Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 20, Number 3 (2024), pp. 423-437

http://www.ripublication.com/gjpam.htm

On $(p,q)^{th}$ generalized relative L-order (α,β) of entire and meromorphic functions

Dibyendu Banerjee[1] **and Ishita Ghosh**[2]

[1]*Department of Mathematics, Visva-Bharati, Santiniketan-731235, India*
[2]*Department of Mathematics, Dr. Bhupendra Nath Dutta Smriti Mahavidyalaya, Hatgobindapur-713407, India*

Abstract

After the works on $(p,q)^{th}$ relative order of entire function with respect to another entire function by Lahiri and Banerjee [13], Banerjee and Jana $\{[3],[4]\}$ studied $(p,q)^{th}$ relative order for meromorphic functions also. Recently Biswas and Biswas [9] studied generalized relative order (α,β). In this paper we attempted to combine the above stated idea of generalized relative order and the idea of L-order given by Somasundaram and Thamizhakasi [20] into $(p,q)^{th}$ generalized relative L-order (α,β). With this definition of order we prove some basic theorems on order which gives the standard results for some particular cases. Further we observe that the finiteness of $(p,q)^{th}$ generalized relative L-order (α,β) is closely connected with the convergence of a certain integral.

Keywords and Phrases: Entire function, Meromorphic function, L-order, $(p,q)^{th}$ relative order, Generalized relative order (α,β).

Mathematics Subject Classification : 30D20, 30D30.

1. INTRODUCTION AND DEFINITIONS

First we consider several types of order starting from classical order of entire and meromorphic functions.

Definition 1.1. Let f be an entire function and $M(r, f) = M_f(r) = \max\{|f(z)| : |z| = r\}$. Then the order of an entire function f is denoted by ρ_f and is defined by

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

When f is meromorphic, then [17]

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

where $T_f(r) = T(r, f)$ is the Nevanlinna characteristic function of f.

Let $L(r)$ is a positive continuous function increasing slowly in the sense of 'Karamata' that is, $L(ar) \sim L(r)$ for every positive constant a. In [20], Somasundaram and Thamizhakasi defined L-order of entire functions.

Definition 1.2. [20] L-order of an entire function f is defined by

$$\rho_L = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log\,[rL(r)]}.$$

In 1963, Sato [18] hinted the definition of generalized order of entire functions.

Definition 1.3. The generalized order of an entire function f is defined by

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r, f)}{\log r}; p \geq 2.$$

When f is meromorphic, then

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r, f)}{\log r}; p \geq 1.$$

Later on, Juneja et. al. [1] introduced $(p, q)^{th}$ order of entire functions.

Definition 1.4. The $(p, q)^{th}$ order of an entire function f is defined by

$$\rho_{p,q} = \limsup_{r\to\infty} \frac{\log^{[p]} M(r, f)}{\log^{[q]} r}; p > q \geq 1.$$

When f is meromorphic, then Bergweiler [5] defined $(p, q)^{th}$ order as

$$\rho_{p,q} = \limsup_{r\to\infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} r}; p \geq q \geq 1.$$

In 1988, Bernal [6] introduced the notion of relative order of an entire function with respect to another entire function.

Definition 1.5. The relative order of an entire function f with respect to another entire function g is defined as

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu); r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log(M_g^{-1}(M_f(r)))}{\log r}.\end{aligned}$$

When f is meromorphic, then the definition of relative order is modified by Lahiri and Banerjee [15] as

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu); r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log(T_g^{-1}(T_f(r)))}{\log r}.\end{aligned}$$

Further when f and g both are meromorphic then the definition of relative order is due to Banerjee [2] as follows.

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < (T_g(r))^\mu; r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log(T_f(r)))}{\log(T_g(r))}.\end{aligned}$$

When $g(z) = e^z$, then the definition of relative order coincides with the classical definition of order.

In 2002, Lahiri and Banerjee [14] introduced the concept of k^{th} generalized relative order.

Definition 1.6. If f and g are entire functions then k^{th} generalized relative order is defined as

$$\begin{aligned}\rho_g^k(f) &= \inf\{\mu > 0 : M_f(r) < M_g(\exp^{[k-1]} r^\mu); r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty} \frac{\log^{[k]}(M_g^{-1}(M_f(r)))}{\log r}\\ &= \limsup_{r\to\infty} \frac{\log^{[k]}(M_g^{-1}(r))}{\log(M_f^{-1}(r))}.\end{aligned}$$

After this in 2005, Lahiri and Banerjee [13] introduced more general type of relative order.

Definition 1.7. If f and g are entire functions then $(p,q)^{th}$ relative order of f with respect to g is defined as [13]

$$\begin{aligned}\rho_{p,q}(f,g) &= \inf\{\mu > 0 : M_f(r) < M_g(\exp^{[p-1]}(\mu \log^{[q]} r)); r > r_0(\mu) > 0\}, p > q \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]}(M_g^{-1}(M_f(r)))}{\log^{[q]} r} \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]}(M_g^{-1}(r))}{\log^{[q]}(M_f^{-1}(r))}, p > q.\end{aligned}$$

Later on, Banerjee and Jana $\{[3],[4]\}$ extended the definition for meromorphic functions.
When f is meromorphic, then $(p,q)^{th}$ relative order is defined by

$$\begin{aligned}\rho_{p,q}(f,g) &= \inf\{\mu > 0 : T_f(r) < T_g(\exp^{[p-1]}(\mu \log^{[q]} r)); r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]}(T_g^{-1}(T_f(r)))}{\log^{[q]} r}, p > q.\end{aligned}$$

Further when f and g both are meromorphic then

$$\begin{aligned}\rho_{p,q}(f,g) &= \inf\{\mu > 0 : T_f(r) < (\exp^{[p-1]}(\mu \log^{[q]} T_g(r))); r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]}(T_f(r))}{\log^{[q]}(T_g(r))}, p > q.\end{aligned}$$

Now let L be a class of continuous non-negative function α on $(-\infty,\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \to \infty$ as $x \to \infty$. For any $\alpha \in L$, we say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to \infty$ for each $c \in (0,\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \leq x \to \infty$ for each $c \in (0,\infty)$. Clearly, $L_2 \subset L_1$.
Considering the above, Sheremeta [19] introduced the concept of generalized order (α,β) of an entire function. For the purpose of further applications, Biswas and Biswas [9] modified the definition of the generalized order (α,β) of entire and meromorphic functions in the following way.

Definition 1.8. [9] Let $\alpha,\beta \in L_1$. The generalized order (α,β) of an entire function is defined as

$$\rho(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha(M_f(r))}{\beta(r)}.$$

The generalized order (α,β) of a meromorphic function is defined as

$$\rho(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \alpha \in L_2, \beta \in L_1.$$

Accordingly generalized relative order is defined as follows.

Definition 1.9. [9] Let $\alpha,\beta \in L_1$. The generalized relative order (α,β) of a meromorphic function f with respect to an entire function g is defined as

$$\rho_g(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha(T_g^{-1}(T_f(r)))}{\beta(r)}.$$

The previous definitions are easily generated as particular cases, e.g. if $g(z) = z$, then Definition 1.9 reduces to Definition 1.8. If $\alpha(r) = \beta(r) = \log r$, then we get the definition of relative order of meromorphic function f with respect to an entire function g introduced by Lahiri and Banerjee [15] and if $g(z) = \exp z$ and $\alpha(r) = \beta(r) = \log r$, then $\rho_g(\alpha,\beta,f) = \rho_f$.

Now we introduce following definitions and using this we prove several results.

Definition 1.10. Let $\alpha,\beta \in L_1$. Then $(p,q)^{th}$ relative order (α,β) of an entire function f with respect to an entire function g is defined by

$$\rho_g^{p,q}(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha(\log^{[p-2]}(M_g^{-1}(M_f(r))))}{\beta(\log^{[q-1]}r)}, p>q.$$

Definition 1.11. Let $\alpha,\beta \in L_1$. Then $(p,q)^{th}$ generalized relative L-order (α,β) of an entire function f with respect to an entire function g is defined as

$$\rho_L^{p,q}(\alpha,\beta,f,g) = \limsup_{r\to\infty} \frac{\alpha(\log^{[p-2]}(M_g^{-1}(M_f(r))))}{\beta(\log^{[q-1]}[rL(r)])}, p>q.$$

When f is meromorphic, then $(p,q)^{th}$ generalized relative L-order (α,β) is defined by

$$\rho_L^{p,q}(\alpha,\beta,f,g) = \limsup_{r\to\infty} \frac{\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))}{\beta(\log^{[q-1]}[rL(r)])}, p>q.$$

Here we give some examples of meromorphic functions whose $(p,q)^{th}$ generalized relative L-order (α,β) varies for different values of p and q.

Example 1.1. Let $f_1(z) = e^{e^z}$, $g(z) = e^z$, $\alpha(r) = \log r$, $\beta(r) = \log\log r$, $L(r) = \log r$. Then $\rho_L^{p,q}(\alpha,\beta,f_1,g)$ may be zero, finite or infinite.
We have $T(r,g) = \dfrac{r}{\pi}$, $T(r,f_1) \sim \dfrac{Ae^r}{r^{\frac{1}{2}}}$, where A is a finite constant. So

$$\rho_L^{p,q}(\alpha,\beta,f_1,g) = \limsup_{r\to\infty} \frac{\log^{[p-1]}\dfrac{\pi Ae^r}{r^{\frac{1}{2}}}}{\log^{[q+1]}[r\log(r)]} = \begin{cases} \infty, & p<q+3 \\ 2, & p=q+3 \\ 0, & p>q+3. \end{cases}$$

Example 1.2. Let $f_2(z) = \dfrac{1}{z-1}$, $g(z) = e^z$, $\alpha(r) = \log r$, $\beta(r) = \log\log r$, $L(r) = \log r$.
Then we have $T(r, g) = \dfrac{r}{\pi}$, $T(r, f_2) = \log r$. So

$$\rho_L^{p,q}(\alpha, \beta, f_2, g) = \limsup_{r\to\infty} \frac{\log^{[p-1]}\pi \log r}{\log^{[q+1]}[r\log (r)]} = \begin{cases} 1, & p = q+1 \\ 0, & p > q+1. \end{cases}$$

2. PROPERTY (A) AND PRELIMINARY RESULTS

In this section we state some results in the form of Lemmas, which will be needed in the subsequent section. First we need the following definition.

Definition 2.1. [6] An entire function g is said to have the property (A) if for any $\sigma > 1$ and for all large r

$$[M_g(r)]^2 \le M_g(r^\sigma)$$

holds.

Examples of entire functions are known [6] which have the property (A) as well as which do not have the property (A).

Lemma 2.1. [6] Let g be an entire function which has the property (A). Then for any positive integer n, for all $\sigma > 1$ and for all large r, the inequality

$$[M_g(r)]^n \le M_g(r^\sigma)$$

holds.

Lemma 2.2. [17] If f is an entire function, then

$$T_f(r) \le \log M_f(r) \le 3T_f(2r), \text{ for all large } r.$$

Lemma 2.3. [16] If $f(z)$ is a transcendental meromorphic function, then

$$T_{f'}(r) \le 2T_f(2r) + O\{T_f(2r)\}, \text{ for all large values of } r.$$

Lemma 2.4. ([12], [21]) Let f be a meromorphic function. Then for all large values of r

$$T_f(r) < C\{T_{f'}(2r) + \log r\}, \text{ where } C \text{ is a constant which is only dependent on } f(0).$$

Lemma 2.5. [3] Let f be a rational meromorphic function, then for all large values of r

$$T_{f'}(r) < T_f(r) + K \log r + O(1), \text{ where } K \text{ is a constant depending on } f \text{ only}.$$

Lemma 2.6. [3] Let f be a transcendental entire and $M_{f'}(r) = \max\{|f'(z)| : |z| = r\}$. Then

$$M_f(r^\lambda) < M_{f'}(r) < M_f(2r), \text{ for } r > 1 \text{ and } \lambda \in (0,1).$$

Lemma 2.7. Let $\alpha, \beta \in L_1$ with β is an increasing function and f be a meromorphic function, g be an entire function. If $\int_{r_0}^{\infty} \frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[rL(r)])\right]^k} dr$ is convergent for some $r_0 > 0$, then

$$\lim_{r\to\infty} \frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[4rL(r)])\right]^k} = 0, \text{ where } 0 < k < \infty.$$

Proof. Given $\epsilon > 0$, there is a number $r'(\epsilon) \geq r_0$ such that for $r > r'(\epsilon) > 1$

$$\int_r^{\infty} \frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(t))))\right)}{\left[\beta(\log^{[q-1]}[tL(t)])\right]^k} dt < \epsilon$$

$$\text{i.e., } \int_r^{2r} \frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(t))))\right)}{\left[\beta(\log^{[q-1]}[tL(t)])\right]^k} dt < \epsilon$$

$$\text{i.e., } \frac{r\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[2rL(2r)])\right]^k} < \epsilon$$

$$\text{i.e., } \frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[2rL(2r)])\right]^k} < \epsilon$$

$$\text{i.e., } \frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[4rL(r)])\right]^k} < \epsilon.$$

Therefore $\lim_{r\to\infty} \frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[4rL(r)])\right]^k} = 0$ and this proves the lemma. □

3. SUM AND PRODUCT THEOREMS

Here we study $(p,q)^{th}$ generalized relative L-order (α,β) of the sum and product of two meromorphic functions with respect to an entire function.

Theorem 3.1. Let $\alpha, \beta \in L_1$ with α and β both are increasing functions and f_1, f_2 be meromorphic functions having $(p,q)^{th}$ generalized relative L-order (α, β) with respect to an entire function g, $\rho_L^{p,q}(\alpha, \beta, f_1, g)$, $\rho_L^{p,q}(\alpha, \beta, f_2, g)$ respectively, where g has the property (A). Then

(i) $\rho_L^{p,q}(\alpha, \beta, f_1 \pm f_2, g) \leq \max\{\rho_L^{p,q}(\alpha, \beta, f_1, g), \rho_L^{p,q}(\alpha, \beta, f_2, g)\}$.

(ii) $\rho_L^{p,q}(\alpha, \beta, f_1 f_2, g) \leq \max\{\rho_L^{p,q}(\alpha, \beta, f_1, g), \rho_L^{p,q}(\alpha, \beta, f_2, g)\}$.

(iii) If $\rho_L^{p,q}(\alpha, \beta, f_1, g) \neq \rho_L^{p,q}(\alpha, \beta, f_2, g),$ then the equality holds in (ii).

(iv) $\rho_L^{p,q}(\alpha, \beta, \frac{f_1}{f_2}, g) \leq \max\{\rho_L^{p,q}(\alpha, \beta, f_1, g), \rho_L^{p,q}(\alpha, \beta, f_2, g)\}$.

(v) If $\rho_L^{p,q}(\alpha, \beta, f_1, g) \neq \rho_L^{p,q}(\alpha, \beta, f_2, g),$ then the equality holds in (iv).

Proof. We may suppose that $\rho_L^{p,q}(\alpha, \beta, f_1, g)$ and $\rho_L^{p,q}(\alpha, \beta, f_2, g)$ both are finite, because if one of $\rho_L^{p,q}(\alpha, \beta, f_1, g), \rho_L^{p,q}(\alpha, \beta, f_2, g)$ or both are infinite, the inequalities are evident. Let

$$\rho_1 = \rho_L^{p,q}(\alpha, \beta, f_1, g), \rho_2 = \rho_L^{p,q}(\alpha, \beta, f_2, g) \text{ and } \rho_1 \leq \rho_2.$$

Then for arbitrary $\epsilon > 0$ and for all large r we have

$$\begin{aligned} T_{f_1}(r) &< T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right) \\ &\leq T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right) \\ \text{and } T_{f_2}(r) &< T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right). \end{aligned}$$

So, for large r

$$\begin{aligned} T_{f_1 \pm f_2}(r) &\leq T_{f_1}(r) + T_{f_2}(r) + O(1) \\ &< 3T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right) \\ &\leq 3\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right), \text{ using Lemma 2.2} \\ &= \frac{1}{3}\log\left[M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right)\right]^9 \\ &\leq \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right)^{\sigma}, \\ &\qquad \text{for every } \sigma > 1, \text{ by Lemma 2.1.} \end{aligned}$$

Since the above inequality is true for any $\sigma > 1$, for simplicity we may choose $\sigma = 2$. Then the above inequality becomes

$$T_{f_1 \pm f_2}(r) \leq \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(2\alpha^{-1}\left((\rho_2+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right)$$

$$< \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right),$$

since α, β are increasing functions

(3.1)

$$\leq T_g\left(2\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right), \text{ by Lemma 2.2}$$
$$\leq T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_2+4\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right).$$

Since ϵ is arbitrary, we obtain

$$\rho_L^{p,q}(\alpha,\beta,f_1\pm f_2,g)\leq \rho_2=\max\{\rho_L^{p,q}(\alpha,\beta,f_1,g),\rho_L^{p,q}(\alpha,\beta,f_2,g)\}$$

which proves (i).

For (ii), since $T_{f_1f_2}(r)\leq T_{f_1}(r)+T_{f_2}(r)$, we obtain similarly as above

$$\rho_L^{p,q}(\alpha,\beta,f_1f_2,g)\leq \max\{\rho_L^{p,q}(\alpha,\beta,f_1,g),\rho_L^{p,q}(\alpha,\beta,f_2,g)\}. \tag{3.2}$$

Now suppose that $\rho_L^{p,q}(\alpha,\beta,f_1,g)\neq\rho_L^{p,q}(\alpha,\beta,f_2,g)$. We show that in (3.2) the equality will hold. For this, let $f=f_1f_2$ and $\rho_L^{p,q}(\alpha,\beta,f_1,g)<\rho_L^{p,q}(\alpha,\beta,f_2,g)$. Then $\rho_L^{p,q}(\alpha,\beta,f,g)\leq\rho_L^{p,q}(\alpha,\beta,f_2,g)$. Again since $f_2=\frac{f}{f_1}$ and $T_{f_1}(r)=T_{\frac{1}{f_1}}(r)+O(1)$, applying (3.2) we have $\rho_L^{p,q}(\alpha,\beta,f_2,g)\leq\max\{\rho_L^{p,q}(\alpha,\beta,f,g),\rho_L^{p,q}(\alpha,\beta,f_1,g)\}$. Since $\rho_L^{p,q}(\alpha,\beta,f_1,g)<\rho_L^{p,q}(\alpha,\beta,f_2,g)$, we have $\rho_L^{p,q}(\alpha,\beta,f_2,g)\leq\rho_L^{p,q}(\alpha,\beta,f,g)$ and so (iii) holds.

Now we prove (iv). Let $f=\frac{f_1}{f_2}$ and suppose $\rho_1\leq\rho_2$. Then $f_1=ff_2$. If possible let $\rho_L^{p,q}(\alpha,\beta,f,g)>\rho_2$. Now applying (3.2) we obtain $\rho_1=\rho_L^{p,q}(\alpha,\beta,f,g)$ and so $\rho_1>\rho_2$, which contradicts our hypothesis. So (iv) holds.

Next suppose $\rho_1<\rho_2$. If possible, let $\rho_L^{p,q}(\alpha,\beta,f,g)<\rho_2$.
Then $\rho_1=\max\{\rho_L^{p,q}(\alpha,\beta,f,g),\rho_2\}=\rho_2$, which is also a contradiction. Therefore (v) is true.
Hence the theorem. □

Example 3.1. Let $f_1(z)=e^{e^z}$, $f_2(z)=\frac{1}{z-1}$, $f=\frac{e^{e^z}}{z-1}$, $g(z)=e^z$, $\alpha(r)=\log r$, $\beta(r)=\log\log r$, $L(r)=\log r$. Then using Theorem 3.1, we get

$$\rho_L^{p,q}(\alpha,\beta,f,g)=\begin{cases}\infty, & p<q+3\\ 2, & p=q+3\\ 0, & p>q+3.\end{cases}$$

If we had considered the classical order we get $\rho_f=\infty$.

4. $(p,q)^{th}$ GENERALIZED RELATIVE L-ORDER (α,β) OF THE DERIVATIVE

Regarding the $(p,q)^{th}$ generalized relative L-order (α,β) of f and its derivative f' with respect to g and g' we prove the following theorems.

Theorem 4.1. Let $\alpha,\beta \in L_1$ with α and β both are increasing functions. Let f be a transcendental meromorphic function and g be an entire function with the property (A). Then $\rho_L^{p,q}(\alpha,\beta,f,g) = \rho_L^{p,q}(\alpha,\beta,f',g)$.

Proof. From Lemmas 2.3 and 2.4, we have for all large r

$$T_{f'}(r) < [K]T_f(2r), \text{ where } K > 1 \tag{4.1}$$

$$\text{and } T_f(r) < [K']T_{f'}(2r), \text{ where } K' > 1. \tag{4.2}$$

Let

$$\rho_1 = \rho_L^{p,q}(\alpha,\beta,f,g), \rho_2 = \rho_L^{p,q}(\alpha,\beta,f',g).$$

Then by Definition 1.11, for arbitrary $\epsilon > 0$ and for all large r we have

$$\begin{aligned} T_f(2r) &< T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+\epsilon)\beta\left(\log^{[q-1]}(2rL(2r))\right)\right)\right)\right) \\ &\le T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right) \end{aligned}$$

Using Lemmas 2.1 and 2.2, for all large r we have from (4.1)

$$\begin{aligned} T_{f'}(r) &< \frac{1}{3}\log\left[M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right)\right]^{3[K]} \\ &\le \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right)^{\sigma}, \text{ for every } \sigma > 1. \end{aligned}$$

Since the above inequality is true for any $\sigma > 1$, for simplicity we may choose $\sigma = 2$. Then the above inequality becomes

$$\begin{aligned} T_{f'}(r) &\le \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(2\alpha^{-1}\left((\rho_1+\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right) \\ &< \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+2\epsilon)\beta\left(\log^{[q-1]}(8rL(r))\right)\right)\right)\right), \\ &\qquad \text{since } \alpha,\beta \text{ are increasing functions} \\ &\le T_g\left(2\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+2\epsilon)\beta\left(\log^{[q-1]}(8rL(r))\right)\right)\right)\right), \text{ by Lemma 2.2} \\ &\le T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\rho_1+4\epsilon)\beta\left(\log^{[q-1]}(16rL(r))\right)\right)\right)\right). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\rho_2 \le \rho_1$.
Similarly using (4.2) we obtain $\rho_1 \le \rho_2$ and this proves the theorem. □

Theorem 4.1 is still true even when f is a rational meromorphic function. Lemma 2.5 gives the assertion.

Theorem 4.2. Let $\alpha, \beta \in L_1$ with α and β both are increasing functions. If f be meromorphic and g be transcendental entire having the property (A), then $\rho_L^{p,q}(\alpha,\beta,f,g) = \rho_L^{p,q}(\alpha,\beta,f,g')$.

Proof. We may suppose that $\rho_L^{p,q}(\alpha,\beta,f,g)$ and $\rho_L^{p,q}(\alpha,\beta,f,g')$ both are finite. Let

$$\rho_1 = \rho_L^{p,q}(\alpha,\beta,f,g) < \mu_1, \rho_2 = \rho_L^{p,q}(\alpha,\beta,f,g') < \lambda_1.$$

Then by Definition 1.11, for arbitrary $\epsilon > 0$ and for all large r similarly like (3.1) we have

$$\begin{aligned}
T_f(r) &< \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\mu_1+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right) \\
&< \frac{1}{3}\log M_{g'}\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\mu_1+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right)^{\frac{1}{\lambda}}, \\
&\qquad \text{where } \lambda \in (0,1), \text{ by Lemma 2.6} \\
&= \frac{1}{3}\log M_{g'}\left(\exp^{[p-2]}\left(\frac{1}{\lambda}\alpha^{-1}\left((\mu_1+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right) \\
&\le \frac{1}{3}\log M_{g'}\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\mu_1+2\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right), \text{ since } \lambda \in (0,1) \\
&\le T_{g'}\left(2\exp^{[p-2]}\left(\alpha^{-1}\left((\mu_1+2\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right), \\
&\qquad \text{using Lemma 2.2, since } g' \text{ is entire} \\
&\le T_{g'}\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\mu_1+4\epsilon)\beta\left(\log^{[q-1]}(8rL(r))\right)\right)\right)\right).
\end{aligned} \tag{4.3}$$

Since $\epsilon > 0$ is arbitrary, we obtain $\rho_2 \le \mu_1$. Now $\mu_1 > \rho_1$ is arbitrary, so we have finally

$$\rho_2 \le \rho_1. \tag{4.4}$$

To obtain the reverse inequality, from Definition 1.11 we have for all large r

$$\begin{aligned}
T_f(r) &< T_{g'}\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right) \\
&\le \log M_{g'}\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right) \\
&\qquad \text{using Lemma 2.2, since } g' \text{ is entire} \\
&< \log M_g\left(2\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+\epsilon)\beta\left(\log^{[q-1]}(rL(r))\right)\right)\right)\right), \text{ by Lemma 2.6}
\end{aligned}$$

$$
\begin{aligned}
&\leq \log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right)\\
&= \frac{1}{3}\log\left[M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right)\right]^3\\
&\leq \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right)^{\sigma},\\
&\qquad\qquad\qquad\qquad \text{for every } \sigma>1, \text{ by Lemma 2.1}
\end{aligned}
$$

Since the above inequality is true for any $\sigma > 1$, for simplicity we may choose $\sigma = 2$. Then the above inequality becomes

$$
\begin{aligned}
T_f(r) &\leq \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(2\alpha^{-1}\left((\lambda_1+2\epsilon)\beta\left(\log^{[q-1]}(2rL(r))\right)\right)\right)\right)\\
&< \frac{1}{3}\log M_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+4\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right),\\
&\qquad\qquad\qquad\qquad \text{since } \alpha,\beta \text{ are increasing functions}\\
&\leq T_g\left(2\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda_1+4\epsilon)\beta\left(\log^{[q-1]}(4rL(r))\right)\right)\right)\right), \text{ by Lemma 2.2}\\
&\leq T_g\left(\exp^{[p-2]}\left(\alpha^{-1}\left((\lambda+4\epsilon)\beta\left(\log^{[q-1]}(8rL(r))\right)\right)\right)\right), \text{ where } \lambda_1<\lambda.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, we obtain $\rho_1 \leq \lambda$. Now $\rho_2 < \lambda$ is arbitrary, so we have finally

$$\rho_1 \leq \rho_2. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\rho_1 = \rho_2$$

and this proves the theorem. □

Starting from the function f' we can similarly show that

$$\rho_L^{p,q}(\alpha,\beta,f',g) = \rho_L^{p,q}(\alpha,\beta,f',g').$$

Thus we have the following theorem.

Theorem 4.3. Let f be meromorphic and g be entire transcendental having the property (A) then

$$\rho_L^{p,q}(\alpha,\beta,f,g) = \rho_L^{p,q}(\alpha,\beta,f',g) = \rho_L^{p,q}(\alpha,\beta,f,g') = \rho_L^{p,q}(\alpha,\beta,f',g').$$

If one or more of the above expression is infinite, then a slight modification of the proof is needed.

5. FINITENESS OF $\rho_L^{p,q}(\alpha,\beta,f,g)$

In this section we show that the $(p,q)^{th}$ generalized relative L-order (α,β) of a meromorphic function f with respect to an entire function g is not greater than a number P defined by certain integral that involves the Nevanlinna characteristics of both functions. As a consequence, $\rho_L^{p,q}(\alpha,\beta,f,g)$ is finite as soon as P is.

Definition 5.1. Let $\alpha,\beta\in L_1$ and let f be meromorphic and g be entire. Then we define the number $P\in[0,\infty]$ as

$$P=\inf\{k\geq 0:\int_{r_0}^{\infty}\frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[rL(r)])\right]^{k}}dr \text{ converges for some } r_0=r_0(k)>0\}.$$

Theorem 5.1. If P is as in Definition 5.1, then $\rho_L^{p,q}(\alpha,\beta,f,g)\leq P$.

Proof. If $P=\infty$ then the result is trivial. Assume now that P is finite. Then for arbitrary $\epsilon(0<\epsilon<1)$, the integral

$$\int_{r_0}^{\infty}\frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[rL(r)])\right]^{P+\epsilon}}dr \text{ is convergent.}$$

Now by Lemma 2.7 we have

$$\lim_{r\to\infty}\frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[4rL(r)])\right]^{P+\epsilon}}=0.$$

So for sufficiently large r, we have

$$\begin{aligned}
&\frac{\exp\left(\alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))\right)}{\left[\beta(\log^{[q-1]}[4rL(r)])\right]^{P+\epsilon}}<\epsilon\\
&\Rightarrow \alpha(\log^{[p-2]}(T_g^{-1}(T_f(r))))<\log\left(\epsilon\left[\beta(\log^{[q-1]}[4rL(r)])\right]^{P+\epsilon}\right)\\
&\qquad=\log\epsilon+(P+\epsilon)\log\left[\beta(\log^{[q-1]}[4rL(r)])\right]\\
&\qquad<\log\epsilon+(P+\epsilon)\beta(\log^{[q-1]}[4rL(r)]).
\end{aligned}$$

Since $\epsilon>0$ is arbitrary, it follows from Definition 1.11 that

$$\rho_L^{p,q}(\alpha,\beta,f,g)\leq P.$$

Hence the theorem. □

6. CONCLUSION

Using this definition of $(p,q)^{th}$ generalized relative L-order (α,β), growth properties of composite entire and meromorphic functions may be studied closely.

REFERENCES

[1] Bajpai, S.K., Juneja, O.P., and Kapor, G.P., On the (p.q)-order and lower (p,q)-order of an entire function, J. Reine Angew. Math. 282 (1976), 53-67.

[2] Banerjee, D., A note on relative order of meromorphic functions, Bull. Cal. Math. Soc., 98(1)(2006), 25-30.

[3] Banerjee, D., Jana, S., Meromorphic functions of relative order (p,q), SOOCHOW JOURNAL OF MATHEMATICS, 33(3) (2007), 343-357.

[4] Banerjee, D., Jana, S., On meromorphic functions of relative order (p,q), Int. Journal of Math. Analysis, 5(28) (2011), 1391-1398.

[5] Bergweiler, W., On the nevanlinna characteristic of a composite function, Complex Variables Theory Appl., 10 (1988), 225-236.

[6] Bernal, L., Orden reletivo de crecimiento de funciones enteras, Collectanea Mathematica, 39 (1988), 209-229.

[7] Biswas, T., Biswas, C. and Saha, B., Sum and product theorems relating to generalized relative order (α,β) and generalized relative type (α,β) of entire functions, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 28(2) (2021), 155-185.

[8] Biswas, T. and Biswas, C., On some growth properties of composite entire functions on the basis of their generalized relative order (α,β), J. of Ramanujan Society of Mathematics and Mathematical Sciences, 9(1) (2021), 11-28.

[9] Biswas, T. and Biswas, C., Some results on generalized relative order (α,β) and generalized relative type (α,β) of meromorphic function with respective to an entire function, Ganita, 70(2) (2020), 239-252.

[10] Biswas, T., Biswas, C. and Biswas, R., A note on generalized growth analysis of composite entire functions, Poincare J. Anal. Appl. 7(2) (2020), 277-286.

[11] Clunie, J., The composition of entire and meromorphic functions, Mathematical Essays dedicated to A. J. Macintyre, Ohio University Press, (1970), 75-92.

[12] Dai, C., Jin, L., Number of deficient values of a class of meromorphic function, Kodia Math. J., 10(1987), 74-82.

[13] Lahiri, B K, Banerjee, D., Entire functions of relative order (p, q), SOOCHOW JOURNAL OF MATHEMATICS, 31(4) (2005), 497-513.

[14] Lahiri, B K, Banerjee, D., Generalized relative order of entire functions, Proc. Nat. Acad. Sci. India, India, 72(A)(IV) (2002), 353-371.

[15] Lahiri, B K, Banerjee, D., Relative order of entire and meromorphic functions, Proc. Nat. Acad. Sci. India Ser. A.,69(A)(III) (1999), 339-354.

[16] Lahiri, I., Generalised order of the derivative of a meromorphic function II, Soochow J. Math., 16(1)(1990), 11-15.

[17] Hayman, W.K., Meromorphic Functions, The Clarendon Press, Oxford (1964).

[18] Sato, D., On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc. 69(1963), 411-414.

[19] Sheremeta, M. N., Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion, Izv. Vyssh. Uchebn. Zaved Mat., 2 (1967), 100-108.

[20] Somasundaram, D., Thamizhakasi, R., A note the entire functions of L-bounded index and L-type, J. Pure Appl. Math., 19(3) (1988), 284-293.

[21] Yang, L., Value Distribution and Its New Research, Beijing 1982.