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The Gamma Function Reflection Formula and Eigenvalues of Hermitian Matrices

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Abstract

The eigenvalues of Hermitian matrices are used to make a "staircase" of the primes similar to that generated by the Riemann zeta function zeros and the Mangoldt function.

Keywords: Riemann zeta function, gamma function, eigenvalues of Hermitian matrices, staircase of the primes, Mangoldt function

1. INTRODUCTION

Equation (3) in section 1.3 of Edward's [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1\cdot 2\cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{1}$$

This equation is valid for all s in the halfplane Re $s > -1$. (Edwards uses the notation $\Pi(s-1)$ instead of $\Gamma(s)$.)

2. THE REFLECTION FORMULA FOR THE GAMMA FUNCTION

Equation 6.5 in Voros's [2] article (the reflection formula for $\Gamma(z)$) is

$$Z(s) = \sum_{k=1}^{\infty} k^{-2s} = \zeta(2s) \tag{2}$$

A function involving the reflection formula is

$$\zeta_1(s,n) = \frac{2\pi\zeta(s-1)}{\Pi(s-1)Z(s-1)} \tag{3}$$

A plot of this expression for the first non-trivial zeta function zero ($s = (0.5, 14.13472514173470)$) and $n \leq 1000$ is

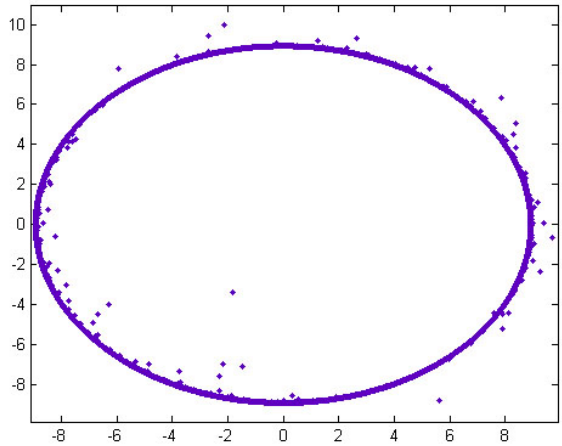

Figure 1

The slopes and intercepts of the logarithms of the n values of the inflection points (where the curve decreases toward the x-axis and then increases) for the first ten zeta function zeros are (0.4458, 0.588), (0.2983, 1.297), (0.2512, 1.35), (0.2068, 1.719), (0.1908, 1.595), (0.1672, 1.902), (0.1537, 1.892), (0.145, 1.941), (0.1311, 1.87), and (0.1262, 1.943). A plot of the logarithms of the n values of the inflection points for the tenth zeta function zero ($s = (0.5, 49.77383247767230)$) and $n \leq 100000$ is

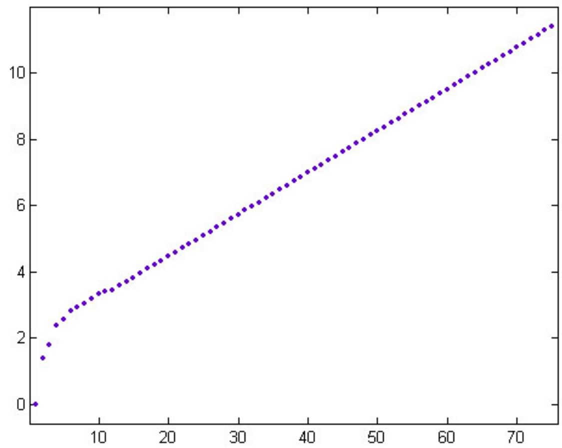

Figure 2

The first sixteen n values will be disregarded in computing the slope. This is unusual for the usual zeta function zeros - at most one value has to be disregarded. In computing the above slopes 1, 2, 6, 6, 7, 9, 9, 9, 13, and 16 n values were discarded respectively. Other than this, the slopes are almost the same as for the usual zeta function zeros. The slopes for the usual zeta function zeros and $n \leq 1000000$ are 0.4444, 0.2988, 0.2512, 0.2064, 0.1909, 0.1673, 0.1535, 0.145, 0.1309, and 0.1263.

A plot of the logarithms of the n values of the inflections points for the first twenty zeta function zeros is

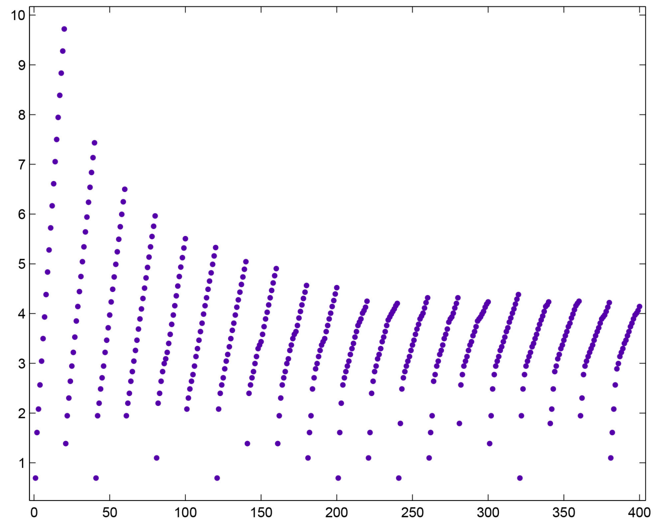

Figure 3

Note that there are 400 elements in the array. A plot of the eigenvalues of a 20x20 Hermitian matrix generated from these values (by averaging the 20x20 matrix with its transpose) is

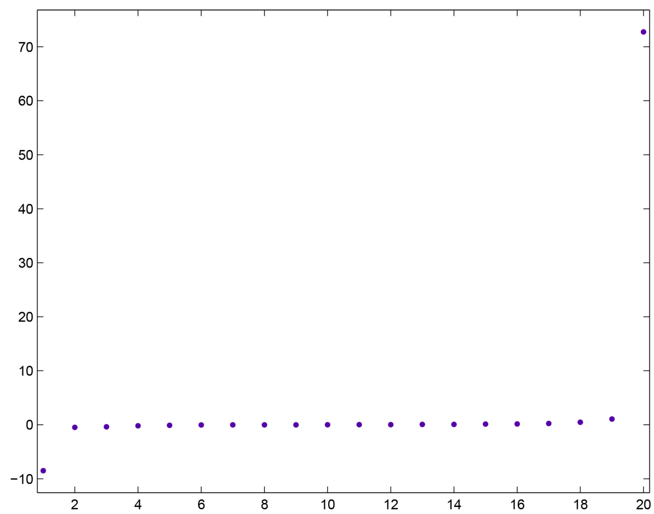

Figure 4

From this perspective, the logarithms of the n values of the inflection points are very uniform.

3. A FUNCTION FOR ANALYZING PROBABILITY DISTRIBUTIONS

A Dirichlet series with exponential terms is

$$D(s) = \sum_{k=1}^{\infty} e^{-ks} \tag{4}$$

where $s = (a,b)$. For $\Re(s) > 0$, the series converges to $e^{-s}/(1-e^{-s})$. The real part can be expressed as $\sum_{k=1}^{\infty} e^{-ka}\cos(kb)$ and the imaginary part can be expressed as $\sum_{k=1}^{\infty} e^{-ka}\sin(kb)$.

This exponentially weighted Dirichlet series will be denoted by $D(n,a,b)$. See the Methods section for the C code.

4. AN APPLICATION TO EIGENVALUES OF HERMITIAN MATRICES

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts for the above eigenvalues is

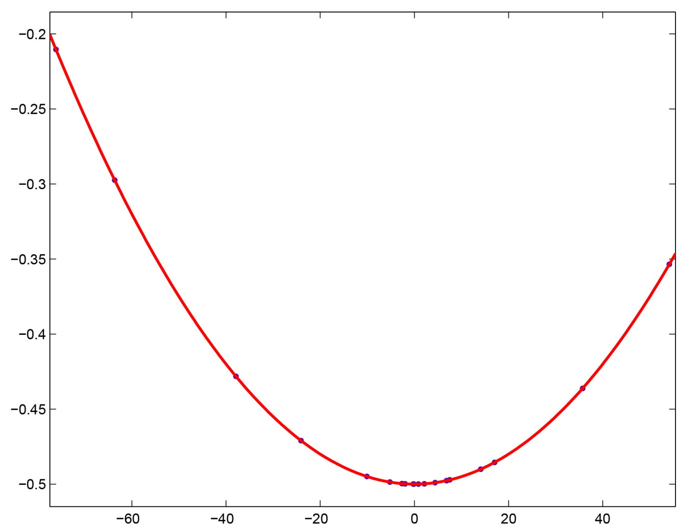

Figure 5

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of $(5.0 \cdot 10^{-5}, 5.0 \cdot 10^{-5})$, $p_2 = -5.464 \cdot 10^{-9}$ with a 95% confidence interval of $(-1.148 \cdot 10^{-8},\ 2.073 \cdot 10^{-10})$, $p_3 = -0.5$ with a 95% confidence interval of

$(-0.5, -0.5)$, SSE=$1.1324 \cdot 10^{-12}$, R-squared=1, and RMSE=$2.877 \cdot 10^{-7}$. The tenth eigenvalue (with a small absolute value of $2.72966475 \cdot 10^{-5}$) is omitted to avoid skewing the distribution. This eigenvalue is not omitted in the following. The expected p_1 parameter is the real part of s and the expected p_3 parameter is -0.5.

A plot of the first component of the Fourier transform of this parabola (neglecting the first point) is

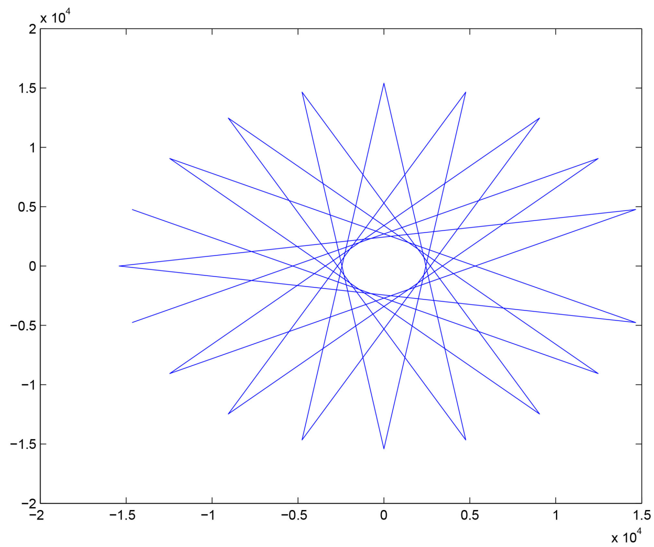

Figure 6

A plot of the second component of the Fourier transform (neglecting the first point) is

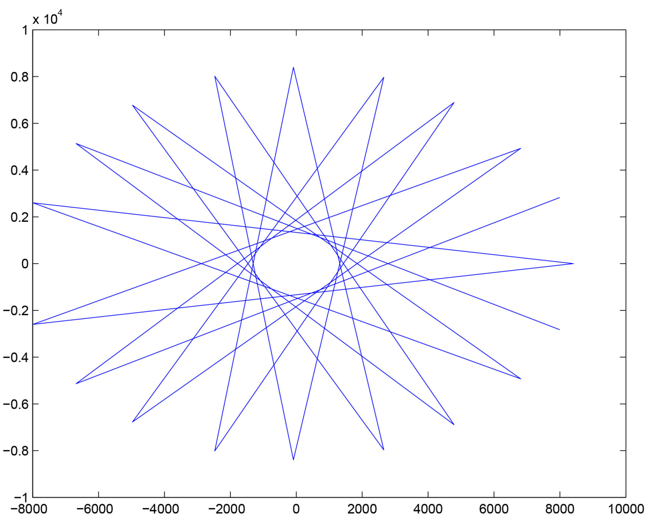

Figure 7

5. A FUNCTION DERIVED FROM $\zeta_1(s,n)$

Let $\alpha(s,n)$ denote $\zeta_1(s,n)\zeta(s,n)$ where the zeta function is defined for $\Re(s) > 1$. A plot of $\alpha(s,n)$ for the first zeta function zero and $n \le 1000$ is

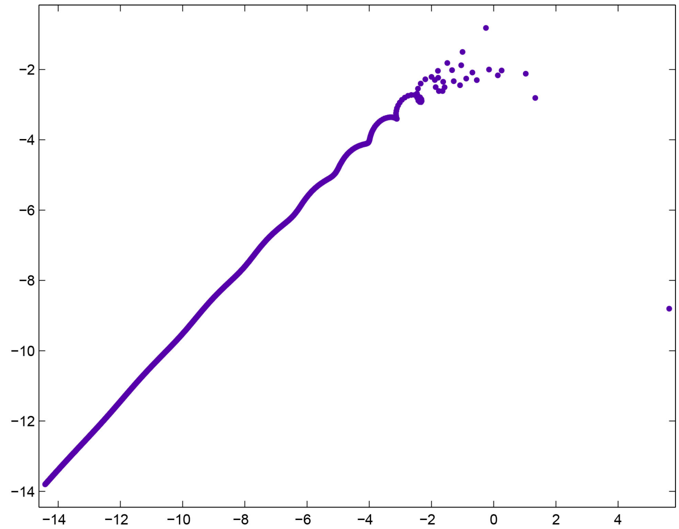

Figure 8

A plot of $\alpha(s,n)$ for $s = (0.5, 13.5)$ and $n \le 1000$ is

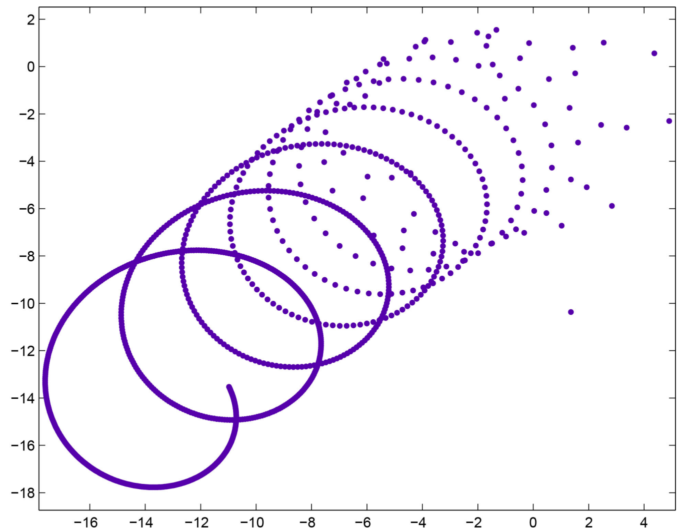

Figure 9

Such curves are typical for $\Re s$ values of 1/2 that are not zeta function zeros.

A plot of the real part of $\alpha(s, n)$ for the first hundred zeta function zeros and $n = 201$ to 300 is

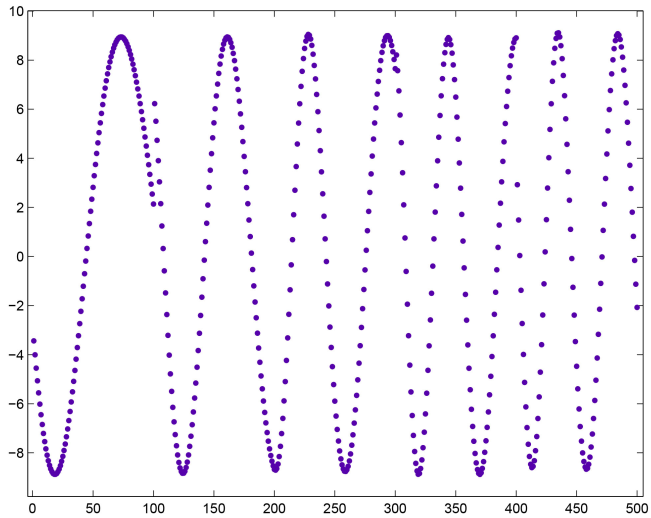

Figure 10

A plot of the real and imaginary components is

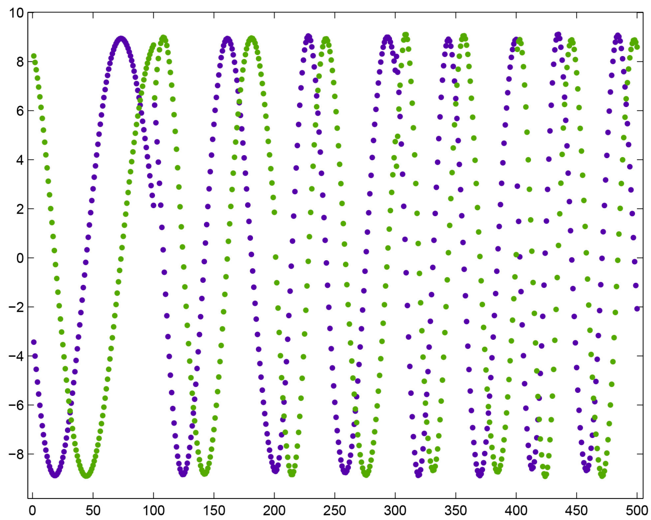

Figure 11

The curves resemble the sine and cosine functions with a decreasing period.

A plot of the eigenvalues of the 100x100 Hermitian matrix generated from the real parts is

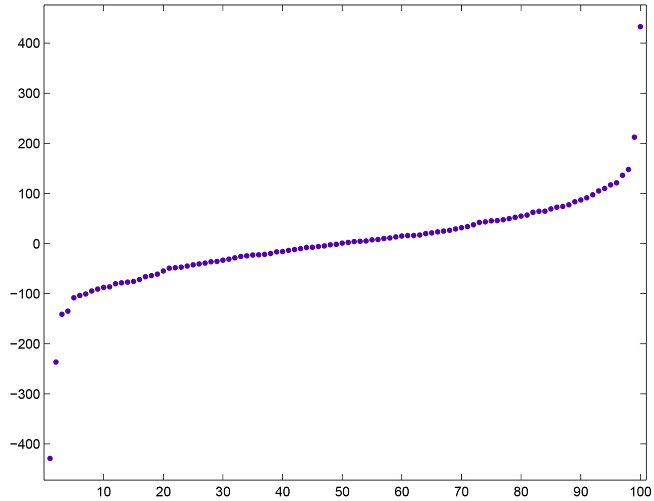

Figure 12

A normal probability plot of the eigenvalues is

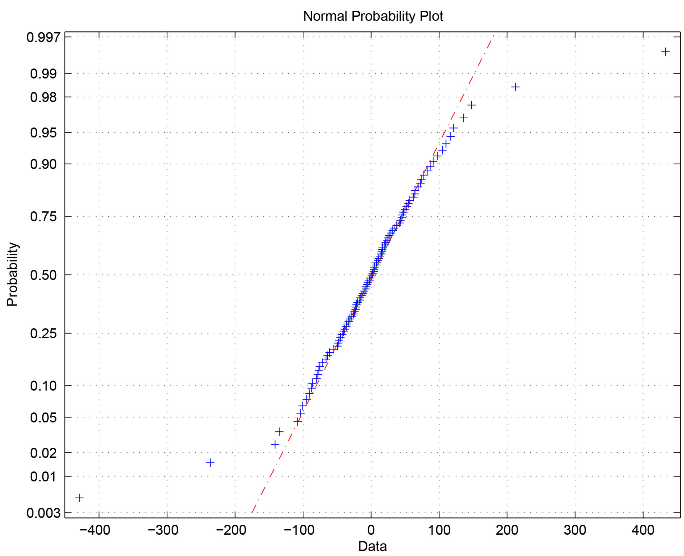

Figure 13

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts for these eigenvalues is

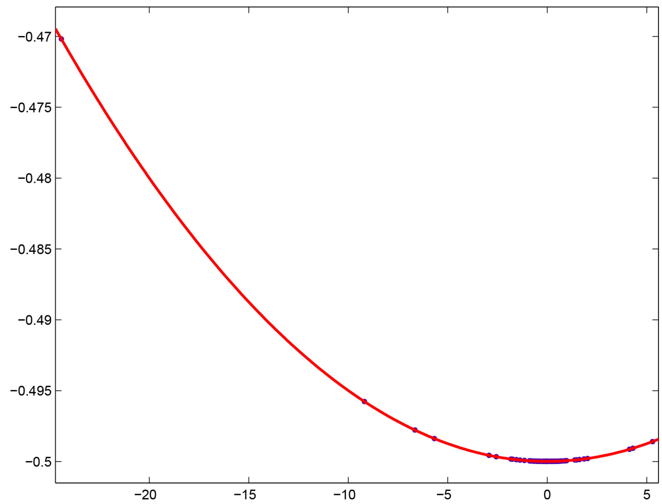

Figure 14

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of $(5.0 \cdot 10^{-5}, 5.0 \cdot 10^{-5})$, $p_2 = 2.806 \cdot 10^{-10}$ with a 95% confidence interval of $(-2.858 \cdot 10^{-10}, 8.469 \cdot 10^{-10})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5, -0.5)$, SSE=$1.741 \cdot 10^{-15}$, R-squared=1, and RMSE=$4.237 \cdot 10^{-9}$.

A plot of the first component of the Fourier transform (neglecting the first point) is

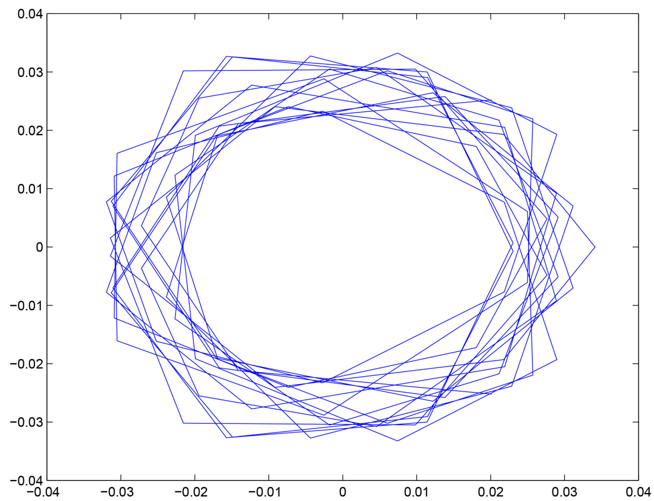

Figure 15

A plot of the second component of the Fourier transform (neglecting the first point) is

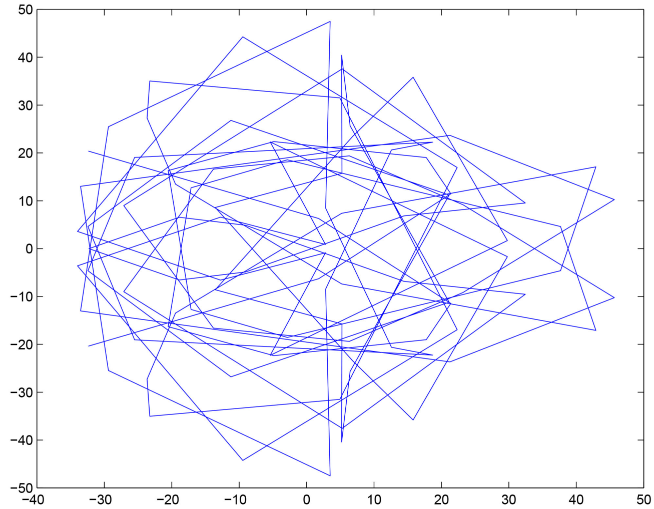

Figure 16

6. A SUM INVOLVING THE MÖBIUS FUNCTION AND THE EIGENVALUES OF HERMITIAN MATRICES

Let w_n denote $\sum_{i|n}(e_{i+1} - e_i)\mu(i)$ where e_i denotes the eigenvalues of a Hermitian matrix. A plot of w_n, $n = 1$ to 99, for the above eigenvalues is

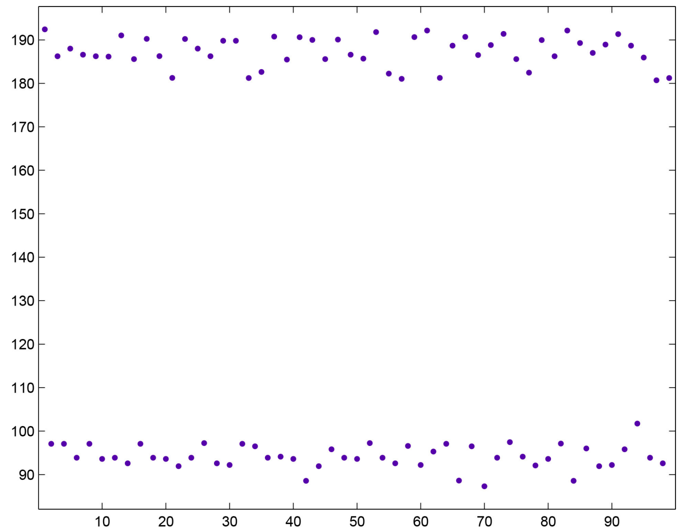

Figure 17

The upper points are for odd n values and the lower points are for even n values. The significance of the patterns in the points will be discussed later.

7. A SCALING FUNCTION INVOLVING THE GAMMA FUNCTION

Let $S(s,n)$ denote

$$\frac{\Gamma_n(-\frac{a}{2})\zeta_n(s)}{2\pi} \tag{5}$$

where the zeta function is defined for $\Re(s) > 1$. A plot of $S(s,n)$ for the first zeta function zero and $n \leq 1000$ is

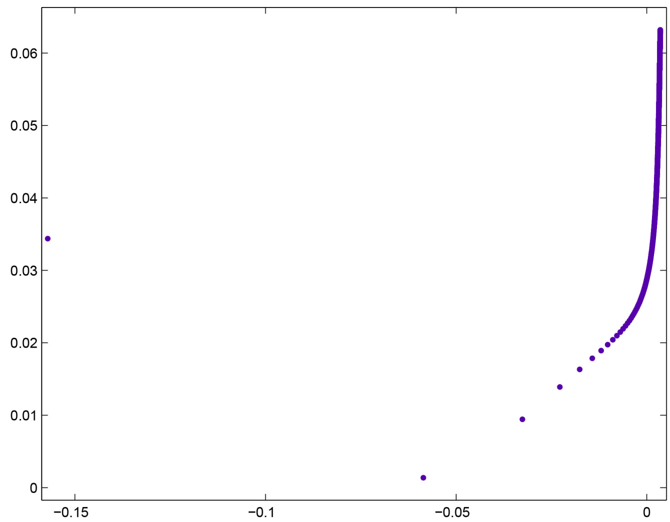

Figure 18

A plot of $S(s,n) \cdot \alpha(s,n)$ is

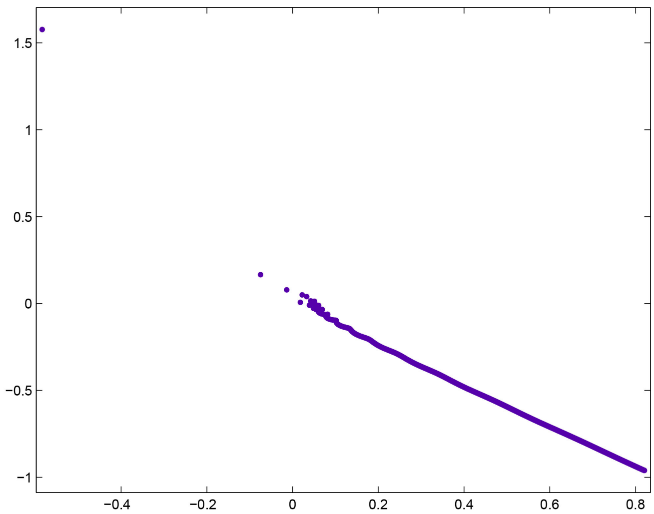

Figure 19

The multiplication changes the size and orientation of $\alpha(s,n)$.

8. A VARIANT OF $\alpha(s, n)$

Let $\alpha'(s, n)$ denote $S(s, n) \cdot \alpha(s, n)$. A plot of the real part of $\alpha'(s, n)$ for the first hundred zeta function zeros and $n = 201$ to 300 is

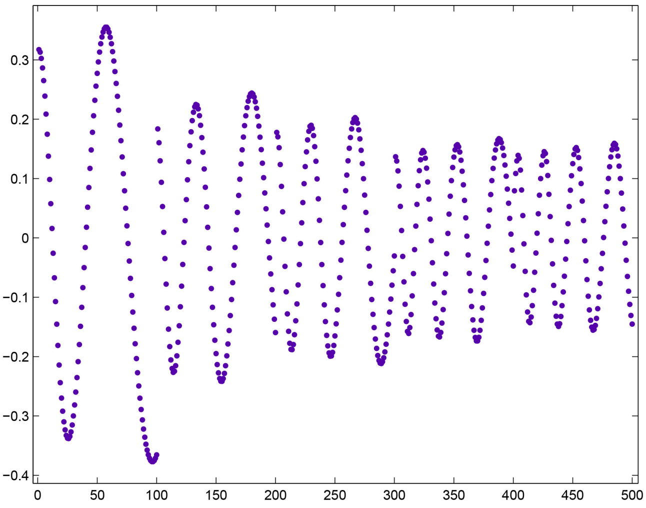

Figure 20

A plot of the real and imaginary components is

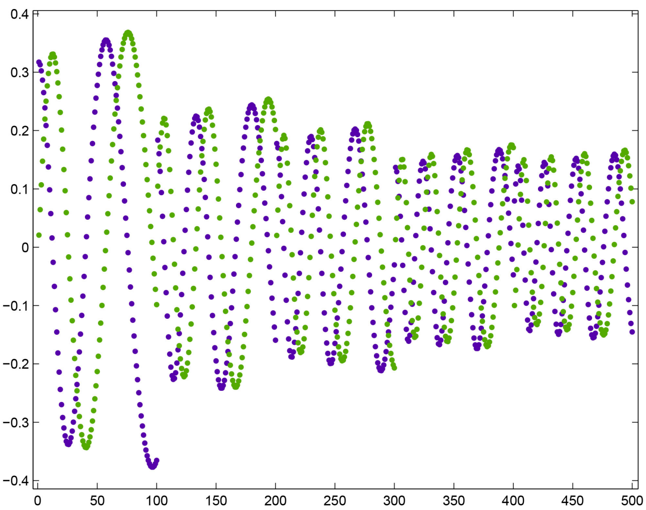

Figure 21

The curves slowly converge.

A plot of the eigenvalues of the 100x100 Hermitian matrix generated from the real parts is

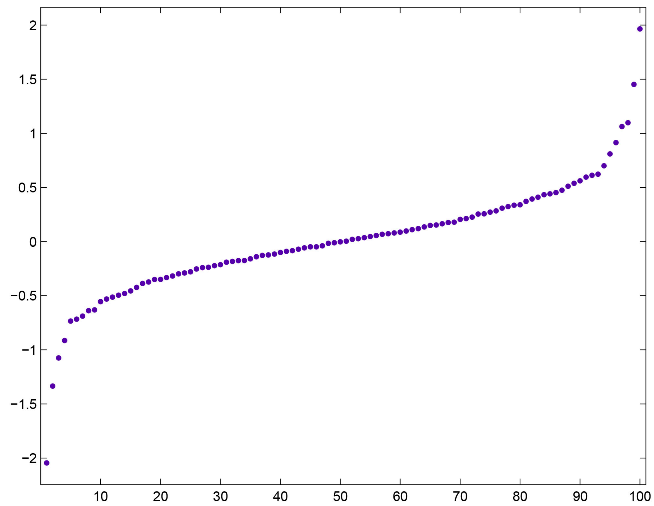

Figure 22

A normal probability plot of the eigenvalues is

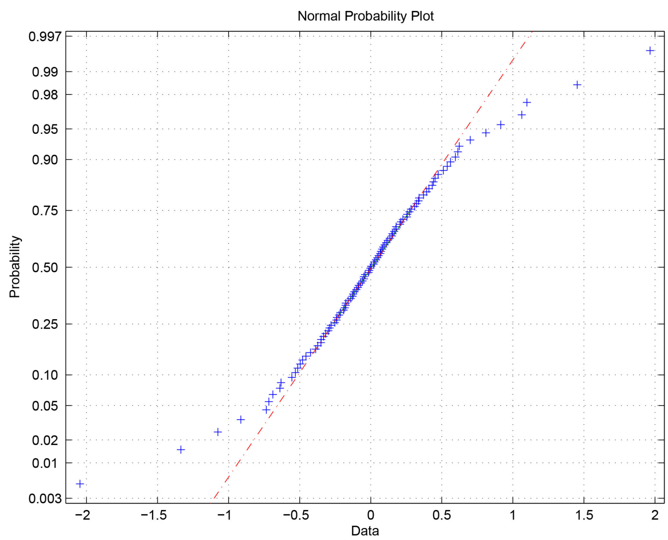

Figure 23

A plot of the convergents of the real parts of $D(n, a, b)$ versus the imaginary parts for these eigenvalues is

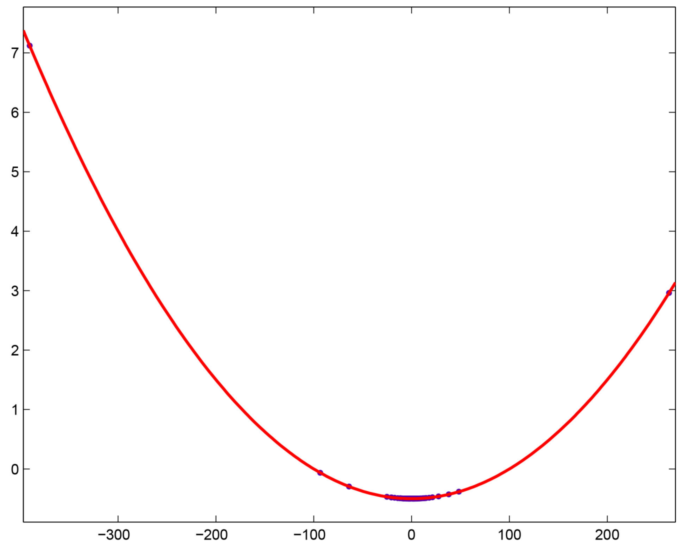

Figure 24

For a quadratic least-squares fit of the curve, $p_1 = 5.002 \cdot 10^{-5}$ with a 95% confidence interval of $(5.001 \cdot 10^{-5}, 5.002 \cdot 10^{-5})$, $p_2 = -1.281 \cdot 10^{-6}$ with a 95% confidence interval of $(-1.538 \cdot 10^{-6}, -1.226 \cdot 10^{-6})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5, -0.5)$, SSE=$1.076 \cdot 10^{-7}$, R-squared=1, and RMSE=$3.331 \cdot 10^{-5}$.

A plot of the first component of the Fourier transform (neglecting the first point) is

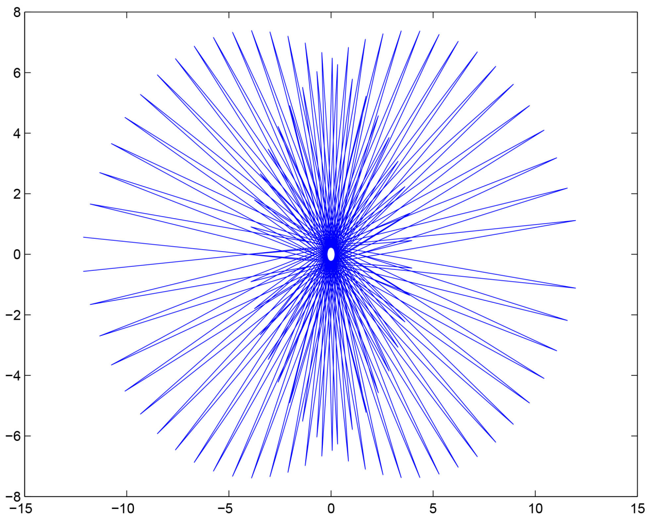

Figure 25

A plot of the real parts is

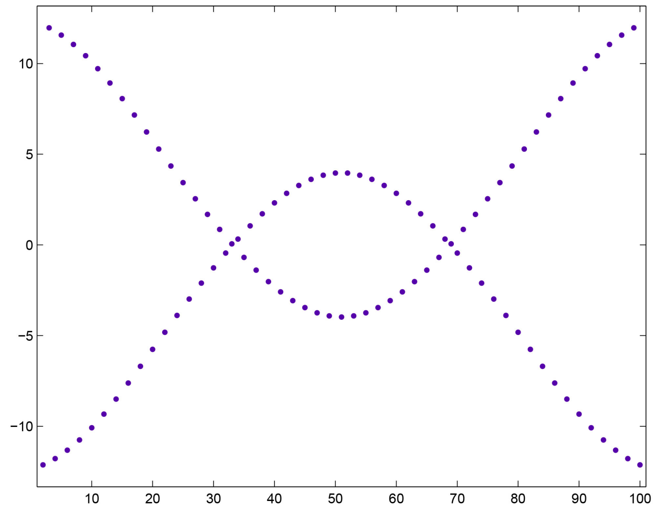

Figure 26

A plot of the second component of the Fourier transform (neglecting the first point) is

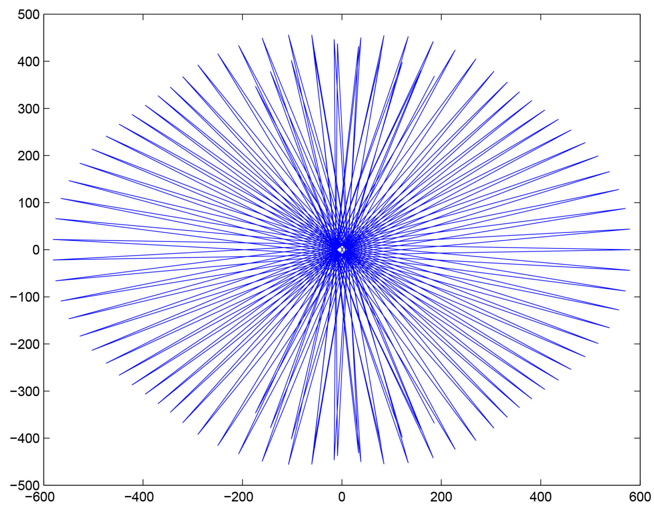

Figure 27

A plot of the real parts is

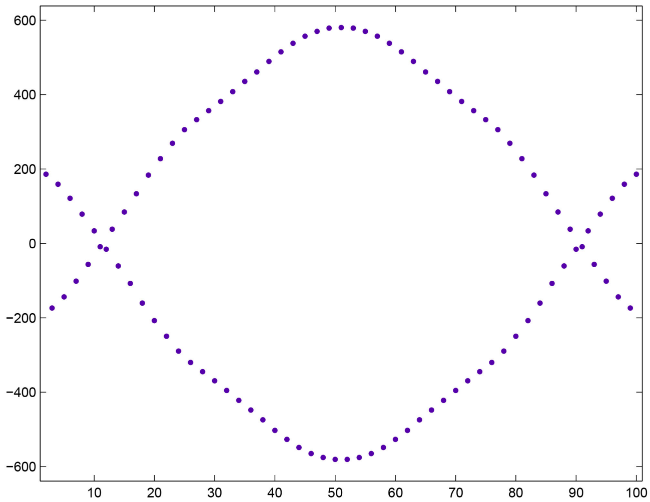

Figure 28

A plot of w_n, $n = 1$ to 99, for the above eigenvalues is

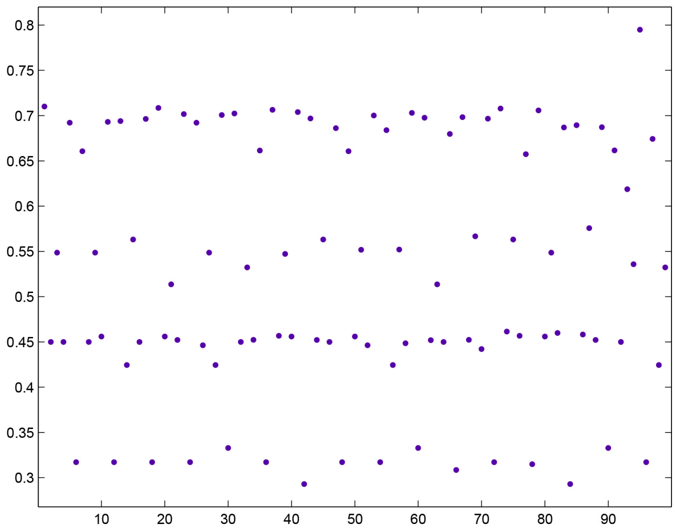

Figure 29

There are now four sets of points. A plot of the “sorted” values is

miscellaneous
1 192.435040
91 191.340060
87 187.006791
93 188.663211

primes
11 186.140896
13 191.023013
17 190.232316
19 186.275871
23 190.208067
29 189.786826
31 189.801191
37 190.750318
41 190.621086
43 189.988155
47 190.065823
53 191.800011
59 190.633991
61 192.143256
67 190.704210
71 188.809909
73 191.375524
79 189.958922
83 192.150558
89 188.916356
97 180.713050

multiples of 5
65 188.674372
85 189.269471
95 185.945651

powers of 5
5 187.997590*
25 187.997590*

powers of 7
7 186.599840*
49 186.599840*

powers of 3
3 186.224570*
9 186.224570*
27 186.224570*
81 186.224570*

multiples of 15
15 185.571680*
45 185.571680*
75 185.571680*

3 times primes
69 186.513597
51 185.698512
39 185.465058
57 181.030569

5 times primes
35 182.623476
55 182.246805

7 times primes
77 182.462698

multiples of 21
21 181.251238*
63 181.251238*

multiples of 11
33 181.228858*
99 181.228858*

powers of 2
2 97.079030*
4 97.079030*
8 97.079030*
16 97.079030*
32 97.079030*
64 97.079030*

2 times primes
22 91.938790,
26 97.240964,
34 96.511680,
38 94.113885,
46 95.833606,
58 96.578375,
62 95.300467,
74 97.467584,
82 97.130950,
86 96.023870,
94 101.747383,

miscellaneous
52 97.240964
68 96.511680
92 95.833606
76 94.113885

multiples of 6
6 93.876100*
12 93.876100*
18 93.876100*
24 93.876100*

36 93.876100*
48 93.876100*
54 93.876100*
72 93.876100*
96 93.876100*

multiples of 10
10 93.618659*
20 93.618659*
40 93.618659*
50 93.618659*
80 93.618659*

multiples of 14
14 92.592230*
28 92.592230*
56 92.592230*

multiples of 30
30 92.224760*
60 92.224760*

miscellaneous
78 92.109831
90 92.224760
98 92.592230

multiples of 22
44 91.938790*
88 91.938790*

multiples of 42
42 88.567632*
84 88.567632*

miscellaneous
66 88.622780
70 87.299191

The asterisks in a group indicate the same value for the n values. Prime powers have the same value. The primes greater than 11 usually have the largest values. The largest value for a non-prime n value occurs for $n = 85$. The value is 189.269471. This is greater than the values for n equal to 71, 89, and 97. In general, the values for n close to 1 and 100 (corresponding to the tails of the eigenvalue distribution) are erratic. Primes and their powers are the basis of the "prime" staircase where the step is determined by the Mangoldt function. The Mangoldt function, denoted by $\Lambda(n)$, equals $\log p$ if $n = p^k$ for some prime p and integer $k \geq 1$ or 0 otherwise. See Mazur and Stein [3] for an introduction to the prime staircase and the Riemann hypothesis.

A plot of the sorted w_n, $n = 1$ to 199, values for the eigenvalues of a 200x200 Hermitian matrix generated from the real parts $\alpha'(s,n)$ where s is the first zeta function zero is

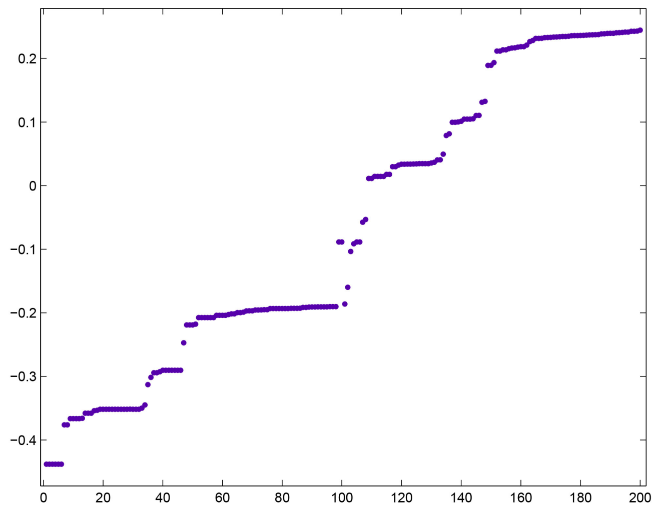

Figure 30

The values less than -0.1 are similar to the values greater than -0.1.

9. $Re(s) = 0.45$

In this section, the imaginary parts of zeta function zeros are used but the real parts of s values are set to 0.45. Hermitian matrices are then generated from the real parts of

$\alpha'(s,n)$ for $n = 201$ to 300. A plot of the eigenvalues of the 100x100 Hermitian matrix is

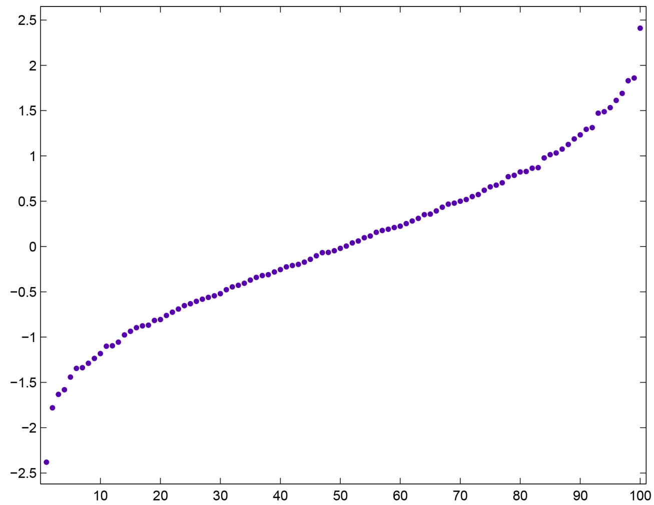

Figure 31

A normal probability plot of the eigenvalues is

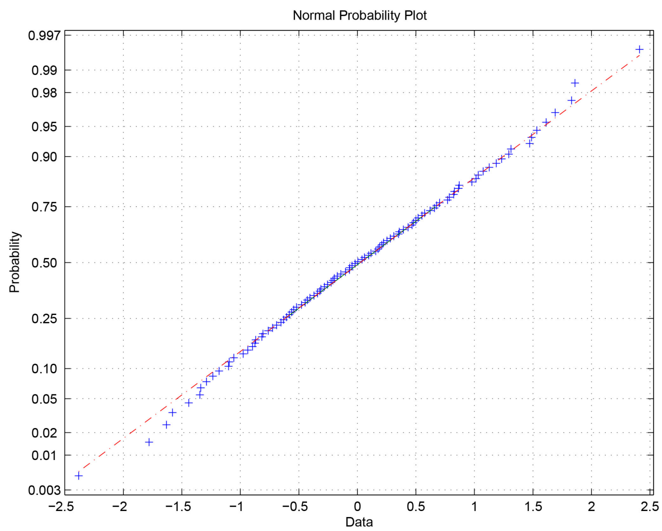

Figure 32

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts for these eigenvalues is

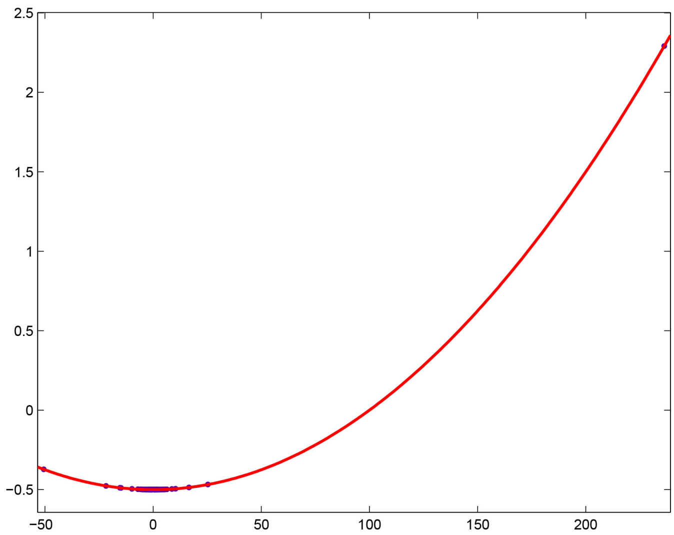

Figure 33

For a quadratic least-squares fit of the curve, $p_1 = 5.001 \cdot 10^{-5}$ with a 95% confidence interval of $(5.001 \cdot 10^{-5}, 5.001 \cdot 10^{-5})$, $p_2 = 1.599 \cdot 10^{-7}$ with a 95% confidence interval of $(1.282 \cdot 10^{-7}, 1.915 \cdot 10^{-7})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5, -0.5)$, SSE=$1.558 \cdot 10^{-10}$, R-squared=1, and RMSE=$1.267 \cdot 10^{-6}$.

A plot of the first component of the Fourier transform (neglecting the first point) is

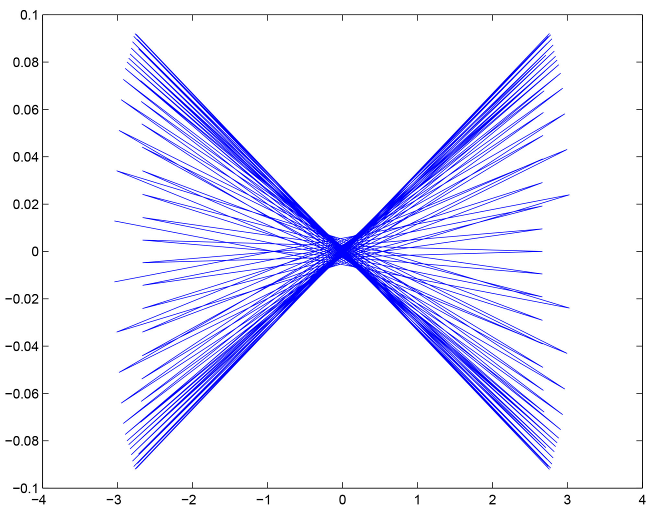

Figure 34

A plot of the real parts is

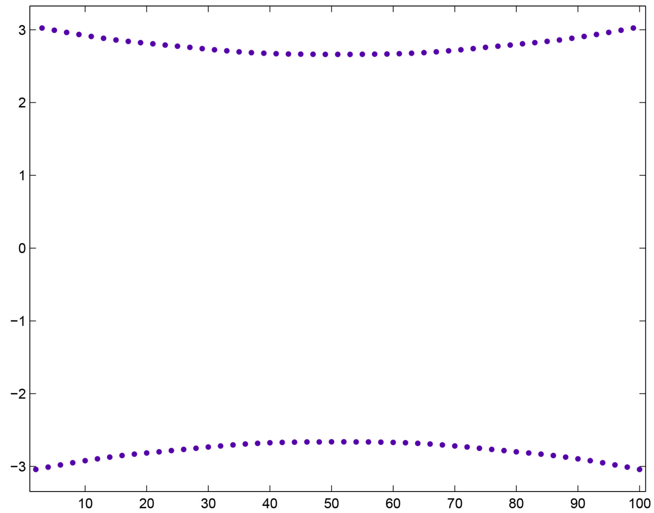

Figure 35

A plot of the second component of the Fourier transform (neglecting the first point) is

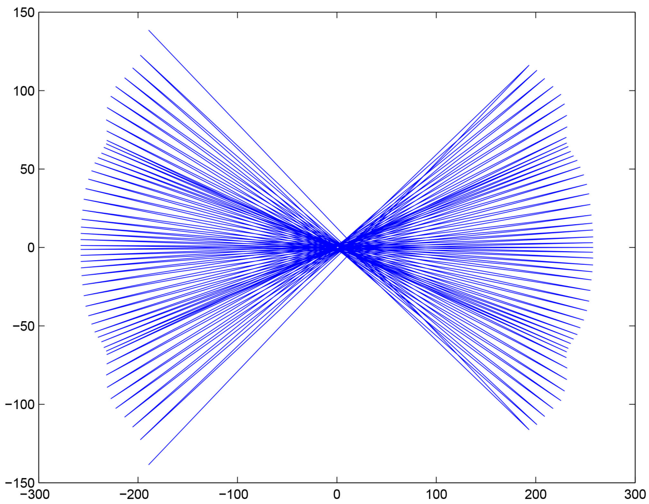

Figure 36

A plot of the real parts is

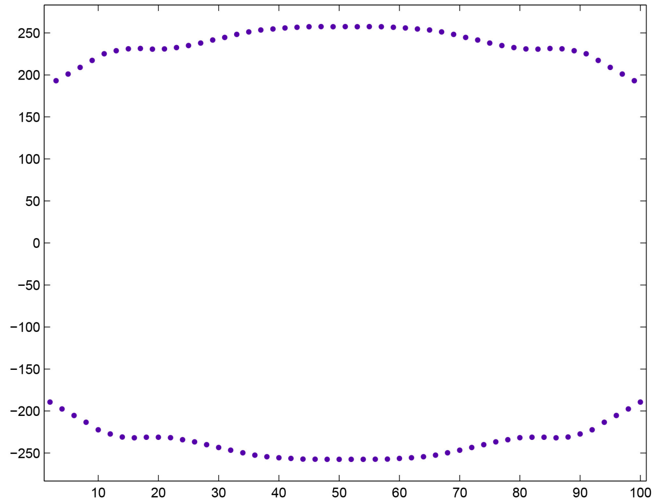

Figure 37

The results are similar to those obtained for the zeta function zeros. The main difference appears to be the real parts of the first and second components of the Fourier transforms.

10. $Re(s) = 0.55$

In this section, the imaginary parts of zeta function zeros are used but the real parts of s values are set to 0.55. Hermitian matrices are then generated from the real parts of $\alpha'(s,n)$ for $n = 201$ to 300. A plot of the eigenvalues of the 100x100 Hermitian matrix is

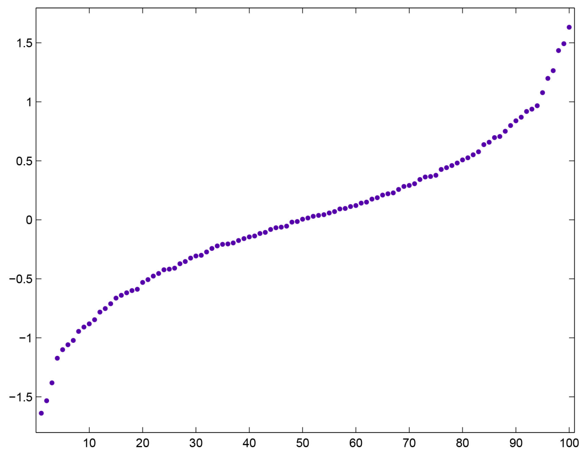

Figure 38

A normal probability plot of the eigenvalues is

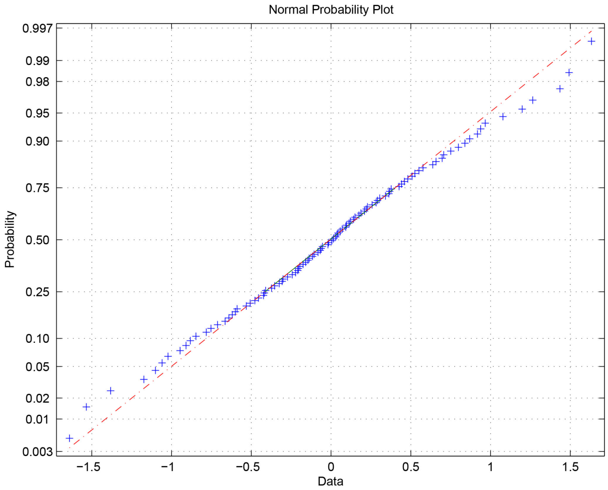

Figure 39

A plot of the convergents of the real parts of $D(n, a, b)$ versus the imaginary parts for these eigenvalues is

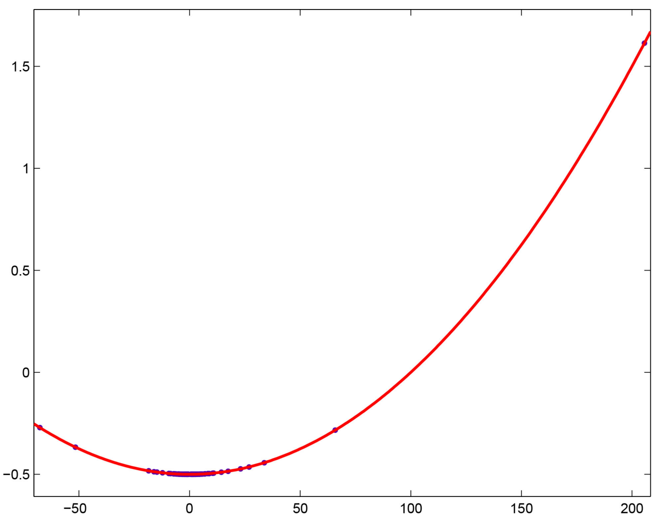

Figure 40

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of $(5.0 \cdot 10^{-5}, 5.0 \cdot 10^{-5})$, $p_2 = 8.048 \cdot 10^{-8}$ with a 95% confidence interval of $(3.617 \cdot 10^{-8}, 1.28 \cdot 10^{-7})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5, -0.5)$, SSE=$9.205 \cdot 10^{-10}$, R-squared=1, and RMSE=$1.308 \cdot 10^{-6}$.

A plot of the first component of the Fourier transform (neglecting the first point) is

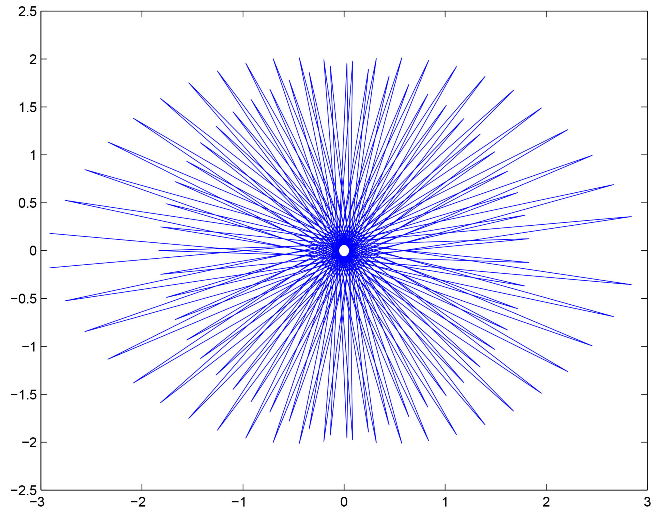

Figure 41

A plot of the real parts is

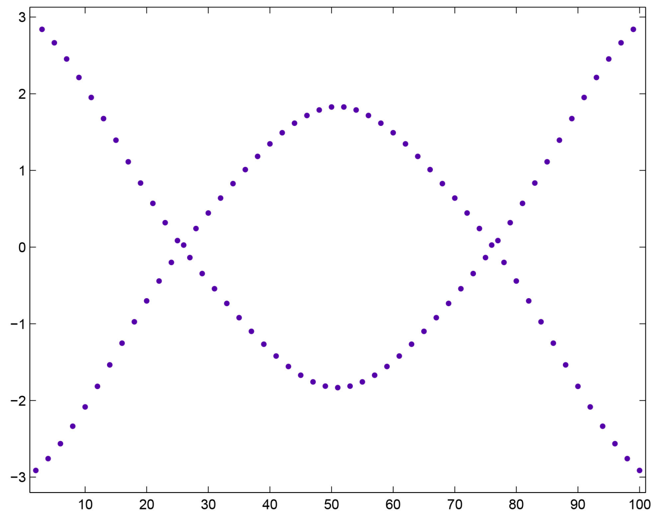

Figure 42

A plot of the second component of the Fourier transform (neglecting the first point) is

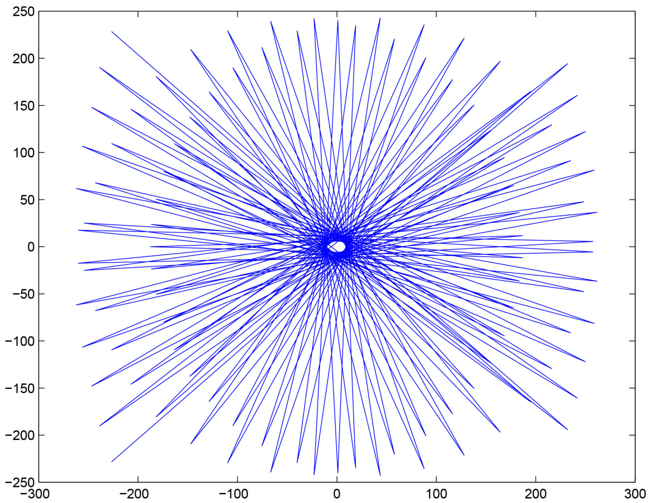

Figure 43

A plot of the real parts is

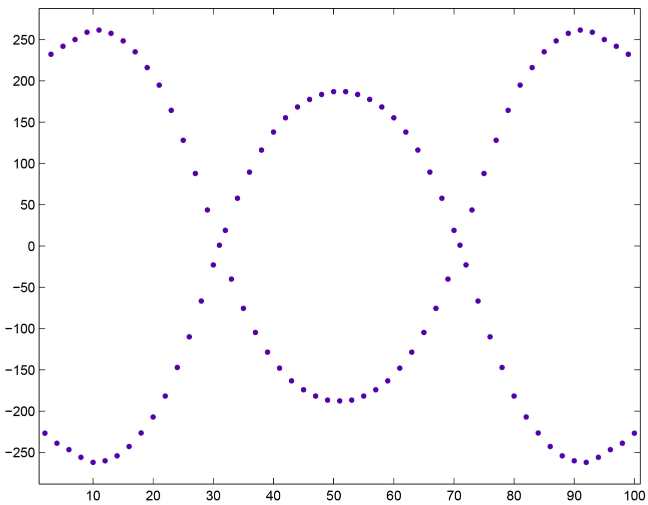

Figure 44

The results are similar to those obtained for the zeta function zeros. The main difference appears to be in the real parts of the components of the Fourier transforms.

A plot of the real parts of the first components for $\alpha'(s,n)$ and $\Re(s)$ equal to 0.45, 0.475, and 0.50 is

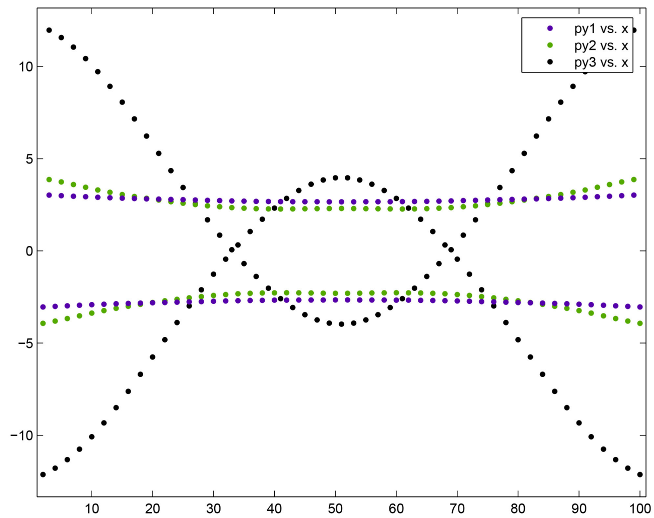

Figure 45

The "py3" curve is for $\Re(s)$ equal to 0.50.

A plot of the real parts of the second components for $\alpha'(s,n)$ and $\Re(s)$ equal to 0.45, 0.475, and 0.50 is

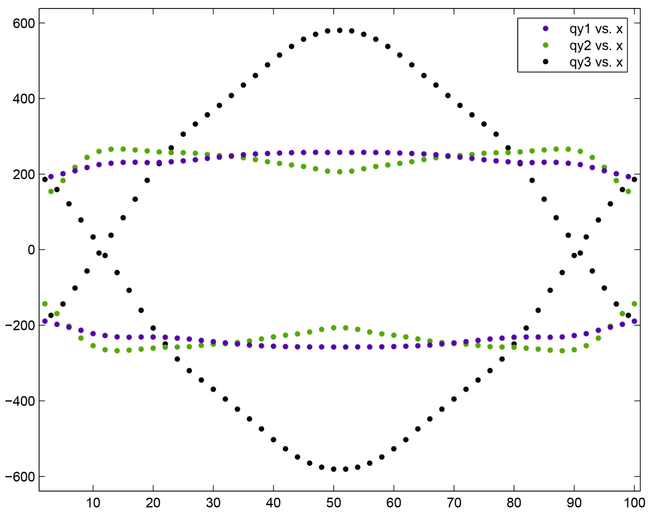

Figure 46

The "qy3" curve is for $\Re(s)$ equal to 0.50.

A plot of the real parts of the first components for $\alpha'(s,n)$ and $\Re(s)$ equal to 0.50, 0.525, and 0.55 is

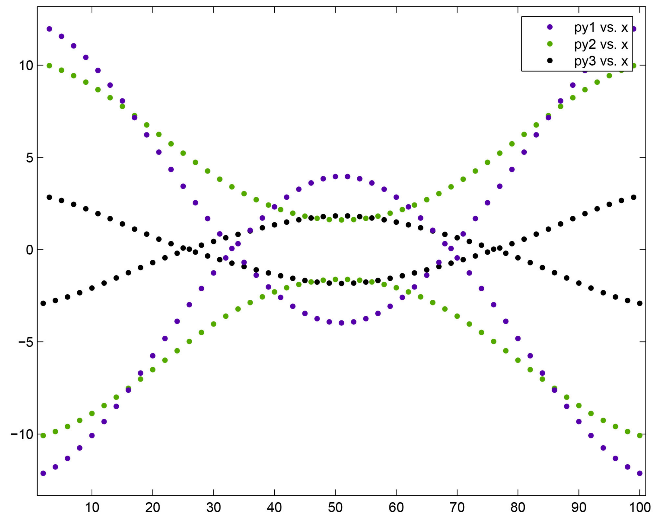

Figure 47

The "py1" curve is for $\Re(s)$ equal to 0.50.

A plot of the real parts of the second components for $\alpha'(s,n)$ and $\Re(s)$ equal to 0.50, 0.525, and 0.55 is

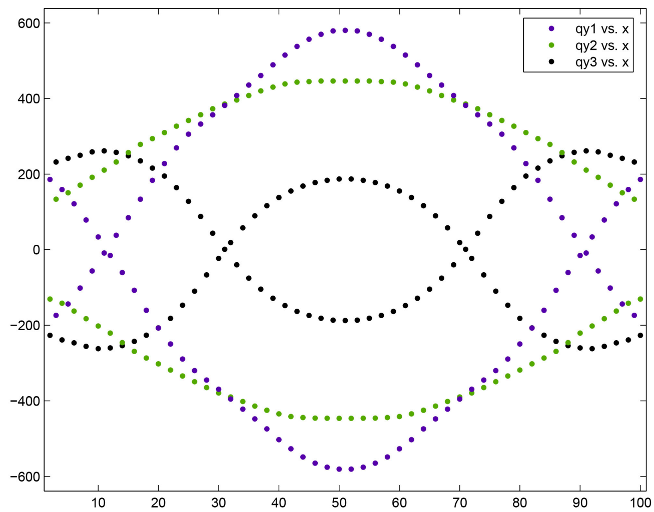

Figure 48

The "qy1" curve is for $\Re(s)$ equal to 0.50.

A plot of the real parts of the first components for $\alpha'(s,n)$ and $\Re(s)$ equal to 0.4997, 0.4998, 0.4999, 0.50, 0.5001, 0.5002, and 0.5003 is

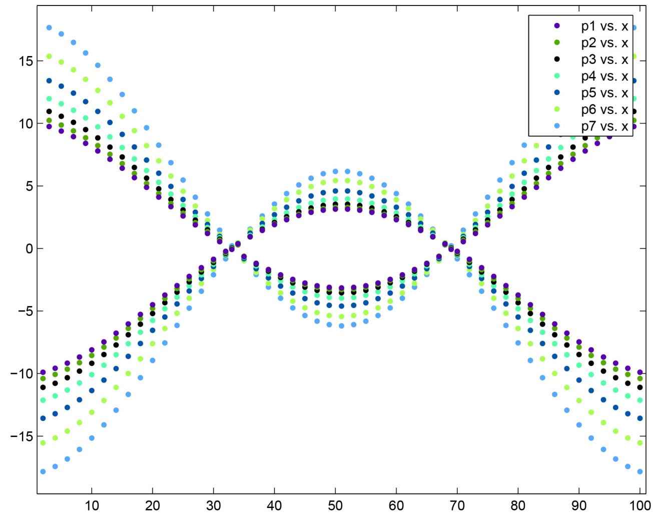

Figure 49

The "p4" curve is for $\Re(s)$ equal to 0.50.

A plot of the real parts of the second components for $\alpha'(s,n)$ and $\Re(s)$ equal to 0.4997, 0.4998, 0.4999, 0.50, 0.5001, 0.5002, and 0.5003 is

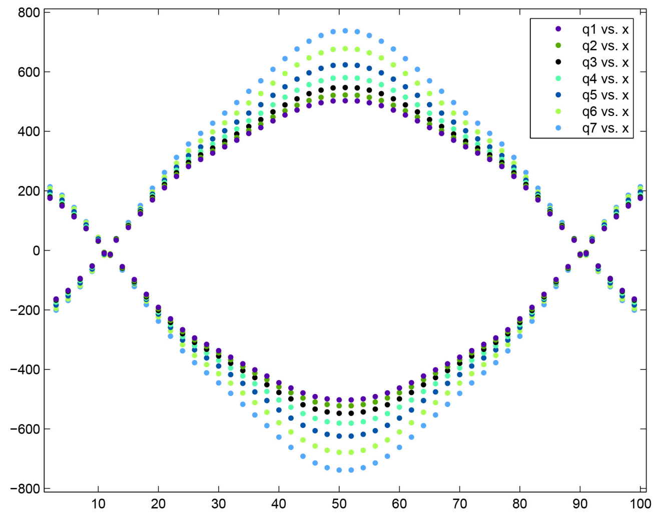

Figure 50

The "q4" curve is for $\Re(s)$ equal to 0.50.

A real part of the 0.50 is unique in that its cuve "contains" the other curves.

See Cox [4] for applications of the eigenvalues of Hermitian matrices to the Barnes G-function and a related function. Se Cox [5] for applications of the eigenvalues of Hermitian matrices to the Hilbert-Polya conjecture. See Cox [6] for more details on the reflection formula of the gamma function. See Cox and Bhattacharjee [7] for a variant gamma function pertaining to the Riemann hypothesis. See Cox [8] for variants of the gamma function and logarithmic spirals.

11. METHODS

The following C code computes the gamma function variant. The prime look-up table contains the primes less than 1500000.

```
//
// compute Mobius function
//
#include <math.h>
#include <stdio.h>
#include "table5.h" // prime look-up table
extern char *malloc();
int mobius(unsigned int a, unsigned int *table, unsigned int tsize) {
unsigned int i,count,p;
if (a==1)
    return(1);
count=0;
for (i=0; i<tsize; i++) {
    p=table[i];
    if (p>a)
        break;
    if (a==(a/p)*p) {
        a=a/p;
        if (a==(a/p)*p)
            return(0);
        count=count+1;
        if (a==1)
            break;
        }
    }
if ((count&1)==0)
    return(1);
```

```
else
    return(-1);
}
//
// compute Euler's phi function
//
int mobius(unsigned int a, unsigned int *t, unsigned int tsize);
unsigned int nueuler(unsigned int n, unsigned int *table,
unsigned int tsize) {
unsigned int d;
int sum;
if (n==1)
    return(1);
sum=0;
for (d=1; d<=n; d++) {
    if (n==(n/d)*d)
        sum=sum+(n/d)*mobius(d, table, tsize);
    }
return((unsigned int)sum);
}
//
// 2πζ(s−1)/(Z(s−1)Π(s−1))
//
unsigned int nueuler(unsigned int a, unsigned int *table, unsigned int tsize);
unsigned int max=100000;
double s=0.50; // usually set to 0.50
//double t=13.5;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
double pi=3.14159265359;
```

```
unsigned int n=1; // select n
unsigned int xmin=0;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
// usually set to 4 when the spiral is not output
unsigned int skip=0; // if set, don't do final multiplication
unsigned int tsize=114155; // size of prime look-up table
void main() {
unsigned int temp,x;
double *rsave,*isave,suma,sumb,sumr,sumi;
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
double R,I;
FILE *Outfp;
Outfp = fopen("transs1.dat","w");
if (max>1500000) {
    printf("max too large \n");
    return;
    }
rsave=(double*) malloc(16000004);
if (rsave==NULL)
    return;
isave=(double*) malloc(16000004);
if (isave==NULL)
    return;
//
// compute Π(s − 1)
//
prods=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    if (n==0)
        temp=nueuler(x,table,tsize);
    else
        temp=x;
    prods=prods*(double)temp/((double)temp+s);
    a=s-1.0;
    if (a>=0.0)
        temp1=pow((double)(x+1),a);
```

```
    else {
        temp1=pow((double)(x+1),-a);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
    b=prods*tempt+temps;
    rsave[x-1]=a;
    isave[x-1]=b;
    }
//
// divide 2π/Z(s − 1) by Π(s − 1)
//
sumr=0.0;
sumi=0.0;
for (x=1; x<=max; x++) {
    if (n==0)
        temp=nueuler(x,table,tsize);
    else
        temp=x;
    a=2.0*(s-1);
    if (a>=0.0)
        temp1=pow((double)temp,a);
    else {
        temp1=pow((double)temp,-a);
        temp1=1.0/temp1;
        }
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    a=sumr;
    b=sumi;
    c=2.0*pi;
    d=2.0*pi;
    temp1=a*a+b*b;
```

```
    e=(a*c+b*d)/temp1;
    f=-(a*d-b*c)/temp1;
    c=rsave[x-1];
    d=isave[x-1];
    temp1=c*c+d*d;
    temps=(c*e+d*f)/temp1;
    tempt=-(c*f-d*e)/temp1;
    rsave[x-1]=temps;
    isave[x-1]=tempt;
    }
//
// multiply by ζ(s − 1) (and ζ(s))
//
olds=0.0;
oldt=0.0;
sumr=0.0;
sumi=0.0;
suma=0.0;
sumb=0.0;
for (x=1; x<=max; x++) {
    if (n==0)
        temp=nueuler(x,table,tsize);
    else
        temp=x;
    a=s-1.0;
    if (a>0.0)
        temp1=pow((double)temp,a);
    else {
        temp1=pow((double)temp,-a);
        temp1=1.0/temp1;
        }
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    if (s>0.0)
        temp1=pow((double)temp,s);
```

```
    else {
        temp1=pow((double)temp,-s);
        temp1=1.0/temp1;
        }
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    suma=suma+R/temp1;
    sumb=sumb-I/temp1;
    c=rsave[x-1];
    d=isave[x-1];
    temps=c*sumr-d*sumi;
    tempt=c*sumi+d*sumr;
    if (skip==0) {
        a=temps;
        b=tempt;
        temps=a*suma-b*sumb;
        tempt=a*sumb+b*suma;
        }
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        if ((out==4)&&((olds<0.0)&&(temps>0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        olds=temps;
        oldt=tempt;
        }
    }
fclose(Outfp);
return;
}
```

```
#include <math.h>
#include <stdio.h>
// #include ”zero1.h” // zeta function zeros
#include ”eign4x.h” // .49996
// #include ”eign3x.h” // .49997
// #include ”eign2x.h” // .49998
// #include ”eign1x.h” // .49999
// #include ”eig0x.h” // .50
// #include ”eigp1x.h” // .50001
// #include ”eigp2x.h” // .50002
// #include ”eigp3x.h” // .50003
// #include ”eigp4x.h” // .50004
//
// D(n,a,b) 10/2/2024 (dkc)
// compute exponentially weighted sum
// similar results are obtained for zeta function zeros
// and eigenvalues of Hermitian matrices
//
unsigned int max=500001;
double a=0.00005; // real part
unsigned int size=100;
void main() {
    unsigned int x,i;
    double esumr,esumi,tempr,b;
    FILE *Outfp;
    Outfp = fopen(”weight.dat”,”w”);
    for (i=1; i<=size; i++) { // number of input values
        b=zero[i-1]; // imaginary part
        esumr=0.0;
        esumi=0.0;
        for (x=1; x<=(max-1); x++) {
            tempr=1.0/exp((double)x*a);
            esumr=esumr+tempr*cos((double)x*b);
            esumi=esumi+tempr*sin((double)x*b);
            }          fprintf(Outfp,” %.10llf, %.10llf, %.10llf,\n”,b,esumr,esumi);
        printf(” %d, %.10llf, %.10llf,          }
    fclose(Outfp);
    return;
```

}

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