpha = b = 1$ and $s = -1$ in Corollary 2.8, we get a recent result due to Altinkaya and Yalçin [12, Corollary 10].

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