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Some Inequalities in Fuzzy 2-inner Product Spaces

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Abstract

In this paper, we study Felbin-type fuzzy inner product spaces, where the inner product is a real-valued fuzzy number. Some inequalities of basic 2-inner product spaces are framed in fuzzy 2-inner product spaces. Results on the refinement of Cauchy-Schwarz's inequality are proved in fuzzy 2-inner product spaces. Also we establish some inequalities in fuzzy 2-inner product spaces in connection to Aczel's inequality.

Keywords: Fuzzy real numbers, fuzzy inner product, fuzzy 2-inner product.
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1. INTRODUCTION

The idea of basic 2- inner product spaces, has been initiated by Diminnie, Gähler and White in [5], [6] and rigorous study of various properties and inequalities based on 2-inner product spaces were investigated in [2], [3], [4], [8]. A. Hasankhani et al [1] introduced the concept of Felbin-type fuzzy inner product space. A new version of Cauchy-Schwarz's inequality in fuzzy inner product space can be found in [9]. In this paper, we have generalized the concept of basic 2-inner product and fuzzy inner product to fuzzy 2-inner product and studied behaviour of various inequalities of [3] in fuzzy 2-inner product spaces.

2. PRELIMINARIES AND NOTATIONS

Definition 2.1. [1] A mapping $\eta : \mathbb{R} \to [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:

1. there exist $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.
2. for each $\alpha \in (0,1]$, there exist real numbers $-\infty < \eta_\alpha^- \leq \eta_\alpha^+ < +\infty$ such that the α-level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^+(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. The real number $\eta_\alpha^- \geq 0$ for all $\eta \in F^+(\mathbb{R})$ and $\alpha \in (0,1]$.

Since each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1, \; if\, t = r \\ 0, \; if\, t \neq r \end{cases} \tag{2.1}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$.
Also α-level set of $\tilde{r}$ is given by $[\tilde{r}]_\alpha = [r, r]$, $0 < \alpha \leq 1$.

Lemma 2.2. [1] *Let* $\eta, \gamma \in F(\mathbb{R})$ *and* $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$. *Then for all* $\alpha \in (0,1]$,

$$\begin{aligned}
&[\eta \oplus \gamma]_\alpha = [\eta_\alpha^- + \gamma_\alpha^-, \eta_\alpha^+ + \gamma_\alpha^+], \\
&[\eta \ominus \gamma]_\alpha = [\eta_\alpha^- - \gamma_\alpha^+, \eta_\alpha^+ - \gamma_\alpha^-]. \\
&[\eta \otimes \gamma]_\alpha = [\eta_\alpha^- \gamma_\alpha^-, \eta_\alpha^+ \gamma_\alpha^+], \forall\, \eta, \gamma \in F^+(\mathbb{R}), \\
&[\tilde{1} \oslash \eta]_\alpha = \left[\frac{1}{\eta_\alpha^+}, \frac{1}{\eta_\alpha^-}\right], \; \textit{if}\; \eta_\alpha^- > 0, \\
&[|\eta|]_\alpha = [\max(0, \eta_\alpha^-, -\eta_\alpha^+), \max(|\eta_\alpha^-|, |\eta_\alpha^+|)].
\end{aligned}$$

Definition 2.3. [1] Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. Define a partial ordering by $\eta \preceq \gamma$ in $F(\mathbb{R})$ if and only if $\eta_\alpha^- \leq \gamma_\alpha^-$ and $\eta_\alpha^+ \leq \gamma_\alpha^+$, for all $\alpha \in (0,1]$.

Remark 2.4. Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. By above definition, if $\eta_\alpha^- = \gamma_\alpha^-$ and $\eta_\alpha^+ = \gamma_\alpha^+$ then $\eta = \gamma$ and vice versa.

Definition 2.5. [1] For a non-negative fuzzy real number η we define $\sqrt{\eta} = \gamma$ where $[\gamma]_\alpha = [\sqrt{\eta_\alpha^-}, \sqrt{\eta_\alpha^+}]$, $\alpha \in (0,1]$.

Lemma 2.6. [1] *Let* $\eta \in F^+(\mathbb{R})$ *and* $\gamma \in F(\mathbb{R})$. *Then*

1. $(\sqrt{\eta})^2 = \eta$,
2. $\gamma \le |\gamma|$.

Lemma 2.7. *Let* X *be a fuzzy* 2*-inner product space, then*

1. $\langle x + ru, y + rv \,|z\rangle^-_\alpha$
$$= \begin{cases} \langle x, y \,|\, z\rangle^-_\alpha + r\langle x, v \,|\, z\rangle^-_\alpha + r\langle y, u \,|\, z\rangle^-_\alpha + r^2\langle u, v \,|\, z\rangle^-_\alpha, & \text{if } r \ge 0\,; \\ \langle x, y \,|\, z\rangle^-_\alpha + r\langle x, v \,|\, z\rangle^+_\alpha + r\langle y, u \,|\, z\rangle^+_\alpha + r^2\langle u, v \,|\, z\rangle^-_\alpha, & \text{if } r < 0. \end{cases}$$
2. $\langle x + ru, y + rv \,|z\rangle^+_\alpha$
$$= \begin{cases} \langle x, y \,|\, z\rangle^+_\alpha + r\langle x, v \,|\, z\rangle^+_\alpha + r\langle y, u \,|\, z\rangle^+_\alpha + r^2\langle u, v \,|\, z\rangle^+_\alpha, & \text{if } r \ge 0\,; \\ \langle x, y \,|\, z\rangle^+_\alpha + r\langle x, v \,|\, z\rangle^-_\alpha + r\langle y, u \,|\, z\rangle^-_\alpha + r^2\langle u, v \,|\, z\rangle^+_\alpha, & \text{if } r < 0. \end{cases}$$

for all $x, y, z, u, v \in X$ *and* $\alpha \in (0, 1]$.

Proof. Let $[\langle x + ru, y + rv \,|z\rangle]_\alpha = [\langle x + ru, y + rv \,|z\rangle^-_\alpha, \langle x + ru, y + rv \,|z\rangle^+_\alpha]$. For $r \ge 0$, we have

$$\begin{aligned} [\langle x + ru, y + rv \,|z\rangle]_\alpha =& [\langle x, y \,|\, z\rangle \oplus \langle x, rv \,|\, z\rangle \oplus \langle ru, y \,|\, z\rangle \oplus \langle ru, rv \,|\, z\rangle]_\alpha \\ =& [\langle x, y \,|\, z\rangle^-_\alpha, \langle x, y \,|\, z\rangle^+_\alpha] + [r\langle x, v \,|\, z\rangle^-_\alpha, r\langle x, v \,|\, z\rangle^+_\alpha] \\ & + [r\langle y, u \,|\, z\rangle^-_\alpha, r\langle y, u \,|\, z\rangle^+_\alpha] + [r^2\langle u, v \,|\, z\rangle^-_\alpha, r^2\langle u, v \,|\, z\rangle^+_\alpha] \\ =& [\langle x, y \,|\, z\rangle^-_\alpha + r\langle x, v \,|\, z\rangle^-_\alpha + r\langle y, u \,|\, z\rangle^-_\alpha + r^2\langle u, v \,|\, z\rangle^-_\alpha, \\ & \langle x, y \,|\, z\rangle^+_\alpha + r\langle x, v \,|\, z\rangle^+_\alpha + r\langle y, u \,|\, z\rangle^+_\alpha + r^2\langle u, v \,|\, z\rangle^+_\alpha]. \end{aligned}$$

And for $r < 0$, we have

$$\begin{aligned} [\langle x + ru, y + rv \,|z\rangle]_\alpha =& [\langle x, y \,|\, z\rangle^-_\alpha, \langle x, y \,|\, z\rangle^+_\alpha] + [r\langle x, v \,|\, z\rangle^+_\alpha, r\langle x, v \,|\, z\rangle^-_\alpha] \\ & + [r\langle y, u \,|\, z\rangle^+_\alpha, r\langle y, u \,|\, z\rangle^-_\alpha] + [r^2\langle u, v \,|\, z\rangle^-_\alpha, r^2\langle u, v \,|\, z\rangle^+_\alpha] \\ =& [\langle x, y \,|\, z\rangle^-_\alpha + r\langle x, v \,|\, z\rangle^+_\alpha + r\langle y, u \,|\, z\rangle^+_\alpha + r^2\langle u, v \,|\, z\rangle^-_\alpha, \\ & \langle x, y \,|\, z\rangle^+_\alpha + r\langle x, v \,|\, z\rangle^-_\alpha + r\langle y, u \,|\, z\rangle^-_\alpha + r^2\langle u, v \,|\, z\rangle^+_\alpha]. \end{aligned}$$

□

Corollary 2.8. *By Lemma 2.7, we have* $\langle x, -y \,|\, z\rangle^+_\alpha = -\langle x, y \,|\, z\rangle^-_\alpha$ *and* $\langle x, -y \,|\, z\rangle^-_\alpha = -\langle x, y \,|\, z\rangle^+_\alpha$.

Definition 2.9. Let X be a vector space over $\mathbb{R}$ and $\dim(X) > 1$. A fuzzy 2-norm on X is a mapping $\|\cdot,\cdot\| : X \times X \to F(\mathbb{R})$ such that for all vectors $x, y, z \in X$, $r \in \mathbb{R}$ and $\alpha \in (0,1]$, we have:

NI.1 $\|x+y, z\| \ \preceq\ \|x,z\| \ \oplus\ \|y,z\|$;

NI.2 $\|rx, z\| \ =\ |\tilde{r}| \ \otimes \|x,z\|$ for all $r \in \mathbb{R}$;

NI.3 $\|x,z\| \ =\ \|z,x\|$;

NI.4 $\|x,z\| \succeq \tilde{0}$;

NI.5 $\|x,z\| = \tilde{0}$ if and only if x and z are linearly dependent;

NI.6 $\inf_{\alpha\in(0,1]} \|x,z\|_{\alpha}^{-} > 0$, if x and z are linearly independent.

Then the vector space X equipped with this fuzzy 2-norm $\|\cdot,\cdot\|$ is called a fuzzy 2-normed space over the field of real numbers $\mathbb{R}$.

Definition 2.10. Let n be a natural number greater than 1 and X be a vector space over $\mathbb{R}$ and $\dim(X) \geq n$. A fuzzy 2-inner product on X is a mapping $\langle\cdot,\cdot|\cdot\rangle : X \times X \times X \to F(\mathbb{R})$ such that for all vectors $x, y, z, x' \in X$, $r \in \mathbb{R}$ and $\alpha \in (0,1]$, we have:

D1) $\langle x+x', y\,|z\rangle \ =\ \langle x, y\,|z\rangle \ \oplus\ \langle x', y\,|z\rangle$;

D2) $\langle rx, y\,|z\rangle \ =\ \tilde{r} \ \otimes \langle x, y\,|z\rangle$ for all $r \in \mathbb{R}$;

D3) $\langle x, y\,|z\rangle \ =\ \langle y, x|z\rangle$;

D4) $\langle x, x\,|z\rangle \ =\ \langle z, z|x\rangle$;

D5) $\langle x, x\,|z\rangle \succeq \tilde{0}$;

D6) $\langle x, x\,|z\rangle = \tilde{0}$ if and only if x, z are linearly dependent.

D7) $\inf_{\alpha\in(0,1]} \langle x, x\,|z\rangle_{\alpha}^{-} > 0$, if x, z are linearly independent.

Then the vector space X equipped with this fuzzy 2-inner product $\langle\cdot,\cdot|\cdot\rangle$ is called a fuzzy 2-inner product space over the field of real numbers $\mathbb{R}$. We denote the α-cut of $\langle\cdot,\cdot|\cdot\rangle$ by $[\langle\cdot,\cdot|\cdot\rangle]_{\alpha} = [\langle\cdot,\cdot|\cdot\rangle_{\alpha}^{-}, \langle\cdot,\cdot|\cdot\rangle_{\alpha}^{+}]$. Note that $\langle\cdot,\cdot|\cdot\rangle_{\alpha}^{-}$ and $\langle\cdot,\cdot|\cdot\rangle_{\alpha}^{+}$ do not satisfy the property of homogeneity for crisp inner product in general.

For any non-zero vectors $x_1, x_2, ..., x_n$ in X, let $V(x_1, x_2, ..., x_n) = \text{span}\{x_1, x_2, ..., x_n\}$ denote the subspace of X, where $x_1, x_2, ..., x_n$ are linearly independent.

Theorem 2.11. *Let $(X, \langle\cdot,\cdot|\cdot\rangle)$ be a fuzzy 2-inner product space and $x, y, z \in X$. If x, z are linearly dependent, then $\langle x, y|z\rangle = \tilde{0}$.*

Proof.
Case 1. y, z are linearly independent. Consider the vector $u = \alpha x - \beta y$, where $\alpha = \langle y, y|z\rangle_\alpha^-$ and $\beta = \langle x, y|z\rangle_\alpha^+$. Now

$$\begin{aligned}0 &\leq \langle u, u|z\rangle_\alpha^- \\ &= \langle \alpha x - \beta y, \alpha x - \beta y|z\rangle_\alpha^- \\ &= \alpha^2\langle x, x|z\rangle_\alpha^- - 2\alpha\beta\langle x, y|z\rangle_\alpha^+ + \beta^2\langle y, y|z\rangle_\alpha^- \\ &= \langle y, y|z\rangle_\alpha^- \left[\langle y, y|z\rangle_\alpha^-\langle x, x|z\rangle_\alpha^- - (\langle x, y|z\rangle_\alpha^+)^2\right] \\ &= -\langle y, y|z\rangle_\alpha^-(\langle x, y|z\rangle_\alpha^+)^2\end{aligned}$$

Since y, z are linearly independent it follows that $\langle y, y|z\rangle_\alpha^- > 0$ and thus $\langle x, y|z\rangle_\alpha^+ = 0$.

Case 2. y, z are linearly dependent. Then also $x + y, z$ are linearly dependent. Because $\langle x, x|z\rangle_\alpha^+ = 0$, $\langle y, y|z\rangle_\alpha^+ = 0$ and $\langle x + y, x + y|z\rangle_\alpha^+ = 0$, from the relation $\langle x + y, x + y|z\rangle_\alpha^+ = \langle x, x|z\rangle_\alpha^+ + 2\langle x, y|z\rangle_\alpha^+ + \langle y, y|z\rangle_\alpha^+$, we get $\langle x, y|z\rangle_\alpha^+ = 0$.

Similarly, using $\langle u, u|z\rangle_\alpha^+ \geq 0$, we can show that $\langle x, y|z\rangle_\alpha^- = 0$ assuming $\alpha = \langle y, y|z\rangle_\alpha^+$ and $\beta = \langle x, y|z\rangle_\alpha^-$. ☐

Let $Y = \text{span}\{z\}$, where z is a non-zero vector and let $X/Y = \{\hat{x} = Y + x : x \in X\}$ be the quotient space. The function $\psi : X/Y \times X/Y \to F(\mathbb{R})$, defined by $\psi(\hat{x}, \hat{y}) = \langle x, y|z\rangle$ is well-defined because if we consider vectors $x, x', y, y' \in X$ such that $(\hat{x'}, \hat{y'}) = (\hat{x}, \hat{y})$, that is $x' - x \in Y$ and $y' - y \in Y$, then

$$\begin{aligned}\psi(\hat{x'}, \hat{y'}) = \langle x', y'|z\rangle &= \langle x' - x + x, y' - y + y|z\rangle \\ &= \langle x' - x,\ y' - y|z\rangle \oplus \langle x' - x,\ y|z\rangle \\ &\quad \oplus \langle x,\ y' - y|z\rangle \oplus \langle x,\ y|z\rangle \\ &= \tilde{0} \oplus \tilde{0} \oplus \tilde{0} \oplus \langle x,\ y|z\rangle (\text{using Theorem 2.11}) \\ &= \psi(\hat{x}, \hat{y}).\end{aligned}$$

If $(X, \langle\cdot, \cdot| \cdot\rangle)$ is a fuzzy 2-inner product space, then ψ satisfies all the properties of fuzzy inner product and so $(X/Y, \psi)$ is a fuzzy inner product space.

Lemma 2.12. [9] In a fuzzy inner product space $(X, \langle\cdot,\cdot\rangle)$, for vectors x, y and for each $\alpha \in (0, 1]$, we have

$$|\langle x, y\rangle|_\alpha^+ \leq \sqrt{\langle x, x\rangle_\alpha^-}\sqrt{\langle y, y\rangle_\alpha^-}. \tag{2.2}$$

Hence, it holds that

$$|\langle x, y\rangle| \preceq \sqrt{\langle x, x\rangle} \otimes \sqrt{\langle y, y\rangle}. \tag{2.3}$$

Theorem 2.13. *In a fuzzy* 2*-inner product space* $(X, \langle\cdot,\cdot|\cdot\rangle)$*, for any* $x, y, z \in X$ *we have*

$$|\langle x, y|z\rangle|_{\alpha}^{+} \leq \sqrt{\langle x, x|z\rangle_{\alpha}^{-}}\ \sqrt{\langle y, y|z\rangle_{\alpha}^{-}}. \tag{2.4}$$

Proof. Since $(X/Y, \psi)$ is a fuzzy inner product space, therefore by Lemma 2.12 for all $\hat{x}, \hat{y} \in X/Y$, we have

$$\begin{aligned} &|\psi(\hat{x}, \hat{y})|_{\alpha}^{+} \leq \sqrt{\psi^{-}(\hat{x}, \hat{x})} \otimes \sqrt{\psi^{-}(\hat{y}, \hat{y})} \\ \Leftrightarrow\ &|\langle x, y|z\rangle|_{\alpha}^{+} \leq \sqrt{\langle x, x|z\rangle_{\alpha}^{-}}\ \sqrt{\langle y, y|z\rangle_{\alpha}^{-}} \end{aligned}$$

□

Remark 2.14. In any fuzzy 2-inner product space $\big(X, \langle\cdot,\cdot\,|\,\cdot\rangle\big)$ inequality (2.4) implies the Cauchy-Schwarz's inequality $\big|\langle x, y\,|\,z\rangle\big| \preceq \sqrt{\langle x,\ x|\ z\rangle} \otimes \sqrt{\langle y,\ y|\ z\rangle}$. Due to the Cauchy-Schwarz's inequality the fuzzy 2-inner product induces a norm defined by $\|x, z\| = \sqrt{\langle x,\ x|\ z\rangle}$.

3. REFINEMENTS OF CAUCHY-SCHWARZ'S INEQUALITY

Let $\big(X, \langle\cdot,\cdot\,|\,\cdot\rangle\big)$ denote a fuzzy 2-inner product space with $\|x, z\| = \sqrt{\langle x, x\,|\,z\rangle}$ for all $x, z \in X$.

Theorem 3.1. *Let* $x, y, z, u, v \in X$ *with* $z \notin V(x, y, u, v)$ *be such that* $\big(\|u, z\|_{\alpha}^{-}\big)^{2} \leq 2\langle x, u|z\rangle_{\alpha}^{+}$ *and* $\big(\|v, z\|_{\alpha}^{-}\big)^{2} \leq 2\langle y, v|z\rangle_{\alpha}^{+}$*. Then*

$$\big|\langle x-u, y-v|z\rangle\big|_{\alpha}^{+} + n_1 q_1 \leq \|x, z\|_{\alpha}^{-}\ \|y, z\|_{\alpha}^{-}$$

where $n_1 = \big\{2\langle x, u|z\rangle_{\alpha}^{+} - (\|u, z\|_{\alpha}^{-})^{2}\big\}^{\frac{1}{2}}$ *and* $q_1 = \big\{2\langle y, v|z\rangle_{\alpha}^{+} - (\|v, z\|_{\alpha}^{-})^{2}\big\}^{\frac{1}{2}}$.

Proof. We know that

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2 \tag{3.1}$$

for every $m, n, p, q \in \mathbb{R}$.

Now

$$\begin{aligned}\|x-u,z\|^2 &= \langle x-u,x-u|z\rangle\\ &= \langle x,x|z\rangle \ominus \langle x,u|z\rangle \ominus \langle u,x|z\rangle \oplus \langle u,u|z\rangle\\ &= \|x,z\|^2 \ominus \tilde{2} \otimes \langle x,u|z\rangle \oplus \|u,z\|^2 \qquad (3.2)\end{aligned}$$

Therefore,

$$\begin{aligned}\langle x-u,x-u|z\rangle_\alpha^- &= \langle x,x|z\rangle_\alpha^- - 2\langle x,u|z\rangle_\alpha^+ + \langle u,u|z\rangle_\alpha^-\\ &= (\|x,z\|_\alpha^-)^2 - n_1^2 \qquad (3.3)\end{aligned}$$

Similarly,

$$\langle y-v,y-v|z\rangle_\alpha^- = (\|y,z\|_\alpha^-)^2 - q_1^2 \qquad (3.4)$$

Using (3.1), we have

$$\big\{(\|x,z\|_\alpha^-)^2 - n_1^2\big\}\big\{(\|y,z\|_\alpha^-)^2 - q_1^2\big\} \le \big\{\|x,z\|_\alpha^-\|y,z\|_\alpha^- - n_1q_1\big\}^2 \qquad (3.5)$$

Now,

$$\begin{aligned}\Big(|\langle x-u,y-v|z\rangle|_\alpha^+\Big)^2 \le&\big(\|x-u,z\|_\alpha^-\big)^2\big(\|y-v,z\|_\alpha^-\big)^2\\ =&\langle x-u,x-u|z\rangle_\alpha^-\langle y-v,y-v|z\rangle_\alpha^-\\ =&\big\{(\|x,z\|_\alpha^-)^2 - n_1^2\big\}\big\{(\|y,z\|_\alpha^-)^2 - q_1^2\big)\big\}\\ \le&\big\{\|x,z\|_\alpha^-\|y,z\|_\alpha^- - n_1q_1\big\}^2 \qquad (3.6)\end{aligned}$$

Equations (3.3) and (3.4) imply $0 \le n_1 = \big(2\langle x,u|z\rangle_\alpha^+ - (\|u,z\|_\alpha^-)^2\big)^{\frac{1}{2}} \le \|x,z\|_\alpha^-$ and $0 \le q_1 = \big(2\langle y,v|z\rangle_\alpha^+ - (\|v,z\|_\alpha^-)^2\big)^{\frac{1}{2}} \le \|y,z\|_\alpha^-$. So, from (3.6), we have

$$\big|\langle x-u,y-v|z\rangle\big|_\alpha^+ + n_1q_1 \le \|x,z\|_\alpha^-\|y,z\|_\alpha^-$$

This completes the proof.

Remark 3.2. Since $\big|\langle x-u,y-v|z\rangle_\alpha^-\big| \le \big|\langle x-u,y-v|z\rangle\big|_\alpha^- \le \big|\langle x-u,y-v|z\rangle\big|_\alpha^+ \le \|x-u,z\|_\alpha^-\|y-v,z\|_\alpha^-$, therefore the inequality in Theorem 3.1 can be rewritten as

$$\big|\langle x-u,y-v|z\rangle_\alpha^-\big| + n_1q_1 \le \|x,z\|_\alpha^-\,\|y,z\|_\alpha^- \qquad (3.7)$$

$$\text{or, } \big|\langle x-u, y-v|z\rangle\big|_\alpha^- + n_1q_1 \leq \|x,z\|_\alpha^- \, \|y,z\|_\alpha^- \tag{3.8}$$

□

Corollary 3.3. *Let* $x, y, z, e \in X$ *be such that* $\|e,z\|_\alpha^- = 1$ *and* $z \notin V(x,y,e)$*. Then*

$$\big|\langle x,y|z\rangle_\alpha^-\big| \leq \big|\langle x,y|z\rangle_\alpha^- - \langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+\big| + \big|\langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+\big| \leq \|x,z\|_\alpha^-\|y,z\|_\alpha^-$$

Proof. If we put $u = \langle x,e|z\rangle_\alpha^+ e$ and $v = \langle y,e|z\rangle_\alpha^+ e$. Then

$$\begin{aligned} n_1^2 &= 2\langle x,u|z\rangle_\alpha^+ - (\|u,z\|_\alpha^-)^2 \\ &= 2\langle x, \langle x,e|z\rangle_\alpha^+ e|z\rangle_\alpha^+ - \big(\|\langle x,e|z\rangle_\alpha^+ e, z\|_\alpha^-\big)^2 \\ &= 2\big(\langle x,e|z\rangle_\alpha^+\big)^2 - \big(\langle x,e|z\rangle_\alpha^+\big)^2\big(\|e,z\|_\alpha^-\big)^2 \\ &= \big(\langle x,e|z\rangle_\alpha^+\big)^2 \geq 0 \,(\text{ since } \|e,z\|_\alpha^- = 1) \end{aligned} \tag{3.9}$$

Similarly, we get

$$q_1^2 = 2\langle y,v|z\rangle_\alpha^+ - (\|v,z\|_\alpha^-)^2 = \big(\langle y,e|z\rangle_\alpha^+\big)^2 \geq 0 \tag{3.10}$$

And

$$\begin{aligned} \big|\langle x-u, y-v|z\rangle_\alpha^-\big| =& \big|\langle x,y|z\rangle_\alpha^- - \langle x,v|z\rangle_\alpha^+ - \langle u,y|z\rangle_\alpha^+ + \langle u,v|z\rangle_\alpha^-\big| \\ =& \big|\langle x,y|z\rangle_\alpha^- - \langle x, \langle y,e|z\rangle_\alpha^+ e|z\rangle_\alpha^+ - \langle\langle x,e|z\rangle_\alpha^+ e, y|z\rangle_\alpha^+ \\ &+ \langle\langle x,e|z\rangle_\alpha^+ e, \langle y,e|z\rangle_\alpha^+ e|z\rangle_\alpha^-\big| \\ =& \big|\langle x,y|z\rangle_\alpha^- - \langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+ - \langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+ \\ &+ \langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+\|e,z\|_\alpha^-\big| \\ =& \big|\langle x,y|z\rangle_\alpha^- - \langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+\big| \,(\text{ since } \|e,z\|_\alpha^- = 1) \end{aligned} \tag{3.11}$$

Now, combining equation (3.11) and Theorem 3.1, we get

$$\begin{aligned} \big|\langle x,y|z\rangle_\alpha^-\big| &\leq \big|\langle x,y|z\rangle_\alpha^- - \langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+\big| + \big|\langle x,e|z\rangle_\alpha^+\langle y,e|z\rangle_\alpha^+\big| \\ &= \big|\langle x-u, y-v|z\rangle_\alpha^-\big| + n_1q_1 \leq \|x,z\|_\alpha^- \, \|y,z\|_\alpha^-. \end{aligned}$$

□

Corollary 3.4. *Let* $x, y, z \in X$ *be such that* $(\|x,z\|_\alpha^-)^2 \leq 2$ *and* $(\|y,z\|_\alpha^-)^2 \leq 2$*. Then*

$$\begin{aligned} &\big|\langle x,y|z\rangle_\alpha^+\big|^2\big(2 - (\|x,z\|_\alpha^-)^2\big)^{\frac{1}{2}}\big(2 - (\|y,z\|_\alpha^-)^2\big)^{\frac{1}{2}} \\ &+ \big|\langle x,y|z\rangle_\alpha^-\big\{1 + (\langle x,y|z\rangle_\alpha^+)^2)\big\} - \langle x,y|z\rangle_\alpha^+\big\{(\|x,z\|_\alpha^+)^2 + (\|y,z\|_\alpha^+)^2\big\}\big| \end{aligned}$$

$$\le \|x,z\|_\alpha^- \|y,z\|_\alpha^-$$

Proof. Let $u = \langle x,y|z\rangle_\alpha^+ y$ and $v = \langle y,x|z\rangle_\alpha^+ x$.
We know from Remark 3.2 that

$$\big|\langle x-u, y-v|z\rangle_\alpha^-\big| + n_1 q_1 \le \|x,z\|_\alpha^- \|y,z\|_\alpha^- \tag{3.12}$$

where $n_1 = \big\{2\langle x,u|z\rangle_\alpha^+ - (\|u,z\|_\alpha^-)^2\big\}^{\frac{1}{2}}$ and $q_1 = \big\{2\langle y,v|z\rangle_\alpha^+ - (\|v,z\|_\alpha^-)^2\big\}^{\frac{1}{2}}$
Now,

$$\begin{aligned} n_1 &= \big\{2\langle x,u|z\rangle_\alpha^+ - (\|u,z\|_\alpha^-)^2\big\}^{\frac{1}{2}} \\ &= \big\{2\langle x, \langle x,y|z\rangle_\alpha^+ y|z\rangle_\alpha^+ - \big(\|\langle x,y|z\rangle_\alpha^+ y, z\|_\alpha^-\big)^2\big\}^{\frac{1}{2}} \\ &= \langle x,y|z\rangle_\alpha^+ \big\{2 - (\|y,z\|_\alpha^-)^2\big\}^{\frac{1}{2}} \end{aligned} \tag{3.13}$$

Similarly,

$$q_1 = \big\{2\langle y,v|z\rangle_\alpha^+ - (\|v,z\|_\alpha^-)^2\big\}^{\frac{1}{2}} = \langle x,y|z\rangle_\alpha^+ \big\{2 - (\|x,z\|_\alpha^-)^2\big\}^{\frac{1}{2}} \tag{3.14}$$

And

$$\begin{aligned} &\big|\langle x-u, y-v|z\rangle_\alpha^-\big| \\ &= \big|\langle x,y|z\rangle_\alpha^- - \langle x,v|z\rangle_\alpha^+ - \langle y,u|z\rangle_\alpha^+ + \langle u,v|z\rangle_\alpha^-\big| \\ &= \big|\langle x,y|z\rangle_\alpha^- - \langle x, \langle y,x|z\rangle_\alpha^+ x|z\rangle_\alpha^+ - \langle y, \langle x,y|z\rangle_\alpha^+ y|z\rangle_\alpha^+ + \langle \langle x,y|z\rangle_\alpha^+ y, \langle y,x|z\rangle_\alpha^+ x|z\rangle_\alpha^-\big| \\ &= \big|\langle x,y|z\rangle_\alpha^- \big\{1 + (\langle x,y|z\rangle_\alpha^+)^2)\big\} - \langle x,y|z\rangle_\alpha^+ \big\{(\|x,z\|_\alpha^+)^2 + (\|y,z\|_\alpha^+)^2\big\}\big| \end{aligned} \tag{3.15}$$

So, from (3.12), we have

$$\begin{aligned} &\big|\langle x,y|z\rangle_\alpha^+\big|^2 \big(2 - (\|x,z\|_\alpha^-)^2\big)^{\frac{1}{2}} \big(2 - (\|y,z\|_\alpha^-)^2\big)^{\frac{1}{2}} \\ &+ \big|\langle x,y|z\rangle_\alpha^- \big\{1 + (\langle x,y|z\rangle_\alpha^+)^2)\big\} - \langle x,y|z\rangle_\alpha^+ \big\{(\|x,z\|_\alpha^+)^2 + (\|y,z\|_\alpha^+)^2\big\}\big| \\ &\le \|x,z\|_\alpha^- \|y,z\|_\alpha^- \end{aligned}$$

□

Theorem 3.5. *Let* $x,y,z,u,v \in X$ *with* $z \notin V(x,y,u,v)$ *be such that* $(\|u,z\|_\alpha^+)^2 \le 2\langle x,u|z\rangle_\alpha^-$ *and* $(\|v,z\|_\alpha^+)^2 \le 2\langle y,v|z\rangle_\alpha^-$. *Then*

$$\big|\langle x-u, y-v|z\rangle\big|_\alpha^+ + n_2 q_2 \le \|x,z\|_\alpha^+ \, \|y,z\|_\alpha^+,$$

where $n_2 = \big\{2\langle x,u|z\rangle_\alpha^- - (\|u,z\|_\alpha^+)^2\big\}^{\frac{1}{2}}$ *and* $q_2 = \big\{2\langle y,v|z\rangle_\alpha^- - (\|v,z\|_\alpha^+)^2\big\}^{\frac{1}{2}}$.

Proof. From equation (3.2), we have

$$\begin{aligned}\langle x-u, x-u|z\rangle_\alpha^+ &= \langle x,x|z\rangle_\alpha^+ - 2\langle x,u|z\rangle_\alpha^- + \langle u,u|z\rangle_\alpha^+ \\ &= (\|x,z\|_\alpha^+)^2 - n_2^2 \qquad (3.16)\end{aligned}$$

Similarly,

$$\langle y-v, y-v|z\rangle_\alpha^+ = (\|y,z\|_\alpha^+)^2 - q_2^2 \qquad (3.17)$$

Using (3.1), we have

$$\Big\{(\|x,z\|_\alpha^+)^2 - n_2^2\Big\}\Big\{(\|y,z\|_\alpha^+)^2 - q_2^2\Big\} \leq \Big\{\|x,z\|_\alpha^+\|y,z\|_\alpha^+ - n_2q_2\Big\}^2 \qquad (3.18)$$

Now,

$$\begin{aligned}\Big(\big|\langle x-u, y-v|z\rangle\big|_\alpha^+\Big)^2 \leq& \big(\|x-u,z\|_\alpha^+\big)^2\big(\|y-v,z\|_\alpha^+\big)^2 \\ =& \langle x-u,x-u|z\rangle_\alpha^+\langle y-v,y-v|z\rangle_\alpha^+ \\ =& \big\{(\|x,z\|_\alpha^+)^2 - n_2^2\big\}\big\{(\|y,z\|_\alpha^+)^2 - q_2^2\big\} \\ \leq& \Big\{\|x,z\|_\alpha^+\|y,z\|_\alpha^+ - n_2q_2\Big\}^2 \qquad (3.19)\end{aligned}$$

Again from (3.16) and (3.17), we get $0 \leq n_2 = \big(2\langle x,u|z\rangle_\alpha^- - (\|u,z\|_\alpha^+)^2\big)^{\frac{1}{2}} \leq \|x,z\|_\alpha^+$ and $0 \leq q_2 = \big(2\langle y,v|z\rangle_\alpha^- - (\|v,z\|_\alpha^+)^2\big)^{\frac{1}{2}} \leq \|y,z\|_\alpha^+$ so,

$$\big|\langle x-u, y-v|z\rangle\big|_\alpha^+ + n_2q_2 \leq \|x,z\|_\alpha^+\|y,z\|_\alpha^+.$$

This completes the proof.

□

Remark 3.6. Since $\big|\langle x-u,y-v|z\rangle_\alpha^+\big| \leq \big|\langle x-u,y-v|z\rangle\big|_\alpha^+$, the inequality of Theorem 3.5 can be rewritten as

$$\big|\langle x-u, y-v|z\rangle_\alpha^+\big| + n_2q_2 \leq \|x,z\|_\alpha^+\|y,z\|_\alpha^+. \qquad (3.20)$$

Corollary 3.7. *Let* $x,y,z,e \in X$ *be such that* $\|e,z\|_\alpha^+ = 1$ *and* $z \notin V(x,y,e)$*. Then*

$$\big|\langle x,y|z\rangle_\alpha^+\big| \leq \big|\langle x,y|z\rangle_\alpha^+ - \langle x,e|z\rangle_\alpha^-\langle y,e|z\rangle_\alpha^-\big| + \big|\langle x,e|z\rangle_\alpha^-\langle y,e|z\rangle_\alpha^-\big| \leq \|x,z\|_\alpha^+\|y,z\|_\alpha^+$$

Proof. Consider $u = \langle x, e|z\rangle^{-}_{\alpha} e$ and $v = \langle y, e|z\rangle^{-}_{\alpha} e$. Then

$$\begin{aligned} n_2^2 &= 2\langle x, u|z\rangle^{-}_{\alpha} - (\|u, z\|^{+}_{\alpha})^2 \\ &= 2\langle x, \langle x, e|z\rangle^{-}_{\alpha} e|z\rangle^{-}_{\alpha} - \big(\|\langle x, e|z\rangle^{-}_{\alpha} e, z\|^{+}_{\alpha}\big)^2 \\ &= 2\big(\langle x, e|z\rangle^{-}_{\alpha}\big)^2 - \big(\langle x, e|z\rangle^{-}_{\alpha}\big)^2 \big(\|e, z\|^{+}_{\alpha}\big)^2 \\ &= \big(\langle x, e|z\rangle^{-}_{\alpha}\big)^2 \geq 0 \,(\text{ since } \|e, z\|^{+}_{\alpha} = 1) \end{aligned} \tag{3.21}$$

Similarly, we get

$$q_2^2 = \langle y, v|z\rangle^{-}_{\alpha} - (\|v, z\|^{+}_{\alpha})^2 = \big(\langle y, e|z\rangle^{-}_{\alpha}\big)^2 \geq 0 \tag{3.22}$$

And

$$\begin{aligned} \big|\langle x-u, y-v|z\rangle^{+}_{\alpha}\big| =& \big|\langle x, y|z\rangle^{+}_{\alpha} - \langle x, v|z\rangle^{-}_{\alpha} - \langle u, y|z\rangle^{-}_{\alpha} + \langle u, v|z\rangle^{+}_{\alpha}\big| \\ =& \big|\langle x, y|z\rangle^{+}_{\alpha} - \langle x, \langle y, e|z\rangle^{-}_{\alpha} e|z\rangle^{-}_{\alpha} - \langle y, \langle x, e|z\rangle^{-}_{\alpha} e|z\rangle^{-}_{\alpha} \\ &+ \langle \langle x, e|z\rangle^{-}_{\alpha} e, \langle y, e|z\rangle^{-}_{\alpha} e|z\rangle^{+}_{\alpha}\big| \\ =& \big|\langle x, y|z\rangle^{+}_{\alpha} - \langle x, e|z\rangle^{-}_{\alpha}\langle y, e|z\rangle^{-}_{\alpha}\big| \end{aligned} \tag{3.23}$$

Now, combining equations (3.23) and (3.20), we get

$$\big|\langle x, y|z\rangle^{+}_{\alpha}\big| \leq \big|\langle x, y|z\rangle^{+}_{\alpha} - \langle x, e|z\rangle^{-}_{\alpha}\langle y, e|z\rangle^{-}_{\alpha}\big| + \big|\langle x, e|z\rangle^{-}_{\alpha}\langle y, e|z\rangle^{-}_{\alpha}\big| \leq \|x, z\|^{+}_{\alpha}\|y, z\|^{+}_{\alpha}.$$

□

4. AćZEL'S INEQUALITY

In this portion we shall point out some results of 2-inner product space to fuzzy 2-inner product space in connection to Aćzel's inequality.

Theorem 4.1. *Let* $\big(X, \langle\cdot, \cdot|\cdot\rangle\big)$ *be a fuzzy* 2*-inner product space,* $M_1, M_2 \in \mathbb{R}$ *and* $x, y, z \in X$ *such that* $\|x, z\|^{-}_{\alpha} \leq |M_1|$, $\|y, z\|^{-}_{\alpha} \leq |M_2|$. *Then*
$\big\{M_1^2 - (\|x, z\|^{-}_{\alpha})^2\big\}\big\{M_2^2 - (\|y, z\|^{-}_{\alpha})^2\big\} \leq \big\{|M_1 M_2| - |\langle x, y|z\rangle^{+}_{\alpha}|\big\}^2$.

Proof. By Lemma 2.2 and Theorem 2.13, we have

$$\big|\langle x, y|z\rangle^{+}_{\alpha}\big| \leq \big|\langle x, y|z\rangle\big|^{+}_{\alpha} \leq \|x, z\|^{-}_{\alpha}\|y, z\|^{-}_{\alpha} \tag{4.1}$$

Now

$$\|x,z\|_\alpha^- \le |M_1| \Rightarrow \big(\|x,z\|_\alpha^-\big)^2 \le M_1^2 \Rightarrow 0 \le M_1^2 - (\|x,z\|_\alpha^-)^2 \tag{4.2}$$

Similarly,

$$0 \le M_2^2 - (\|y,z\|_\alpha^-)^2 \tag{4.3}$$

and so

$$\begin{aligned} 0 &\le \big\{M_1^2 - (\|x,z\|_\alpha^-)^2\big\}\big\{M_2^2 - (\|y,z\|_\alpha^-)^2\big\} \\ &\le \big\{|M_1M_2| - \|x,z\|_\alpha^-\|y,z\|_\alpha^-\big\}^2 (\text{ using } (3.1)) \\ &\le \big\{|M_1M_2| - |\langle x,y|z\rangle_\alpha^+|\big\}^2 (\text{ using } (4.1)) \end{aligned}$$

□

Corollary 4.2. *Let* $\big(X, \langle \cdot, \cdot | \cdot \rangle\big)$ *be a fuzzy* 2*-inner product space,* $M_1, M_2 \in \mathbb{R}$ *and* $x, y, z \in X$ *such that* $\|x,z\|_\alpha^- \le |M_1|$, $\|y,z\|_\alpha^- \le |M_2|$. *Then*
$\big\{M_1^2 - (\|x,z\|_\alpha^-)^2\big\}\big\{M_2^2 - (\|y,z\|_\alpha^-)^2\big\} \le \big\{|M_1M_2| - \langle x,y|z\rangle_\alpha^+\big\}^2$, *since* $\langle x,y|z\rangle_\alpha^+ \le |\langle x,y|z\rangle_\alpha^+|$.

In fuzzy 2-inner product space the counterpart of Cauchy-Schwarz's inequality can be stated as follows:

Corollary 4.3. *Let* $\big(X, \langle \cdot, \cdot | \cdot \ \rangle\big)$ *be a fuzzy* 2*-inner product space and if* $M > 0$, $x, y, z \in X$ *such that* $\|x,z\|_\alpha^-, \|y,z\|_\alpha^- \le M$. *Then*

$$0 \le \big(\|x,z\|_\alpha^-\big)^2\big(\|y,z\|_\alpha^-\big)^2 - \big(\langle x,y|z\rangle_\alpha^+\big)^2 \le M^2\|x-y,z\|_\alpha^-.$$

Proof. By Corollary 4.2 and taking $M_1 = M_2 = M$, we get

$$\begin{aligned} &\big\{M^2 - (\|x,z\|_\alpha^-)^2\big\}\big\{M^2 - (\|y,z\|_\alpha^-)^2\big\} \le \big\{M^2 - \langle x,y|z\rangle_\alpha^+\big\}^2 \\ &\Rightarrow M^4 - M^2(\|x,z\|_\alpha^-)^2 - M^2(\|y,z\|_\alpha^-)^2 + (\|x,z\|_\alpha^-)^2(\|y,z\|_\alpha^-)^2 \\ &\le M^4 - 2M^2\big(\langle x,y|z\rangle_\alpha^+\big) + \big(\langle x,y|z\rangle_\alpha^+\big)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} 0 &\le \big(\|x,z\|_\alpha^-\big)^2\big(\|y,z\|_\alpha^-\big)^2 - \big(\langle x,y|z\rangle_\alpha^+\big)^2 \\ &\le M^2\big\{(\|x,z\|_\alpha^-)^2 + (\|y,z\|_\alpha^-)^2 - 2\big(\langle x,y|z\rangle_\alpha^+\big)\big\} = M^2\|x-y,z\|_\alpha^- \end{aligned}$$

□

Theorem 4.4. *Let $(X, \langle\cdot,\cdot|\cdot\rangle)$ be a fuzzy 2-inner product space, $M_1, M_2 \in \mathbb{R}$ and $x, y, z \in X$ such that $\|x,z\|_\alpha^- \le |M_1|$, $\|y,z\|_\alpha^- \le |M_2|$. Then*
$\big(|M_1| - \|x,z\|_\alpha^-\big)^{\frac{1}{2}}\big(|M_2| - \|y,z\|_\alpha^-\big)^{\frac{1}{2}} \le |M_1M_2|^{\frac{1}{2}} - |\langle x,y|z\rangle_\alpha^+|^{\frac{1}{2}}$

Proof. Using (3.1) and considering $m = \sqrt{|M_1|}, n = \sqrt{\|x,z\|_\alpha^-}, p = \sqrt{|M_2|}, q = \sqrt{\|y,z\|_\alpha^-}$, we get

$$\begin{aligned}
&\big(|M_1| - \|x,z\|_\alpha^-\big)\big(|M_2| - \|y,z\|_\alpha^-\big) \le \big\{\sqrt{|M_1M_2|} - \sqrt{\|x,z\|_\alpha^-\|y,z\|_\alpha^-}\big\}^2\\
&\Rightarrow\ \big(|M_1| - \|x,z\|_\alpha^-\big)^{\frac{1}{2}}\big(|M_2| - \|y,z\|_\alpha^-\big)^{\frac{1}{2}} \le |M_1M_2|^{\frac{1}{2}} - |\langle x,y|z\rangle_\alpha^+|^{\frac{1}{2}}
\end{aligned}$$

□

Corollary 4.5. *Let $(X, \langle\cdot,\cdot|\cdot\rangle)$ be a fuzzy 2-inner product space and if $M > 0$, $x, y, z \in X$ such that $\|x,z\|_\alpha^-, \|y,z\|_\alpha^- \le M$. Then*
$0 \le \|x,z\|_\alpha^-\|y,z\|_\alpha^- - \big|\langle x,y|z\rangle_\alpha^+\big| \le M\big\{\|x,z\|_\alpha^- + \|y,z\|_\alpha^- - 2\big|\langle x,y|z\rangle_\alpha^+\big|^{\frac{1}{2}}\big\}$

Proof. By Theorem 4.4 and taking $M_1 = M_2 = M$, we get

$$\begin{aligned}
&\{M - \|x,z\|_\alpha^-\}^{\frac{1}{2}}\{M - \|y,z\|_\alpha^-\}^{\frac{1}{2}} \le M - |\langle x,y|z\rangle_\alpha^+|^{\frac{1}{2}}\\
&\Rightarrow M^2 - M\|x,z\|_\alpha^- - M\|y,z\|_\alpha^- + \|x,z\|_\alpha^-\|y,z\|_\alpha^- \le M^2 - 2M|\langle x,y|z\rangle_\alpha^+|^{\frac{1}{2}} + \big|\langle x,y|z\rangle_\alpha^+\big|
\end{aligned}$$

Therefore, we have
$0 \le \|x,z\|_\alpha^-\|y,z\|_\alpha^- - \big|\langle x,y|z\rangle_\alpha^+\big| \le M\big\{\|x,z\|_\alpha^- + \|y,z\|_\alpha^- - 2\big|\langle x,y|z\rangle_\alpha^+\big|^{\frac{1}{2}}\big\}$. □

5. CONCLUSION

The study extended certain inequalities from basic 2-inner product spaces to the context of fuzzy 2-inner product spaces. The investigation delved into the refinement of Cauchy-Schwarz's inequality within this fuzzy context, shedding light on the intricacies of fuzzy 2-inner product spaces. Additionally, a set of inequalities was established in fuzzy 2-inner product spaces, specifically in relation to Aćzel's inequality.

6. FUTURE SCOPE

The inequalities discussed above can be extended to encompass fuzzy n-inner product spaces, offering a broader and more versatile framework. Further exploration can involve refining the triangle inequality specifically for fuzzy 2-norms generated by fuzzy 2-inner products.

Moreover, there is potential for delving into the study of infinite series within the context of fuzzy inner products. Analyzing infinite series in fuzzy inner product spaces can contribute to a more comprehensive understanding of their mathematical properties and applications.

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