Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 20, Number 3 (2024), pp. 497-517

https://dx.doi.org/10.37622/GJPAM/20.3.2024.497-517

The Hilbert-Pólya Conjecture

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Abstract

Evidence is presented that the Hilbert-Pólya conjecture (postulating that there is a connection between the distribution of non-trival Riemann zeta function zeros and that of the eigenvalues of Hermitian matrices) is true. Square-free numbers similar to colossally abundant numbers are derived using the Möbius function and zeta function zeros (or the eigenvalues of Hermitian matrices).

Keywords: Riemann zeta function, Riemann hypothesis, exponentially weighted Dirichlet series, eigenvalues of Hermitian matrices, Gaussian unitary ensembles, colossally abundant numbers

1. INTRODUCTION

The Riemann zeta function $\zeta(s)$ for $0 < Re(s) < 1$ can be computed from the η function;

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{1}$$

Let $C(n, a, b)$ denote

$$\frac{2 \cdot n^{-a}}{1 - 2^{1-s}} \cdot (\sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j^s} \cdot \cos(b \cdot (\ln(\frac{n}{j})))) \tag{2}$$

where $s = (a, b)$.

A Dirichlet series with exponential terms is

$$D(s) = \sum_{k=1}^{\infty} e^{-ks} \tag{3}$$

where $s = (a, b)$. For $\Re(s) > 0$, the series converges to $e^{-s}/(1 - e^{-s})$. The real part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \cos(kb)$ and the imaginary part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \sin(kb)$.

2. AN APPLICATION OF $C(n, a, b)$

In an example, the $C(n, a, b)$ function is computed for j values of inflection points (where the curve crosses the x-axis from above) that are at least three greater than the previous j value, $n = 500001$, and $s = (0.00005, 500.30908494169051)$ (a zeta function zero when $\Re s = 0.50$). A plot of the logarithms of the j values of the inflection points is

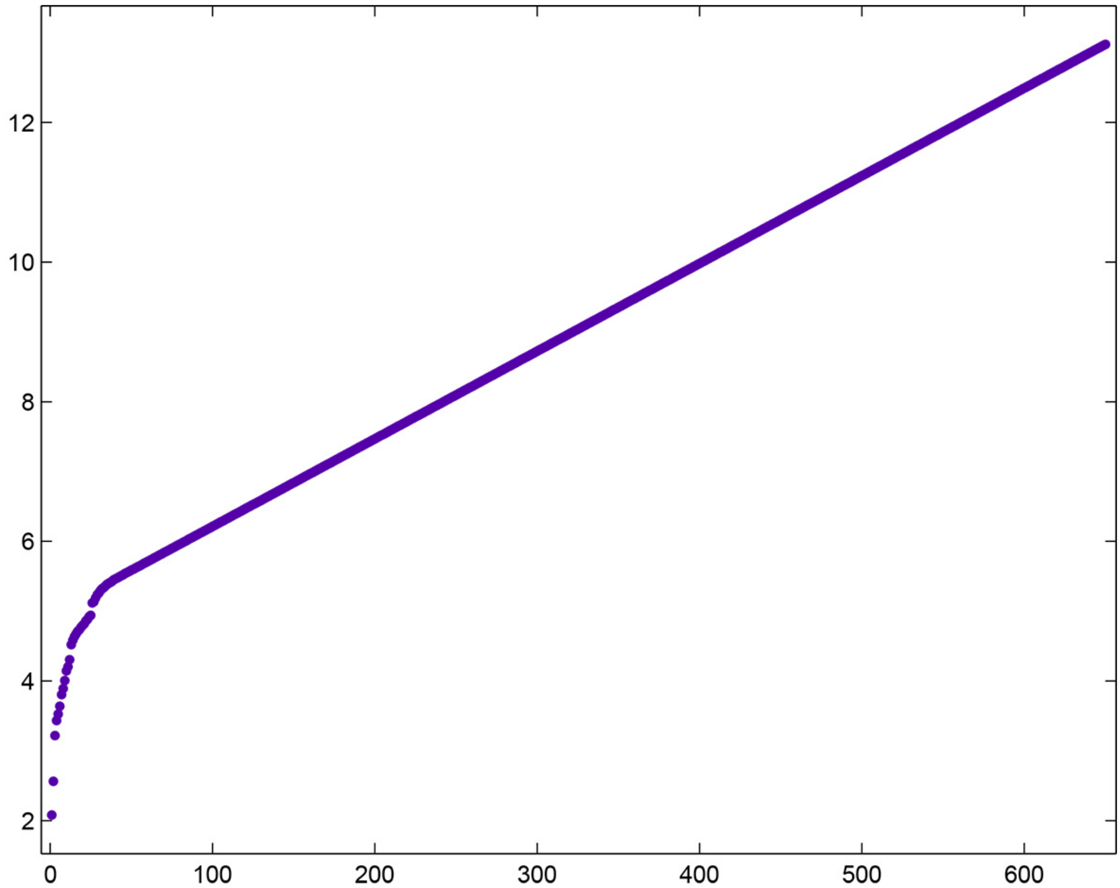

Figure 1

A plot of the real and imaginary parts of the exponentially weighted Dirichlet series computed at these j values versus the logarithms of the j values of the inflection points is

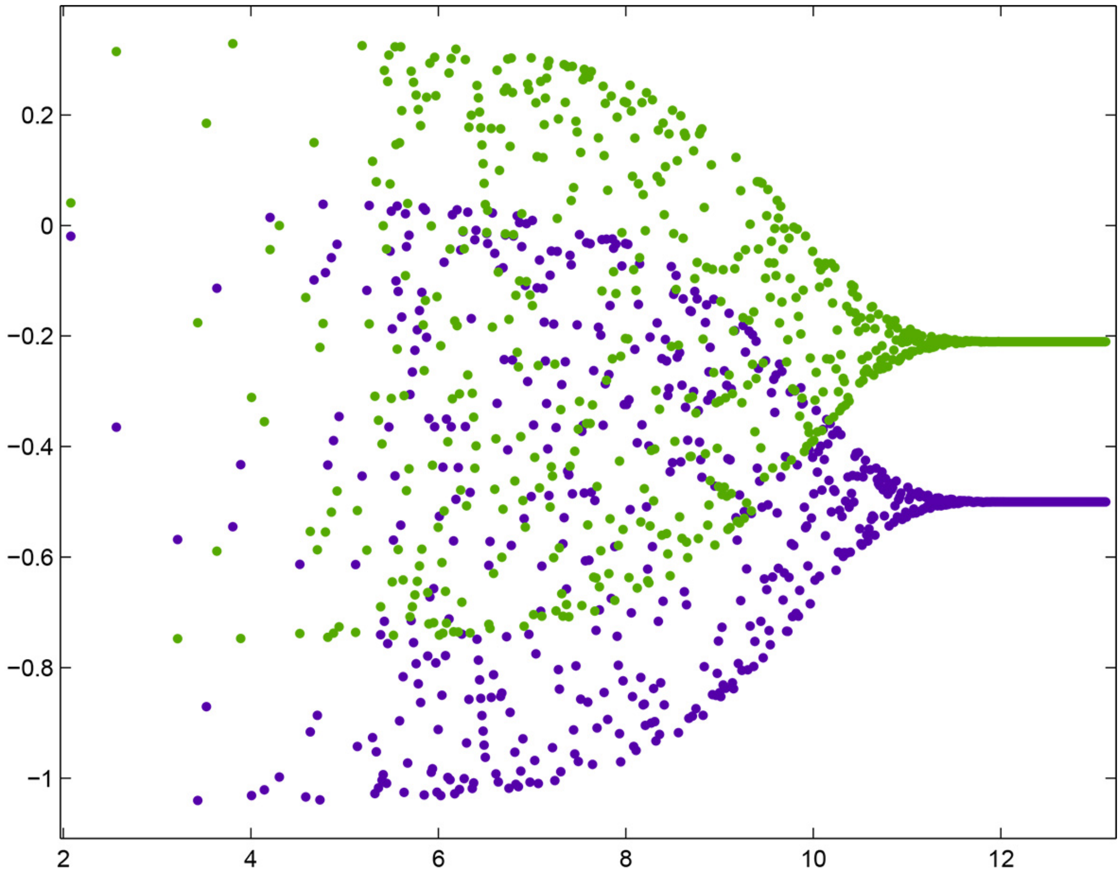

Figure 2

This exponentially weighted Dirichlet series will be denoted by $D(n, a, b)$ to distinguish it from the D series. The series converges to about $(-0.5000,\ -0.2102)$.

A plot of the convergents of the real parts of $D(n, a, b)$ versus the imaginary parts for the first three hundred zeta function zeros is

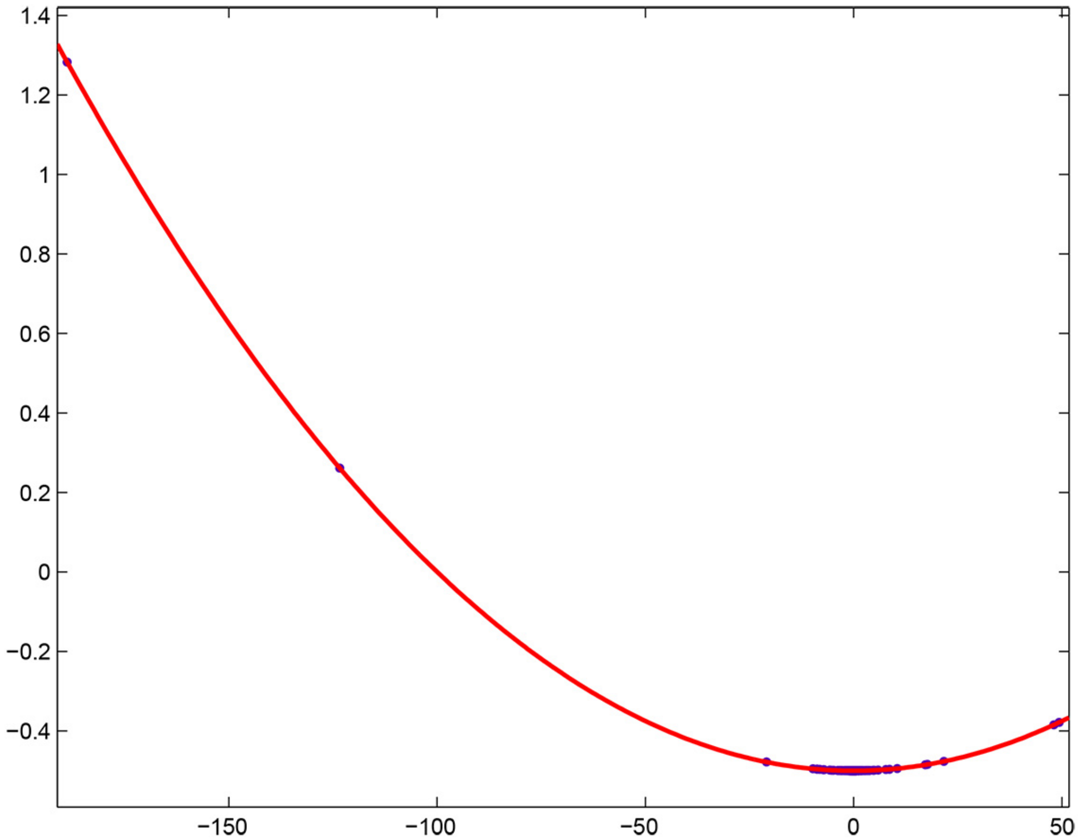

Figure 3

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of $(5.0 \cdot 10^{-5}$, $5.0 \cdot 10^{-5})$, $p_2 = 2.204 \cdot 10^{-8}$ with a 95% confidence interval of $(-1.738 \cdot 10^{-8}$, $6.146 \cdot 10^{-8})$, $p_3 = -0.50$ with a 95% confidence interval of $(-0.50$, $-0.50)$, SSE=$1.435 \cdot 10^{-9}$, R-squared=1, and RMSE=$2.198 \cdot 10^{-6}$. The expected value of the p_1 parameter is the real value of s and the expected value of the p_3 parameter is -0.50.

A plot of the first component of the Fourier transform of this parabola (neglecting the first point) is

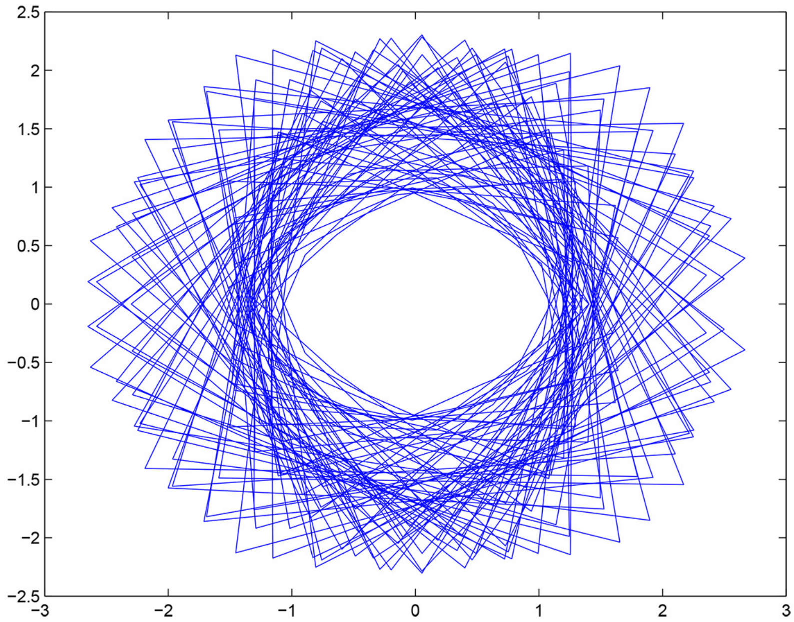

Figure 4

A plot of the second component of the Fourier transform (neglecting the first point) is

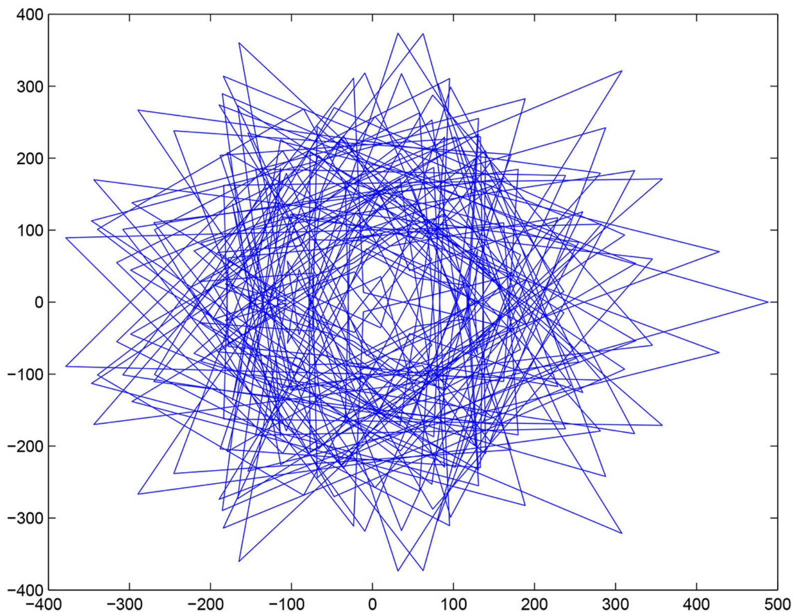

Figure 5

3. EIGENVALUES OF HERMITIAN MATRICES

A plot of the convergents of the real parts of $D(n, a, b)$ versus the imaginary parts (that is, $C(n, a, b)$ weighted by $D(n, a, b)$) for the first five hundred eigenvalues of a Hermitian matrix (where the diagonal was scaled by a factor of $\sqrt{2}$) is

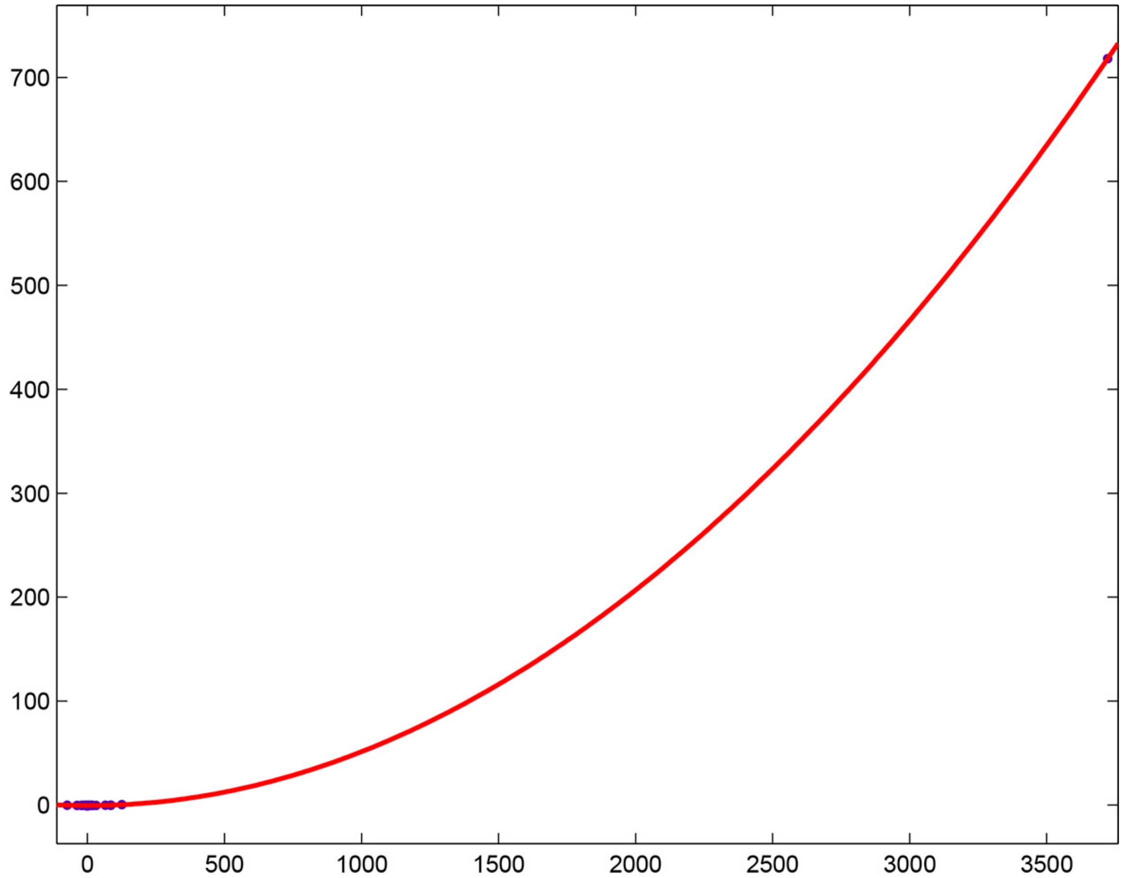

Figure 6

For a quadratic least-squares fit of the curve, $p_1 = 5.19 \cdot 10^{-5}$ with a 95% confidence interval of $(5.189 \cdot 10^{-5}, 5.19 \cdot 10^{-5})$, $p_2 = -0.0001225$ with a 95% confidence interval of $(-0.0001337, -0.00011112)$, $p_3 = -0.5001$ with a 95% confidence interval of $(-0.5002, -0.5)$, SSE=0.0007241, R-squared=1, and RMSE=0.001207. The distribution is skewed by one point.

A plot of the first component of the Fourier transform of this one-sided parabola (neglecting the first point) is

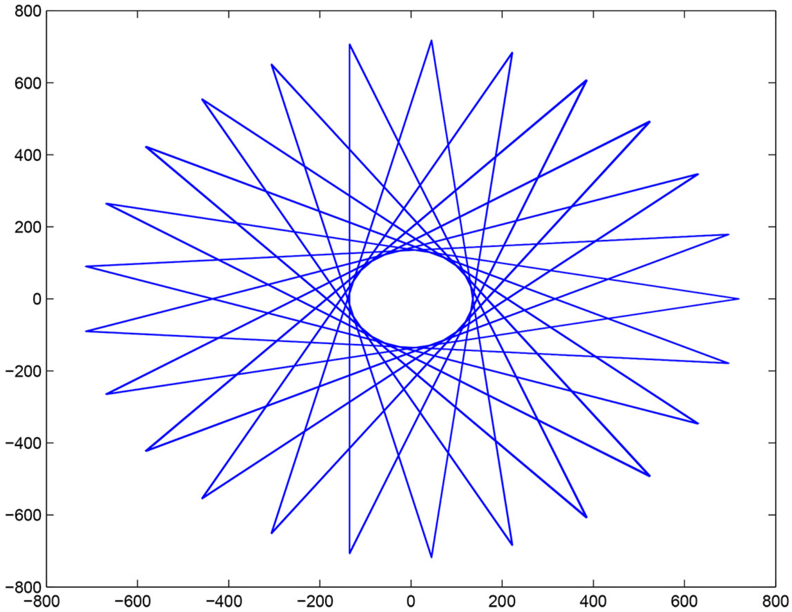

Figure 7

A plot of the second component of the Fourier transform (neglecting the first point) is

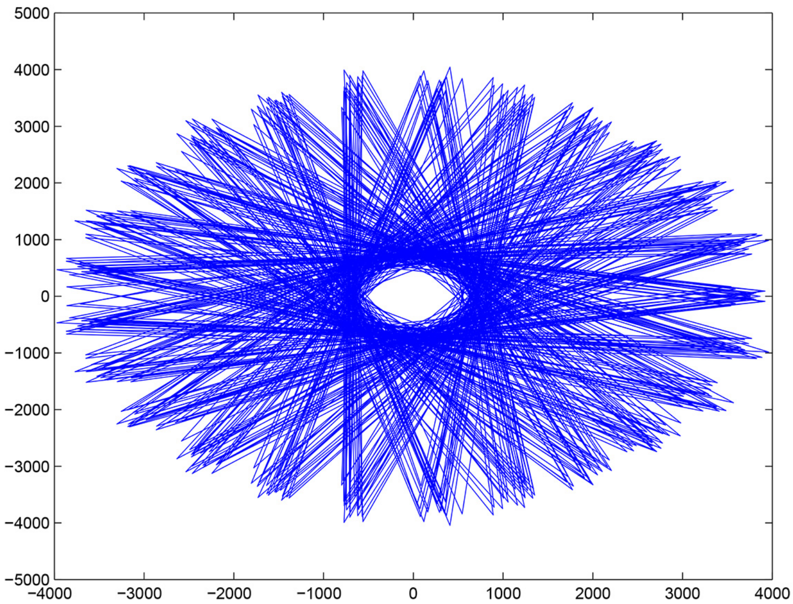

Figure 8

A plot of the real parts of the first component is

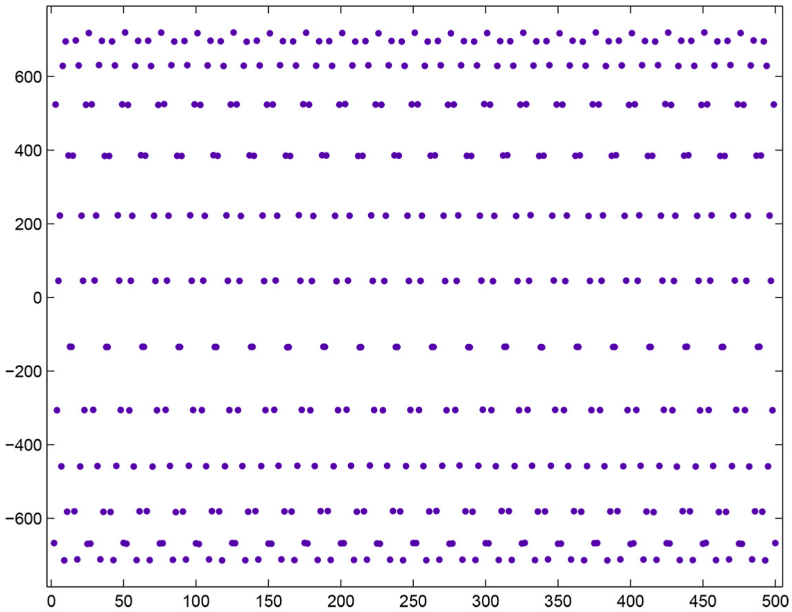

Figure 9

A plot of the real parts of the second component is

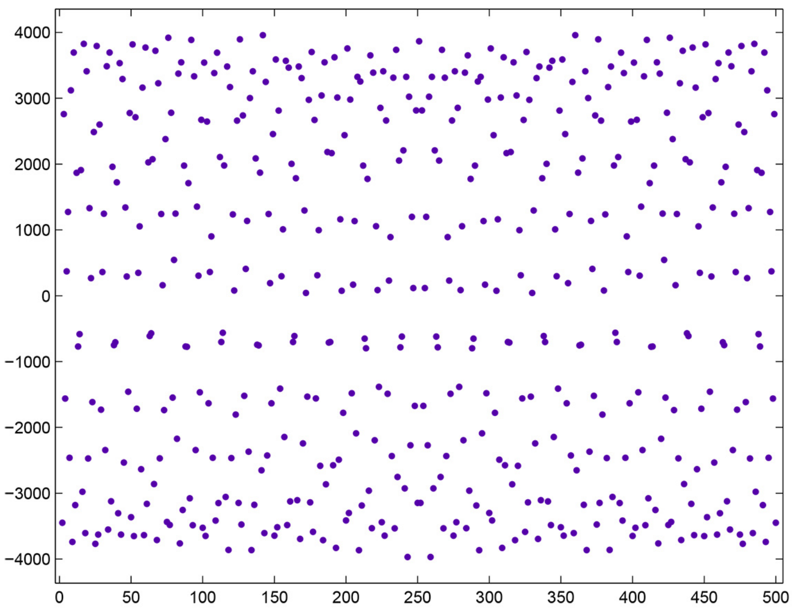

Figure 10

The Fourier transform of the eigenvalues of Hermitian matrices is easier to analyze than the Fourier transform of the zeta function zeros.

4. GAUSSIAN UNITARY ENSEMBLES

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts (that is, $C(n,a,b)$ weighted by $D(n,a,b)$) for the first three hundred elements of a GUE is

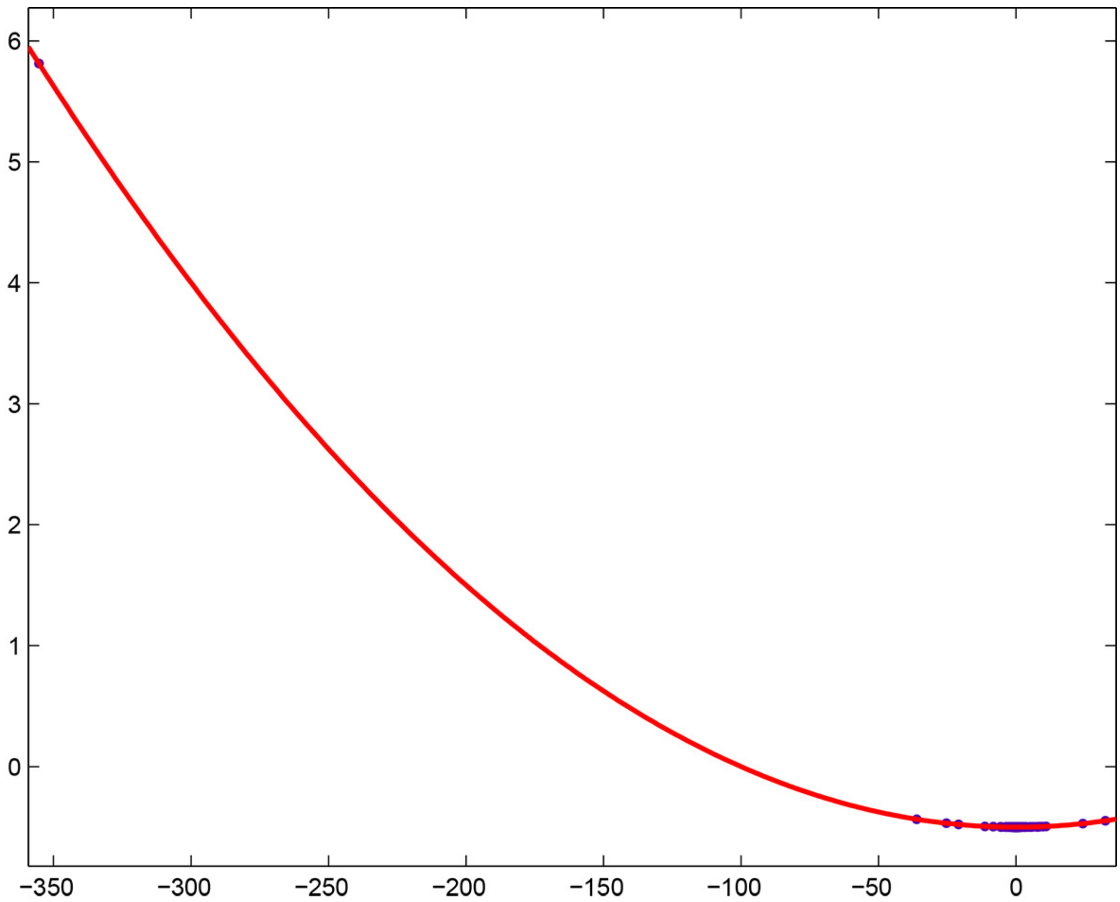

Figure 11

For a quadratic least-squares fit of the curve, $p_1 = 5.002 \cdot 10^{-5}$ with a 95% confidence interval of $(5.002 \cdot 10^{-5}$, $5.002 \cdot 10^{-5})$, $p_2 = 3.451 \cdot 10^{-8}$, with a 95% confidence interval of $(-1.353 \cdot 10^{-8}$, $8.255 \cdot 10^{-8})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5$, $-0.5)$, SSE=$9.122 \cdot 10^{-10}$, R-squared=1, and RMSE=$1.753 \cdot 10^{-6}$.

A plot of the first component of the Fourier transform of this parabola (neglecting the first point) is

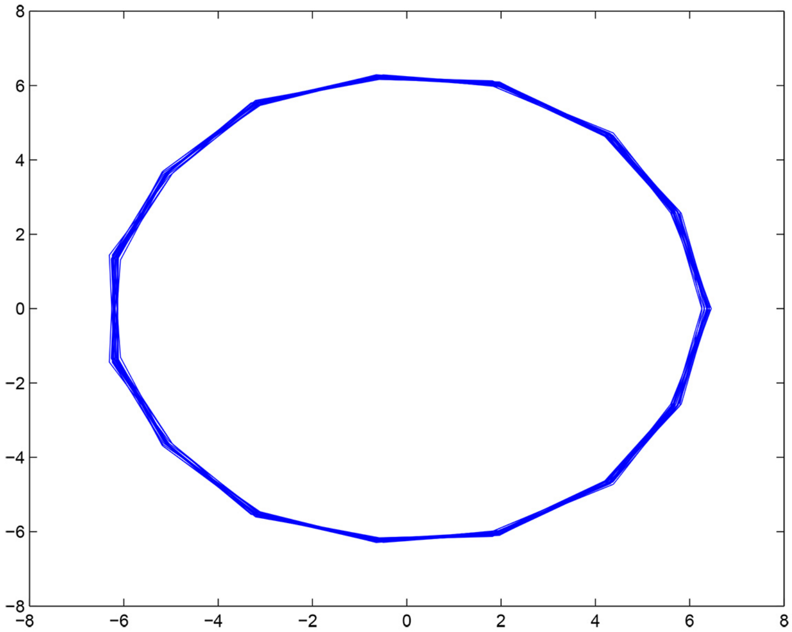

Figure 12

A plot of the real parts of the first component is

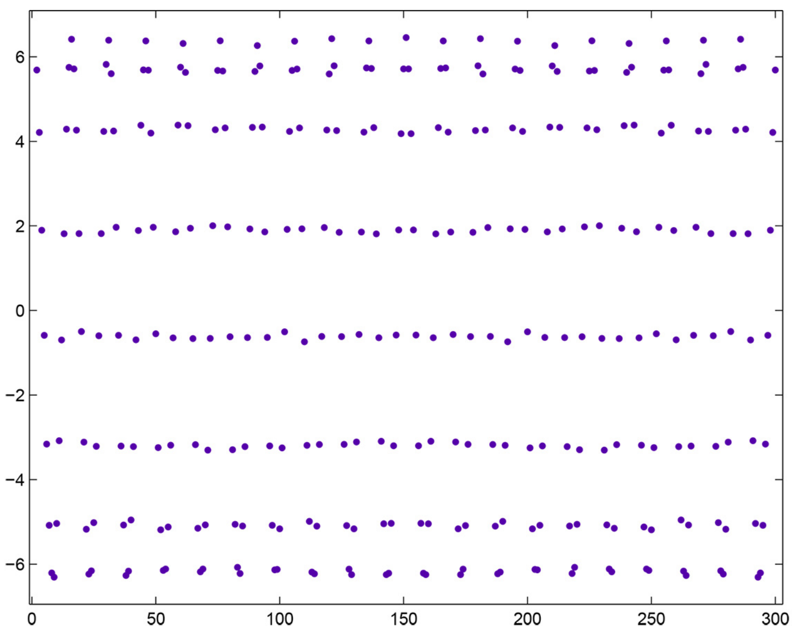

Figure 13

A plot of the second component of the Fourier transform (neglecting the first point) is

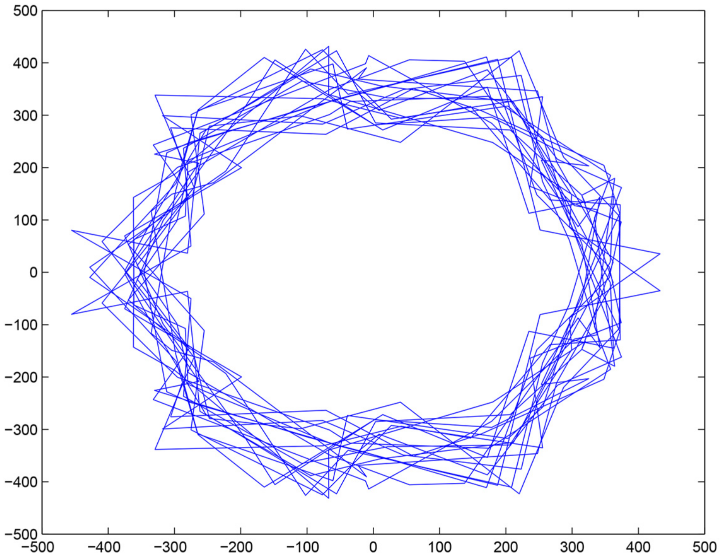

Figure 14

5. A SUM INVOLVING THE MÖBIUS FUNCTION AND THE ZETA FUNCTION ZEROS

Let v_n denote $\sum_{i|n}(b_{i+1} - b_i)\mu(i)$ where μ_i denotes the Möbius function. A normal probablity plot of v_n, $n \leq 300$ is

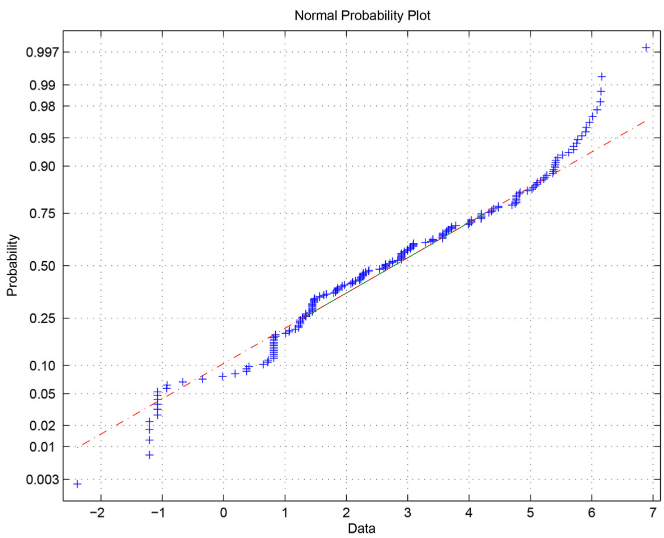

Figure 15

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts (that is, $C(n,a,b)$ weighted by $D(n,a,b)$) for the first three hundred values of v_n is

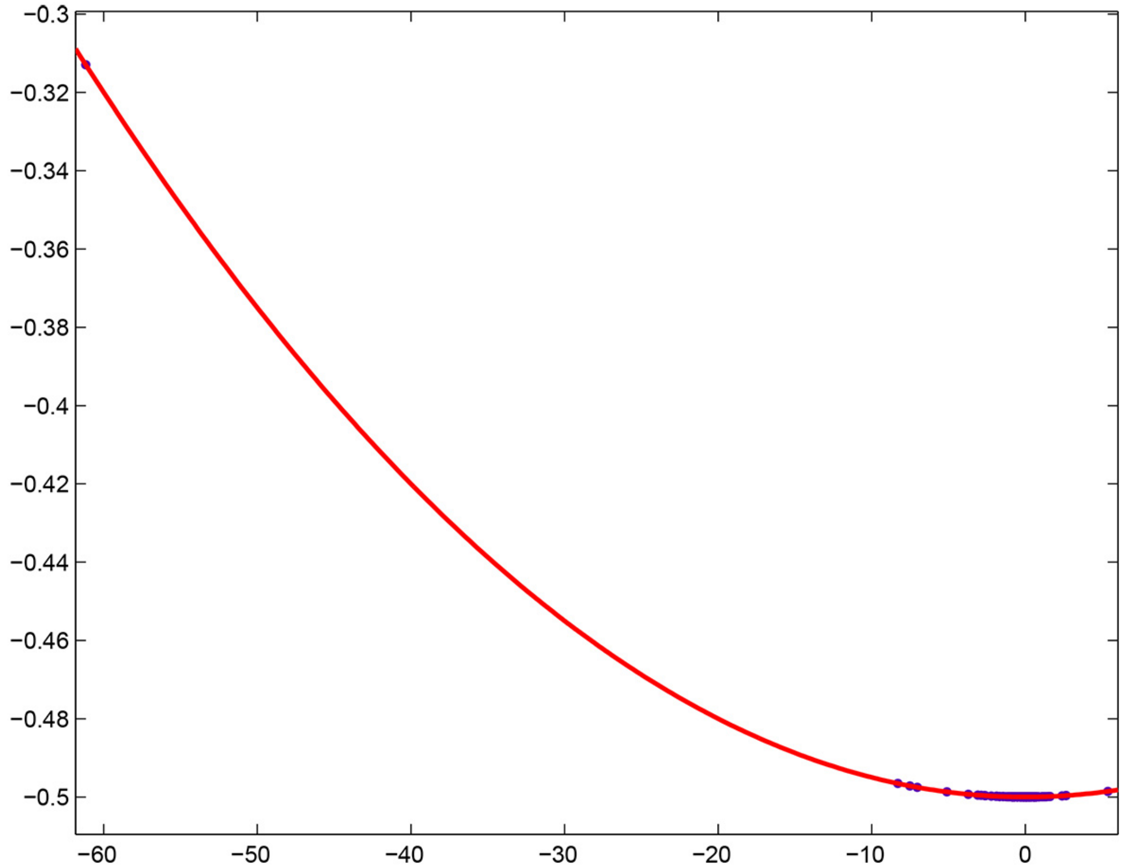

Figure 16

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of $(5.0 \cdot 10^{-5}$, $5.0 \cdot 10^{-5}$, $p_2 = 1.584 \cdot 10^{-9}$ with a 95% confidence interval of $(1.276 \cdot 10^{-9}$, $1.892 \cdot 10^{-9})$, $p_3 = -0.50$ with a 95% confidence interval of $(-0.50$, $-0.50)$, SSE=$1.655 \cdot 10^{-15}$, R-squared=1, and RMSE=$2.898 \cdot 10^{-9}$.

A plot of the first component of the Fourier transform of this parabola (neglecting the first point) is

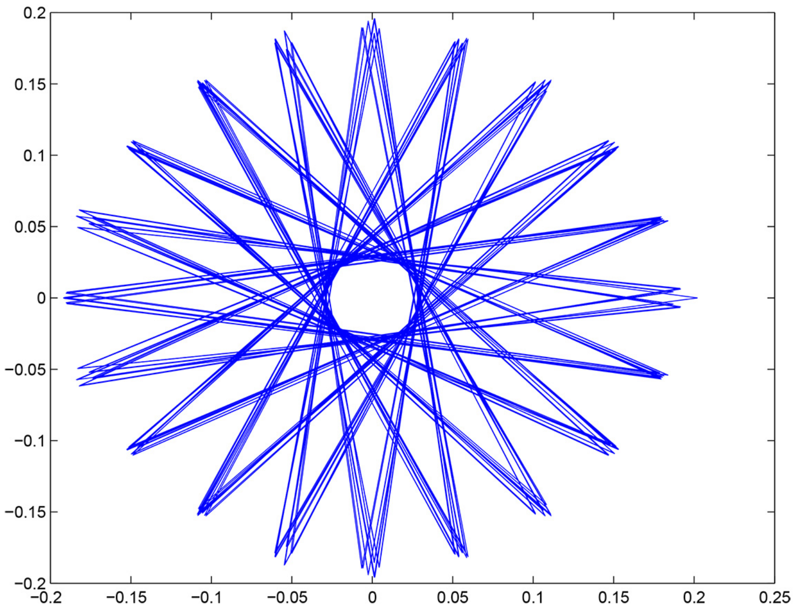

Figure 17

A plot of the real parts of the first component is

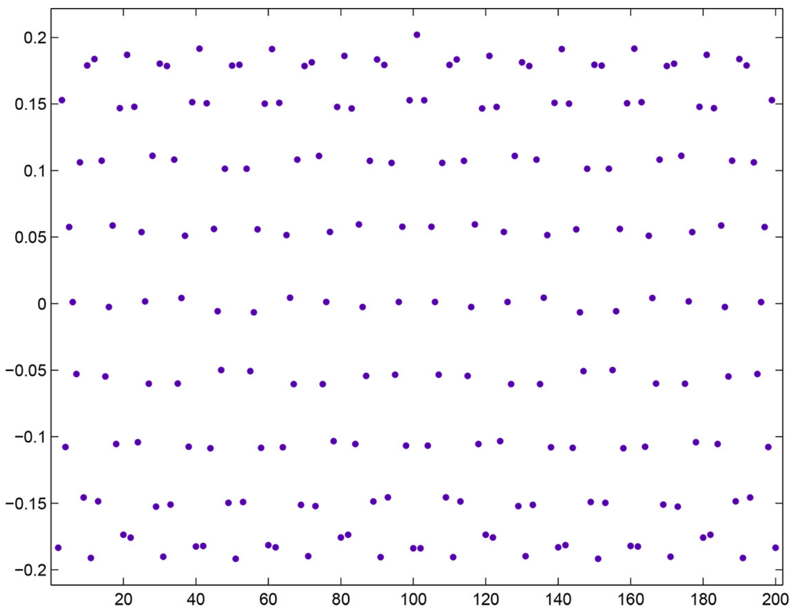

Figure 18

A plot of the second component of the Fourier transform (neglecting the first point) is

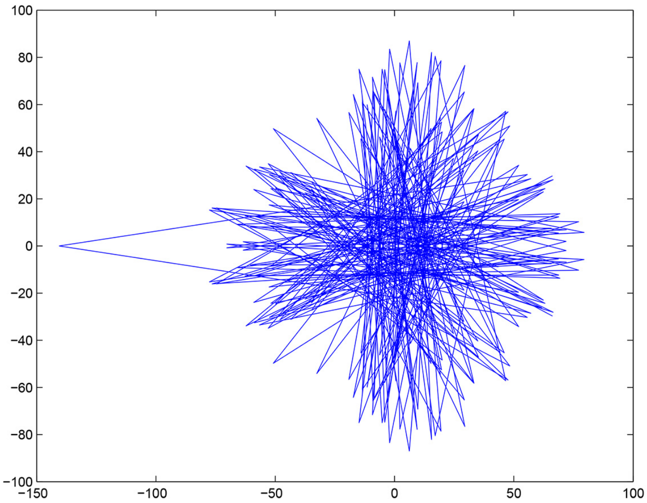

Figure 19

6. A SUM INVOLVING THE MÖBIUS FUNCTION AND THE EIGENVALUES OF HERMITIAN MATRICES

Let w_n denote $\sum_{i|n}(e_{i+1} - e_i)\mu(i)$ where e_i denotes the eigenvalues of a Hermitian matrix. A normal probablity plot of w_n, $n \leq 300$ is

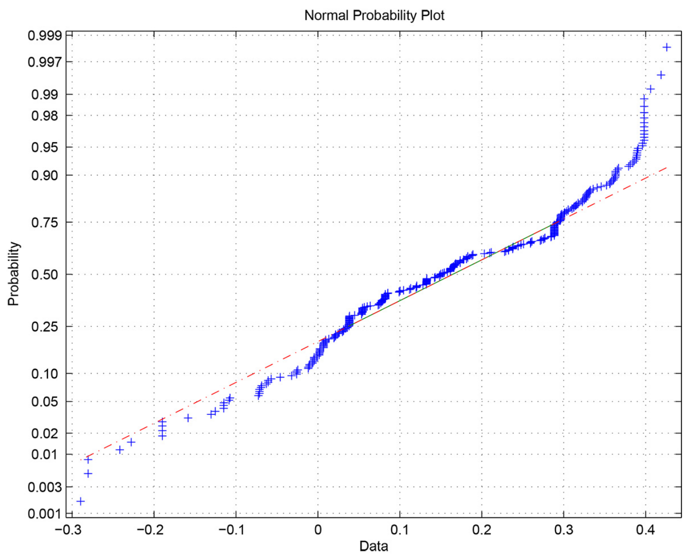

Figure 20

A plot of the convergents of the real parts of $D(n, a, b)$ versus the imaginary parts (that is, $C(n, a, b)$ weighted by $D(n, a, b)$) for the first three hundred values of w_n is

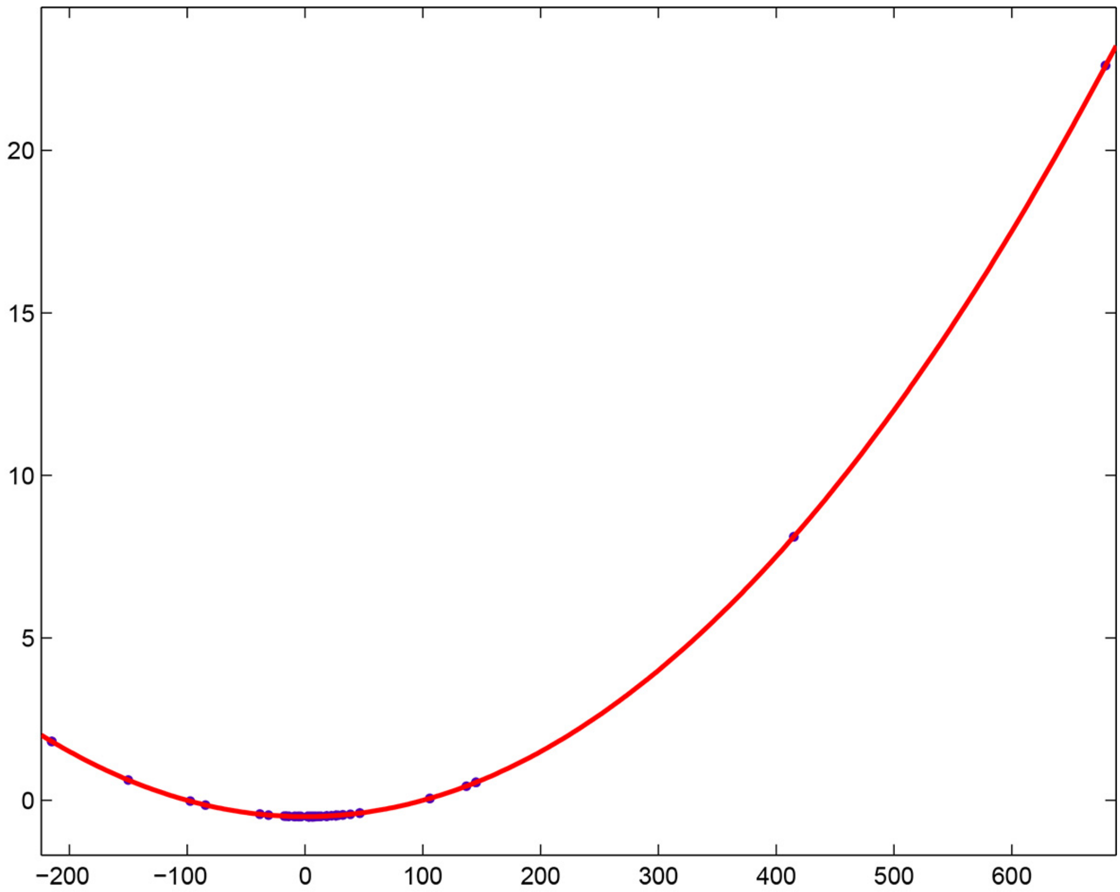

Figure 21

For a quadratic least-squares fit of the curve, $p_1 = 5.007 \cdot 10^{-5}$ with a 95% confidence interval of $(5.007 \cdot 10^{-5}$, $5.007 \cdot 10^{-5})$, $p_2 = -1.171 \cdot 10^{-5}$ with a 95% confidence interval of $(-1.276 \cdot 10^{-5}$, $-1.066 \cdot 10^{-5})$, $p_3 = -0.5001$ with a 95% confidence interval of $(-0.5002$, $-0.50)$, SSE=0.0002457, R-squared=1, and RMSE=0.0009096.

A plot of the first component of the Fourier transform of this parabola (after sorting the w_n values in ascending order) is

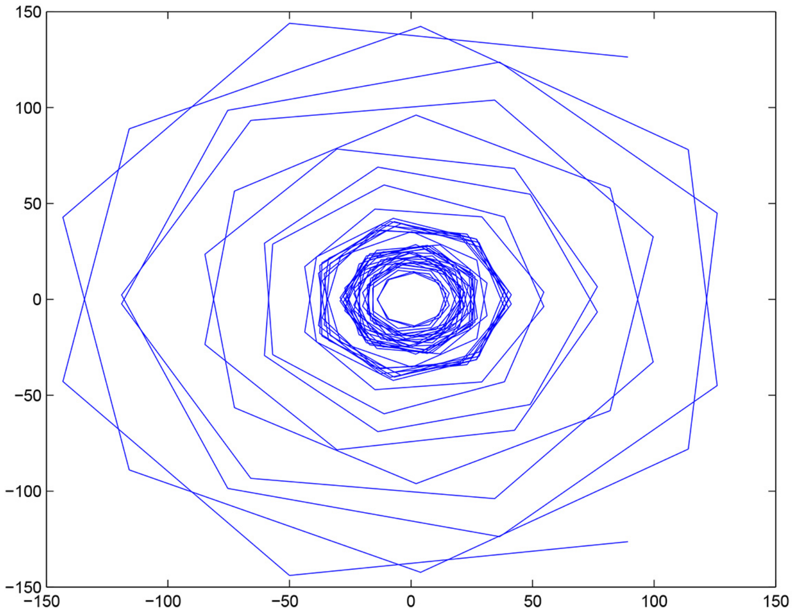

Figure 22

A plot of the real parts of the first component is

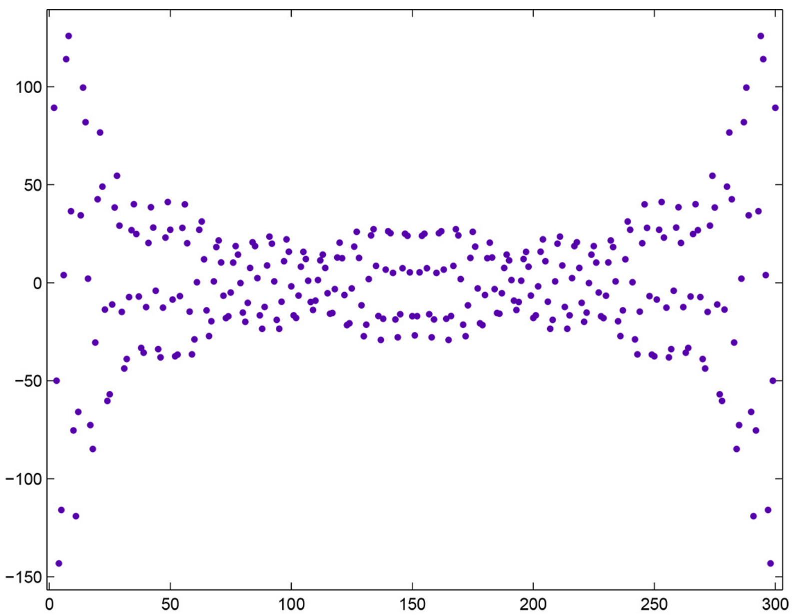

Figure 23

The peculiar shape is due to the sorting.

A plot of the second component of the Fourier transform is

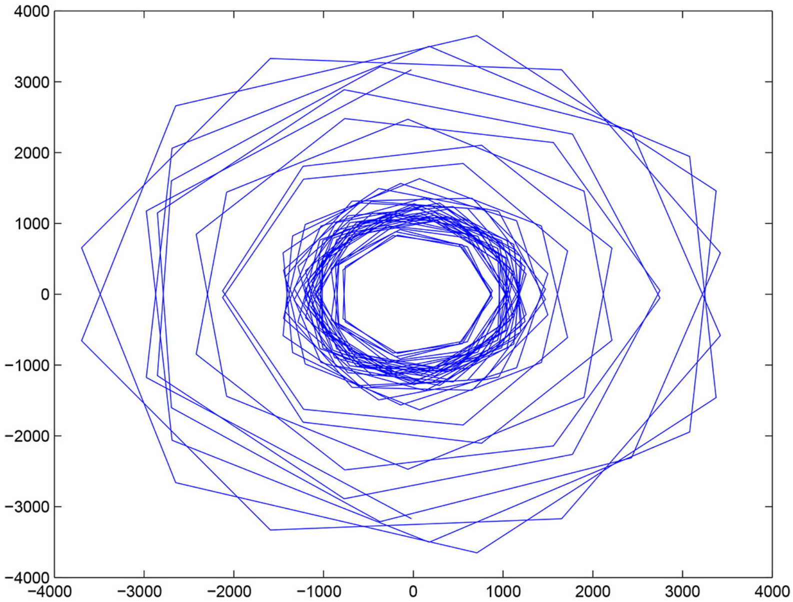

Figure 24

7. MINIMA IN v_n AND w_n SEQUENCES

Let v'_n denote $\sum_{i|n}(b_{i+2} - b_i)\mu(i)$. Minimum values (including negative values) in the sequence v'_2, v'_3, v'_4,. . ., $v'_{2000000}$ are considered. If the current value is smaller than that all previous values, the index of the minimum is output. The indices are 2, 6, 26, 30, 78, 390, 1326, 4758, 6006, 6630, 30030, 65130, 122930, 186186, and 366366. These are square-free numbers consisting of the product of small primes, similar to colossally abundant numbers.

Minimum values in the sequence w_2, w_3, w_4,. . ., w_{4000} are considered. The indices of the minima are 2, 10, 38, 110, 190, 310, 418, 2090, and 3994. Let w'_n denote $\sum_{i|n}(e_{i+2} - e_i)\mu(i)$. Minimum values in the sequence w'_2, w'_3, w'_4,. . ., w'_{4000} are considered. The indices of the minima are 2, 6, 10, 70, 110, 130, 230, 390, 1610, 1978, 2090, and 3994. These numbers vary according to the random Hermitian matrix used.

8. METHODS

```
#include <math.h>
#include <stdio.h>
//
// compute C(n,a,b), 09/20/2024 (dkc)
//
unsigned int max=200001;
double a=0.0001;
double b=14.13472514173470;
//double b=21.02203963877156;
//double b=25.01085758014569;
//double b=30.42487612585951;
//double b=32.93506158773919;
//double b=37.58617815882568;
//double b=40.91871901214750;
//double b=43.32707328091500;
//double b=48.00515088116716;
//double b=49.77383247767230;
//double b=52.97032147771446;
//double b=56.44624769706339;
//double b=59.34704400260235;
//double b=60.83177852460981;
//double b=65.11254404808160;
//double b=67.07981052949417;
//double b=69.54640171117399;
//double b=72.06715767448191;
//double b=75.70469069908393;
//double b=77.14484006887480;
//double b=79.33737502024937;
//double b=84.73549298051705;
//double b=87.42527461312523;
//double b=88.80911120763446;
//double b=92.49189927055849;
//double b=94.65134404051989;
//double b=95.87063422824531;
//double b=98.83119421819369;
//double b=101.31785100573138;
```

```
unsigned int xmin=0; // usually set to 0
unsigned int out=2; // usually 1, 2 for inflection points
unsigned int out3p=1; // usually 0, 1 for differences in j values >=2
unsigned int polar=1; // set to use polar coordinates
void main() {
unsigned int x,oldx;
double sumr,sumi,R,I,temp1,oldsumr,oldsumi,temp,tempa,tempb,y,e,f,g;
double tempr,esumr,esumi;
FILE *Outfp;
Outfp = fopen(”c2nab2f.dat”,”w”);
y=1.0-a;
if (y>=0.0)
    temp1=pow((double)2,y);
else {
    temp1=pow((double)2,-y);
    temp1=1.0/temp1;
    }
e=temp1*(cos(b*log(2)));
f=temp1*(sin(b*log(2)));
e=1.0-e;
f=-f;
y=-a;
if (y>=0.0)
    temp1=pow((double)max,y);
else {
    temp1=pow((double)max,-y);
    temp1=1.0/temp1;
    }
y=2.0*temp1;
oldx=0;
sumr=0.0;
sumi=0.0;
oldsumi=0.0;
oldsumr=0.0;
esumr=0.0;
esumi=0.0;
for (x=1; x<=(max-1); x++) {
    tempr=1.0/exp((double)x*a);
```

```
esumr=esumr+tempr*cos((double)x*b);
esumi=esumi+tempr*sin((double)x*b);
temp1=pow((double)x,a);
R=temp1*cos(b*log((double)x));
I=temp1*sin(b*log((double)x));
temp1=R*R+I*I;
if (x!=(x/2)*2) {
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    }
else {
    sumr=sumr-R/temp1;
    sumi=sumi+I/temp1;
    }
temp=cos(b*log((double)max/(double)x));
tempa=sumr*temp;
tempb=sumi*temp;
tempa=tempa*y;
tempb=tempb*y;
g=tempa*e-tempb*f;
tempb=tempa*f+tempb*e;
tempa=g;
if (x>xmin) {
    if (out==1)
        fprintf(Outfp," %.10llf, %.10llf \n",tempa,tempb);
    if ((out==2)&&((oldsumr¿0.0)&&(tempa¡0.0))) {
        if (out3p==0)
            fprintf(Outfp," %d %.10llf %.10llf \n",x,tempa,tempb);
        if ((x-oldx)>2) {
            printf(" %d %d %.10llf %.10llf \n",x,oldx,oldsumr,tempa);
            if (out3p!=0) {
                if (polar==0)
                    fprintf(Outfp," %d %d \n",x,oldx);
                else
                    fprintf(Outfp," %d %d %.10llf %.10llf %.10llf %.10llf
\n",x,oldx,
                        sqrt(tempa*tempa+tempb*tempb),atan2(tempb,tempa),esumr,esumi);
                }
```

```
                }
            oldx=x;
            }
        oldsumr=tempa;
        oldsumi=tempb;
        }
    }
fclose(Outfp);
return;
}
```