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A Note on the Distribution of the Riemann Zeta Function Zeros

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Abstract

An exponentially weighted Dirichlet series is used to map the Riemann zeta function zeros to a parabola. The distribution of the zeta function zeros in this mapping is investigated. The Fourier transform is used to analyze the mapping of the zeta function zeros to the parabola.
Keywords: Riemann zeta function, Riemann hypothesis, exponentially weighted Dirichlet series.

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ for $0 < Re(s) < 1$ can be computed from the η function;

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{1}$$

Let $C(n, a, b)$ denote

$$\frac{2 \cdot n^{-a}}{1 - 2^{1-s}} \cdot (\sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j^s} \cdot \cos(b \cdot (\ln(\frac{n}{j})))) \tag{2}$$

where $s = (a, b)$.
A Dirichlet series with exponential terms is

$$D(s) = \sum_{k=1}^{\infty} e^{-ks} \tag{3}$$

where $s = (a, b)$. For $\Re(s) > 0$, the series converges to $e^{-s}/(1 - e^{-s})$. The real part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \cos(kb)$ and the imaginary part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \sin(kb)$.

2. AN APPLICATION OF $C(n, a, b)$

In an example, the $C(n, a, b)$ function is computed for j values of inflection points (where the curve crosses the x-axis from above) that are at least three greater than the previous j value, $n = 500001$, and $s = (0.00005, 500.30908494169051)$ (a zeta function zero when $\Re s = 0.50$). A plot of the logarithms of the j values of the inflection points is

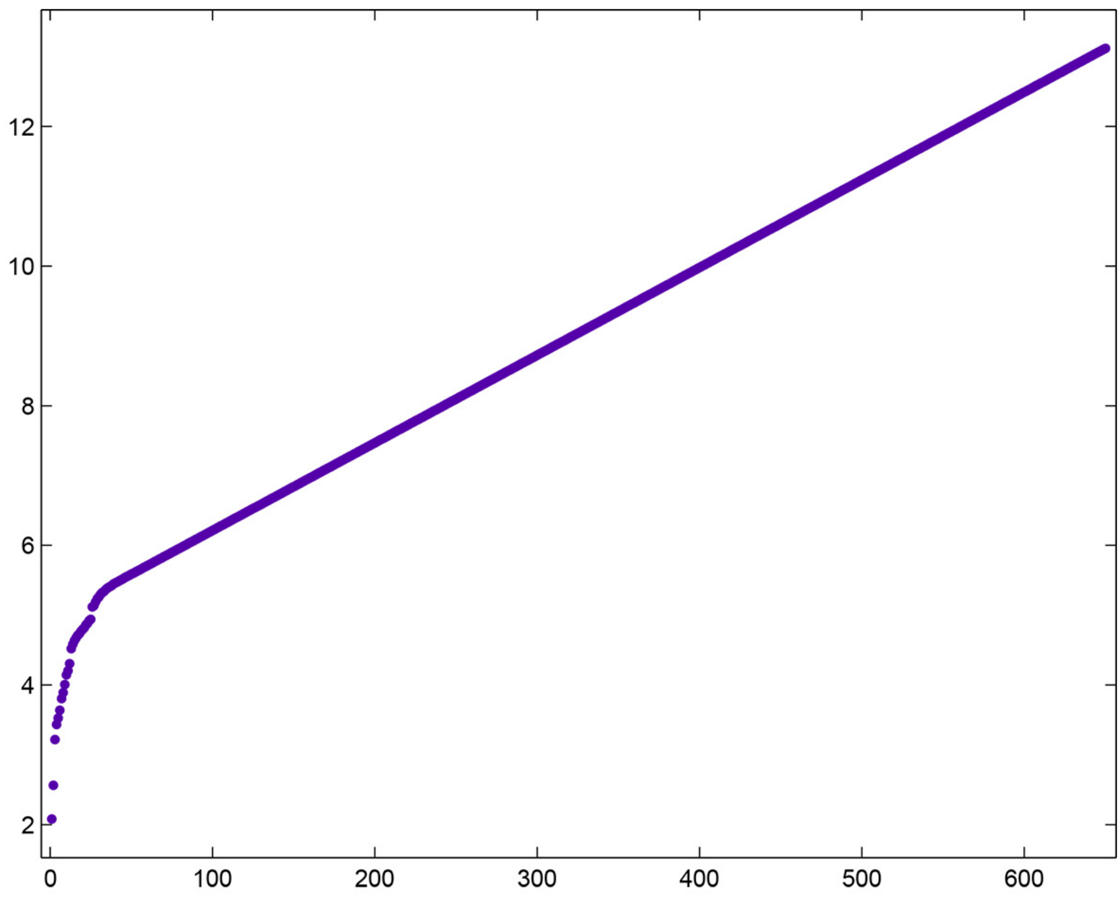

Figure 1

A plot of the real and imaginary parts of the exponentially weighted Dirichlet series computed at these j values versus the logarithms of the j values of the inflection points is

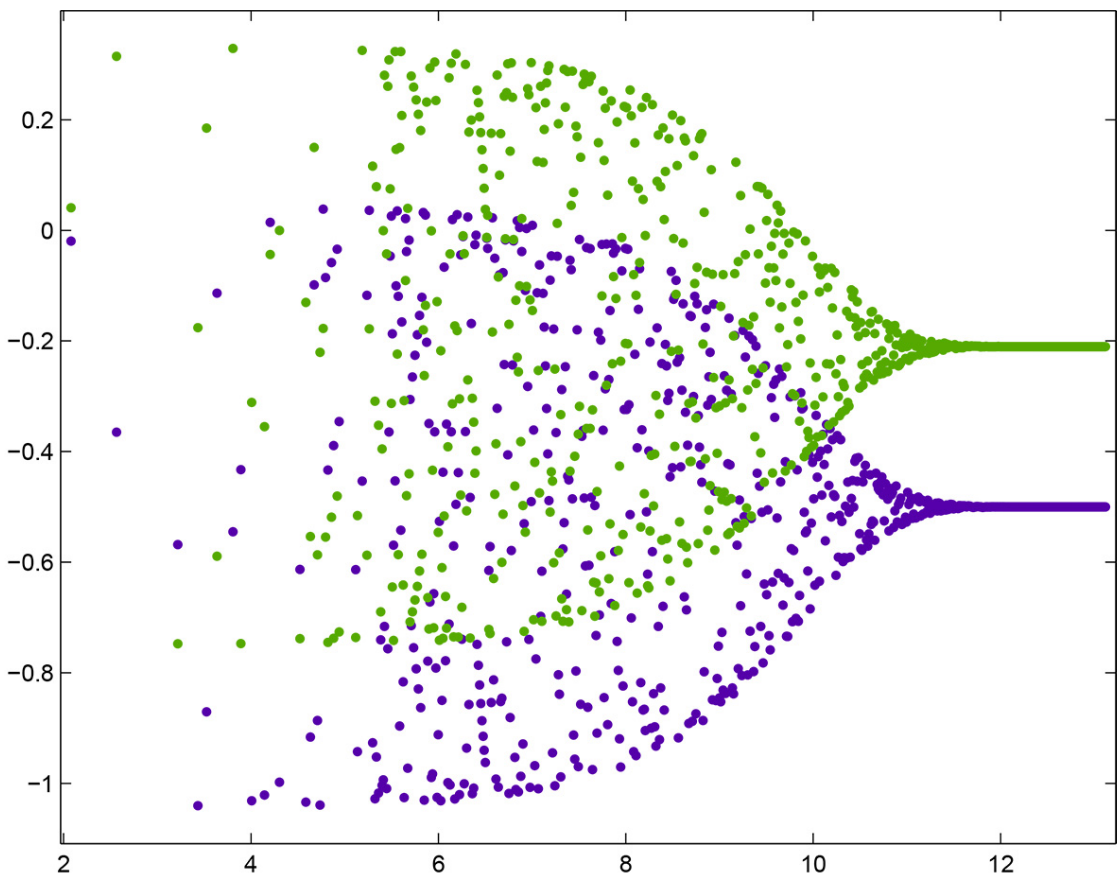

Figure 2

This exponentially weighted Dirichlet series will be denoted by $D(n, a, b)$ to distinguish it from the D series. The series converges to about $(-0.5000, -0.2102)$.

3. THE DISTRIBUTION OF ZEROS IN A MAPPING TO $D(n,a,b)$

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts for the first twenty-five thousand zeta function zeros is

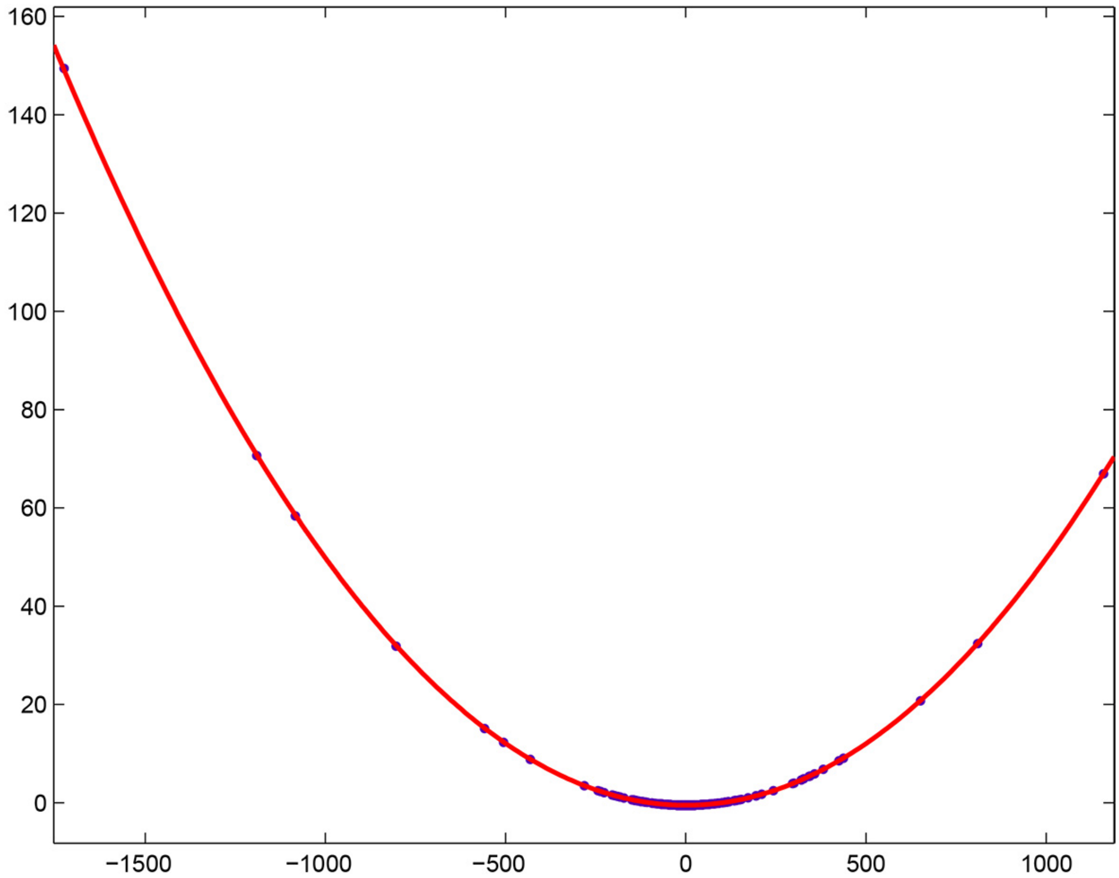

Figure 3

For a quadratic least-squares fit of the curve, $p_1 = 5.026 \cdot 10^{-5}$ with a 95% confidence interval of $(5.026 \cdot 10^{-5}$, $5.026 \cdot 10^{-5})$, $p_2 = -4.204 \cdot 10^{-5}$ with a 95% confidence interval of $(-4.379 \cdot 10^{-5}$, $-4.0282 \cdot 10^{-5})$, $p_3 = -0.50$ with a 95% confidence interval of $(-0.5001$, $-0.5)$, SSE=0.2031, R-squared=1, and RMSE=0.002851. The expected value of the p_1 parameter is the real value of s and the expected value of the p_3 parameter is -0.50. See Caceras [1] for a partial explanation of this.

In a histogram of the differences between 256.0 and the imaginary parts of $D(n,a,b)$ convergents (when the imaginary parts of the $D(n,a,b)$ convergent values are positive and less than 256.0), the last nine histogram values are 43, 73, 97, 131, 210, 293, 655, 1740, and 8814. A plot of the logarithms of these values is

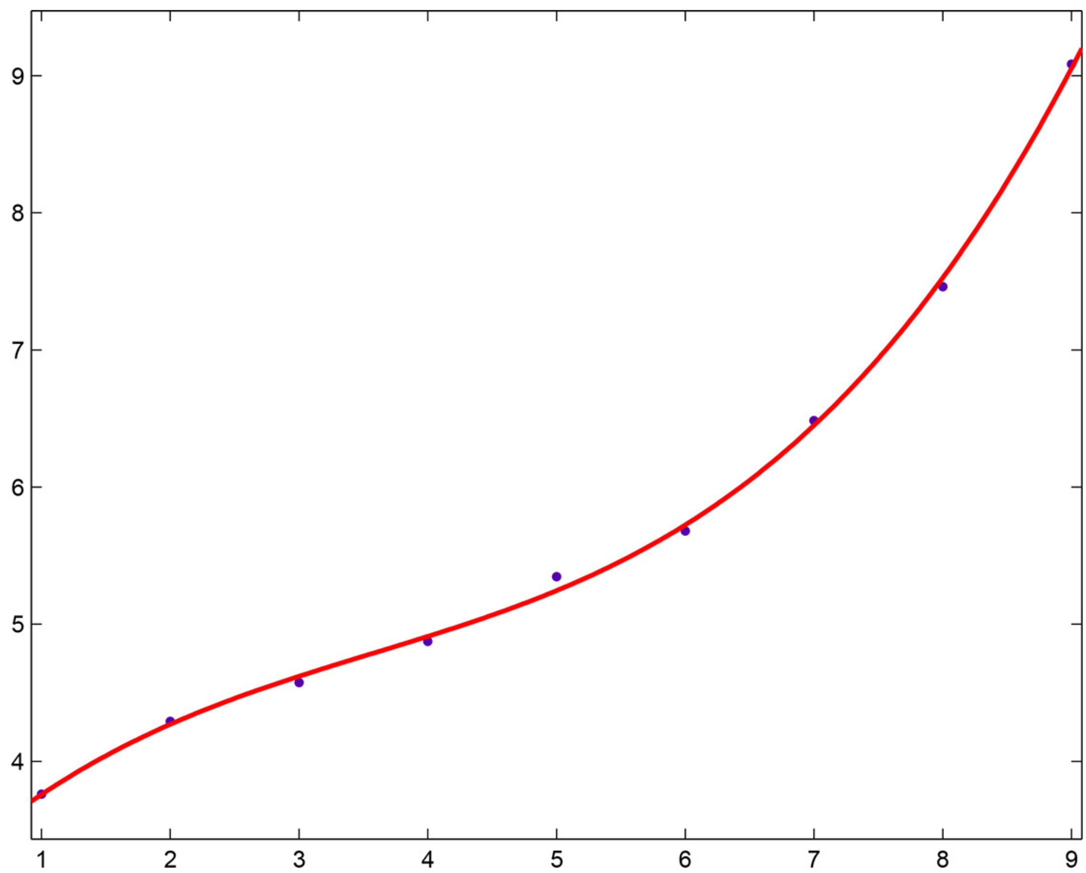

Figure 4

For a cubic least-squares fit of the curve, $p_1 = 0.01704$ with a 95% confidence interval of (0.01249, 0.01216), $p_2 = -0.1831$ with a 95% confidence interval of $(-0.2521, -0.114)$, $p_3 = 0.9418$ with a 95% confidence interval of (0.6368, 1.247), $p_4 = 2.982$ with a 95% confidence interval of (2.61, 3.354), SSE=0.0224, R-squared=0.999, and RMSE=0.06693.

In a histogram of the differences between the imaginary parts of $D(n, a, b)$ convergents and -256.0 (when the imaginary parts of the $D(n, a, b)$ convergent values are negative and greater than -256.0). the last nine histogram values are 52, 65, 90, 141, 185, 312, 635, 1758, and 8798. A plot of the logarithms of these values is

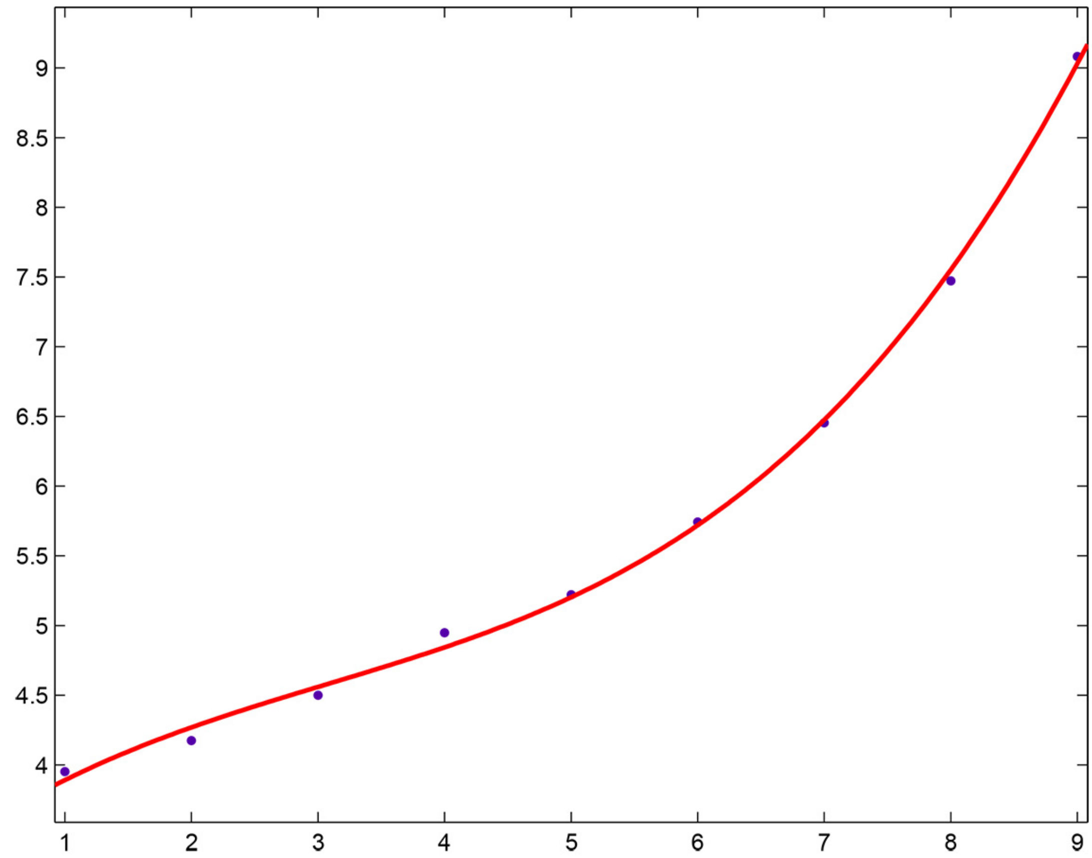

Figure 5

For a cubic least-squares fit of the curve, $p_1 = 0.01364$ with a 95% confidence of (0.007733, 0.01955), $p_2 = -0.1259$ with a 95% confidence interval of $(-0.2155,$ $-0.03645)$, $p_3 = 0.6609$ with a 95% confidence interval of (0.2655, 1.056), $p_4 =$ 3.342 with a 95% confidence of (2.86, 3.824), SSE=0.03763, R-squared=0.9983, and RMSE=0.08675.

There are 24979 imaginary parts of $D(n,a,b)$ convergents between -256.0 and 256.0. A plot of the corresponding real parts of $D(n,a,b)$ convergents versus these imaginary parts is

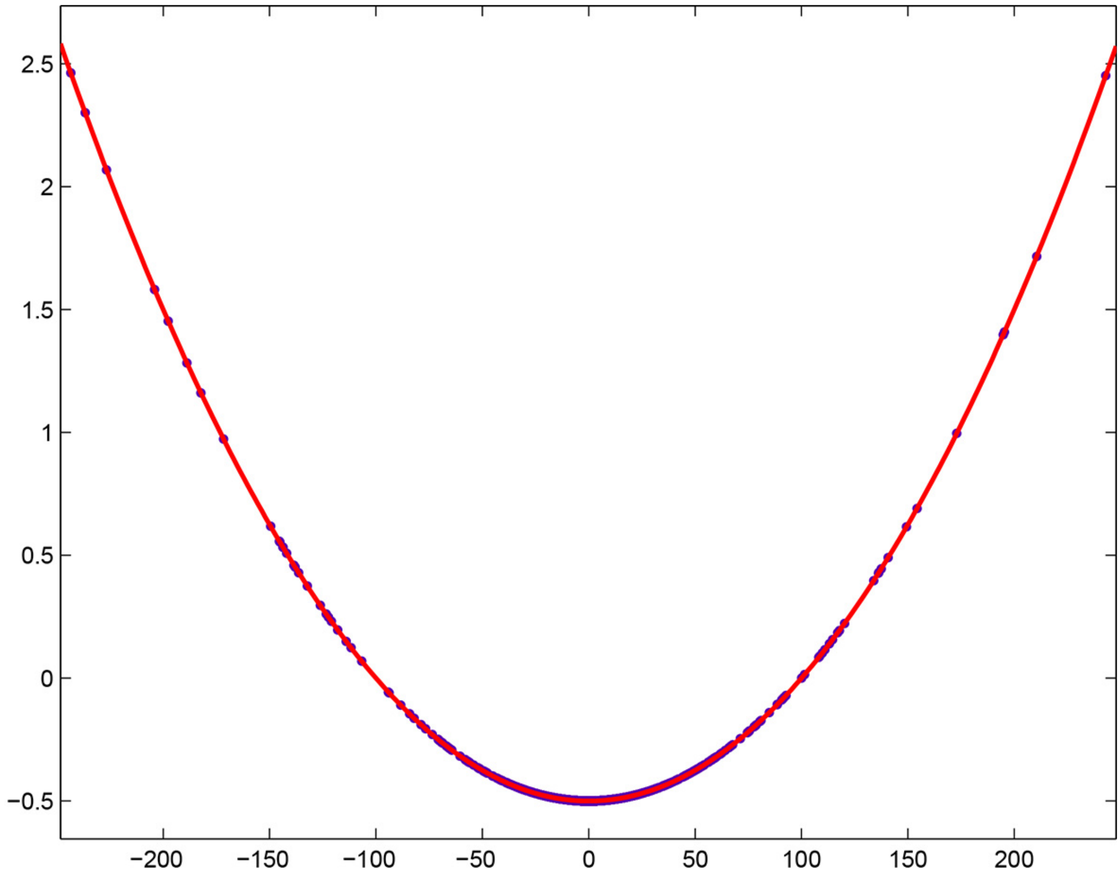

Figure 6

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of ($5.0 \cdot 10^{-5}$, $5.0 \cdot 10^{-5}$), $p_2 = -2.785 \cdot 10^{-8}$) with a 95% confidence interval of ($-3.172 \cdot 10^{-8}$, $-2.399 \cdot 10^{-8}$), $p_3 = -0.5$ with a 95% confidence interval of ($-.5$, $-.5$), SSE=$1.793 \cdot 10^{-7}$, R-squared=1, and RMSE=$2.68 \cdot 10^{-6}$. This curve more closely resembles a parabola.

4. FOURIER TRANSFORM OF PARABOLA

A plot of the Fourier transform of the first one hundred $D(n, a, b)$ convergents of the original parabola is

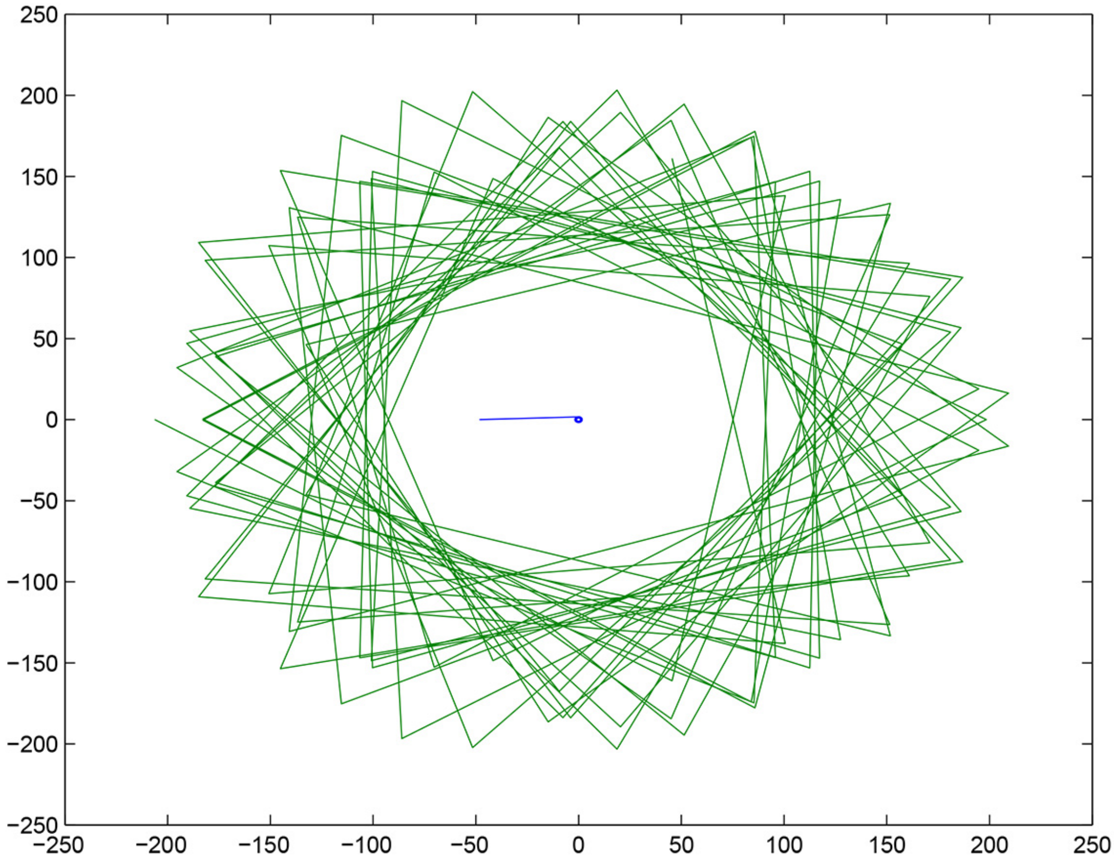

Figure 7

One component starts at about -50 $(100 \cdot -0.50)$ and ends in a small spiral around (0, 0). A plot of the first component of the Fourier transform (complex numbers) is

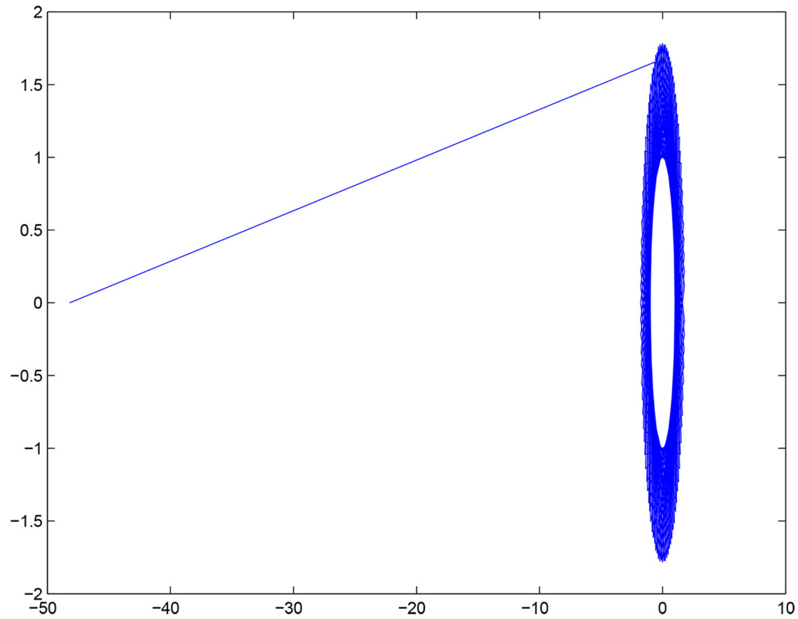

Figure 8

A plot of the real parts of the first component is

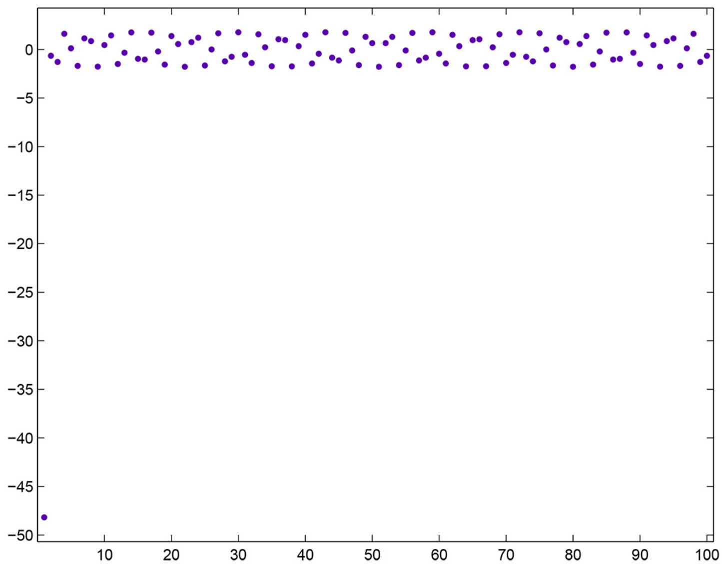

Figure 9

The curve consists of three overlapping sinusoidal waves.

A plot of the first component without the first point is

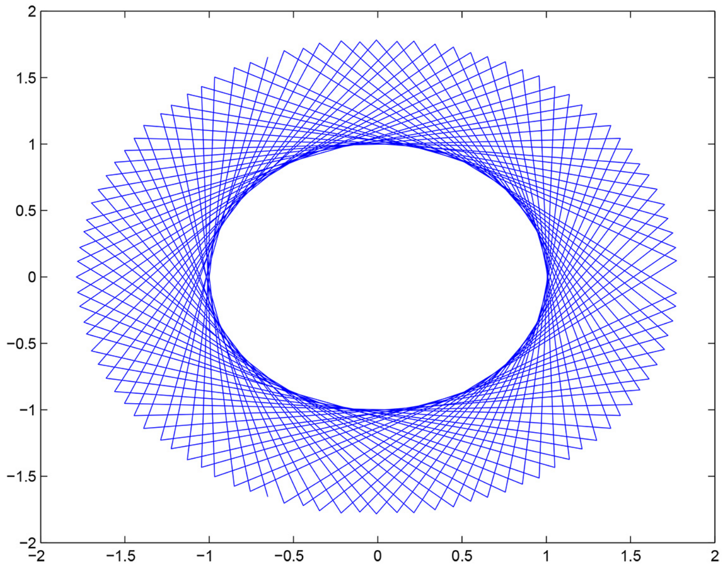

Figure 10

A plot of the real parts of the first component without the first point is

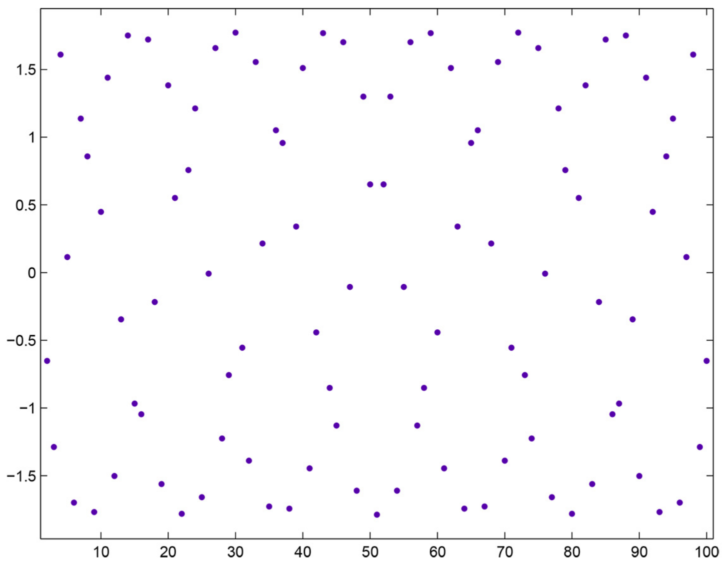

Figure 11

From this perpective, the curve consists of crisscrossing lines.

5. METHODS

```
#include <math.h>
#include <stdio.h>
//
// compute C(n,a,b), 09/20/2024 (dkc)
//
unsigned int max=200001;
double a=0.0001;
double b=14.13472514173470;
//double b=21.02203963877156;
//double b=25.01085758014569;
//double b=30.42487612585951;
//double b=32.93506158773919;
//double b=37.58617815882568;
//double b=40.91871901214750;
//double b=43.32707328091500;
//double b=48.00515088116716;
//double b=49.77383247767230;
//double b=52.97032147771446;
//double b=56.44624769706339;
//double b=59.34704400260235;
//double b=60.83177852460981;
//double b=65.11254404808160;
//double b=67.07981052949417;
//double b=69.54640171117399;
//double b=72.06715767448191;
//double b=75.70469069908393;
//double b=77.14484006887480;
//double b=79.33737502024937;
//double b=84.73549298051705;
//double b=87.42527461312523;
//double b=88.80911120763446;
//double b=92.49189927055849;
//double b=94.65134404051989;
//double b=95.87063422824531;
//double b=98.83119421819369;
//double b=101.31785100573138;
```

```
unsigned int xmin=0; // usually set to 0
unsigned int out=2; // usually 1, 2 for inflection points
unsigned int out3p=1; // usually 0, 1 for differences in j values >=2
unsigned int polar=1; // set to use polar coordinates
void main() {
unsigned int x,oldx;
double sumr,sumi,R,I,temp1,oldsumr,oldsumi,temp,tempa,tempb,y,e,f,g;
double tempr,esumr,esumi;
FILE *Outfp;
Outfp = fopen("c2nab2f.dat","w");
y=1.0-a;
if (y>=0.0)
    temp1=pow((double)2,y);
else {
    temp1=pow((double)2,-y);
    temp1=1.0/temp1;
    }
e=temp1*(cos(b*log(2)));
f=temp1*(sin(b*log(2)));
e=1.0-e;
f=-f;
y=-a;
if (y>=0.0)
    temp1=pow((double)max,y);
else {
    temp1=pow((double)max,-y);
    temp1=1.0/temp1;
    }
y=2.0*temp1;
oldx=0;
sumr=0.0;
sumi=0.0;
oldsumi=0.0;
oldsumr=0.0;
esumr=0.0;
esumi=0.0;
for (x=1; x<=(max-1); x++) {
    tempr=1.0/exp((double)x*a);
```

```
    esumr=esumr+tempr*cos((double)x*b);
    esumi=esumi+tempr*sin((double)x*b);
    temp1=pow((double)x,a);
    R=temp1*cos(b*log((double)x));
    I=temp1*sin(b*log((double)x));
    temp1=R*R+I*I;
    if (x!=(x/2)*2) {
        sumr=sumr+R/temp1;
        sumi=sumi-I/temp1;
        }
    else {
        sumr=sumr-R/temp1;
        sumi=sumi+I/temp1;
        }
    temp=cos(b*log((double)max/(double)x));
    tempa=sumr*temp;
    tempb=sumi*temp;
    tempa=tempa*y;
    tempb=tempb*y;
    g=tempa*e-tempb*f;
    tempb=tempa*f+tempb*e;
    tempa=g;
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp,” %.10llf, %.10llf \n”,tempa,tempb);
        if ((out==2)&&((oldsumr¿0.0)&&(tempa¡0.0))) {
            if (out3p==0)
                fprintf(Outfp,” %d %.10llf %.10llf \n”,x,tempa,tempb);
            if ((x-oldx)>2) {
                printf(” %d %d %.10llf %.10llf \n”,x,oldx,oldsumr,tempa);
                if (out3p!=0) {
                    if (polar==0)
                        fprintf(Outfp,” %d %d \n”,x,oldx);
                    else
                        fprintf(Outfp,”   %d   %d   %.10llf   %.10llf   %.10llf   %.10llf
\n”,x,oldx,
                            sqrt(tempa*tempa+tempb*tempb),atan2(tempb,tempa),esumr,esumi);
                    }
```

```
                }
            oldx=x;
            }
        oldsumr=tempa;
        oldsumi=tempb;
        }
    }
fclose(Outfp);
return;
}
```

REFERENCES

[1] Caceres, P., Ellipse Symmetry in Dirichlet Series: A Duality Between Analytic (Riemann Zeta) and Geometric Forms, DOI 10.13140/RG.2.2.17565.52960