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The Gamma Function Reflection Formula and Eigenvalues of Hermitian Matrices

Darrell Cox[1]

[1] *Department of Mathematics, Grayson County College, United States*

Abstract

The eigenvalues of Hermitian matrices are used to make a "staircase" of the primes similar to that generated by the Riemann zeta function zeros and the Mangoldt function.

Keywords: Riemann zeta function, gamma function, eigenvalues of Hermitian matrices, staircase of the primes, Mangoldt function

1. INTRODUCTION

Equation (3) in section 1.3 of Edward's [1] book is

$$\Pi(s) = \lim_{N\to\infty} \frac{1\cdot 2\cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s \tag{1}$$

This equation is valid for all s in the halfplane Re $s > -1$. (Edwards uses the notation $\Pi(s-1)$ instead of $\Gamma(s)$.)

2. THE REFLECTION FORMULA FOR THE GAMMA FUNCTION

Equation 6.5 in Voros's [2] article (the reflection formula for $\Gamma(z)$) is

$$Z(s) = \sum_{k=1}^{\infty} k^{-2s} = \zeta(2s) \tag{2}$$

A function involving the reflection formula is

$$\zeta_1(s,n) = \frac{2\pi\zeta(s-1)}{\Pi(s-1)Z(s-1)} \tag{3}$$

A plot of this expression for the first non-trivial zeta function zero ($s = (0.5, 14.13472514173470)$) and $n \leq 1000$ is

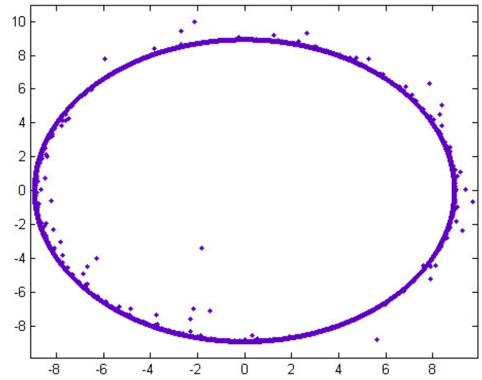

Figure 1

The slopes and intercepts of the logarithms of the n values of the inflection points (where the curve decreases toward the x-axis and then increases) for the first ten zeta function zeros are (0.4458, 0.588), (0.2983, 1.297), (0.2512, 1.35), (0.2068, 1.719), (0.1908, 1.595), (0.1672, 1.902), (0.1537, 1.892), (0.145, 1.941), (0.1311, 1.87), and (0.1262, 1.943). A plot of the logarithms of the n values of the inflection points for the tenth zeta function zero ($s = (0.5, 49.77383247767230)$) and $n \leq 100000$ is

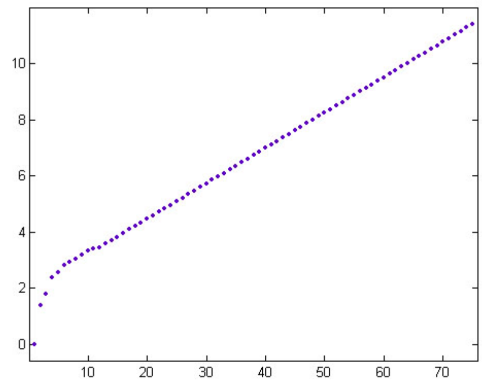

Figure 2

The first sixteen n values will be disregarded in computing the slope. This is unusual for the usual zeta function zeros - at most one value has to be disregarded. In computing the above slopes 1, 2, 6, 6, 7, 9, 9, 9, 13, and 16 n values were discarded respectively. Other than this, the slopes are almost the same as for the usual zeta function zeros. The slopes for the usual zeta function zeros and $n \leq 1000000$ are 0.4444, 0.2988, 0.2512, 0.2064, 0.1909, 0.1673, 0.1535, 0.145, 0.1309, and 0.1263.

A plot of the logarithms of the n values of the inflections points for the first twenty zeta function zeros is

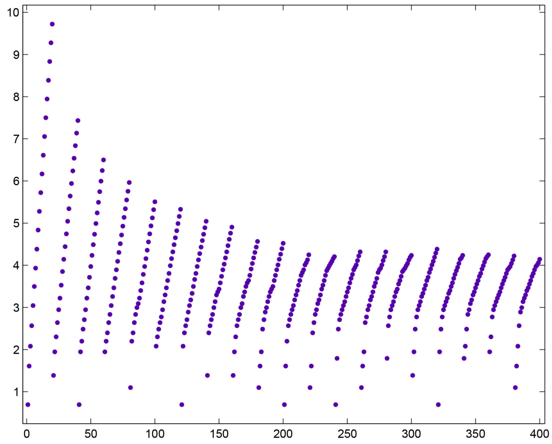

Figure 3

Note that there are 400 elements in the array. A plot of the eigenvalues of a 20x20 Hermitian matrix generated from these values (by averaging the 20x20 matrix with its transpose) is

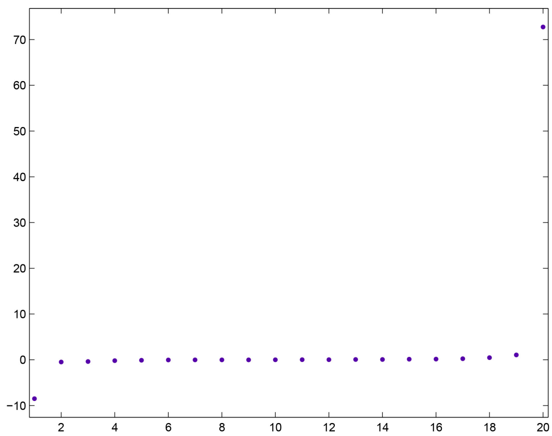

Figure 4

From this perspective, the logarithms of the n values of the inflection points are very uniform.

3. A FUNCTION FOR ANALYZING PROBABILITY DISTRIBUTIONS

The Riemann zeta function $\zeta(s)$ for $0 < Re(s) < 1$ can be computed from the η function;

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) \tag{4}$$

Let $C(n, a, b)$ denote

$$\frac{2 \cdot n^{-a}}{1 - 2^{1-s}} \cdot \left(\sum_{j=1}^{n-1} \frac{(-1)^{j+1}}{j^s} \cdot \cos\left(b \cdot \left(\ln\left(\frac{n}{j}\right)\right)\right)\right) \tag{5}$$

where $s = (a, b)$.

A Dirichlet series with exponential terms is

$$D(s) = \sum_{k=1}^{\infty} e^{-ks} \tag{6}$$

where $s = (a, b)$. For $\Re(s) > 0$, the series converges to $e^{-s}/(1 - e^{-s})$. The real part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \cos(kb)$ and the imaginary part can be expressed as $\sum_{k=1}^{\infty} e^{-ka} \sin(kb)$.

In an example, the $C(n, a, b)$ function is computed for j values of inflection points (where the curve crosses the x-axis from above) that are at least three greater than the previous j value, $n = 500001$, and $s = (0.00005, 500.30908494169051)$ (a zeta function zero when $\Re s = 0.50$). A plot of the logarithms of the j values of the inflection points is

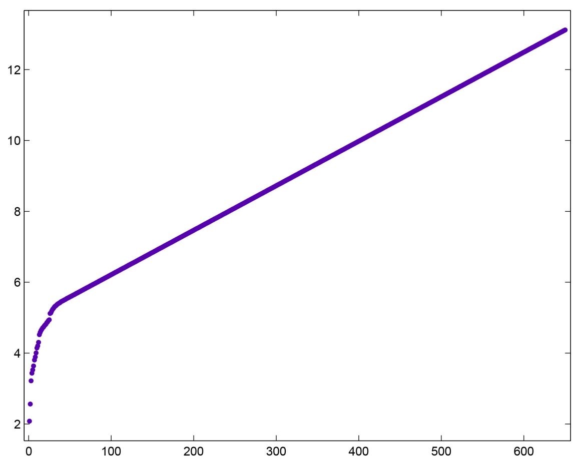

Figure 5

A plot of the real and imaginary parts of the exponentially weighted Dirichlet series computed at these j values versus the logarithms of the j values of the inflection points is

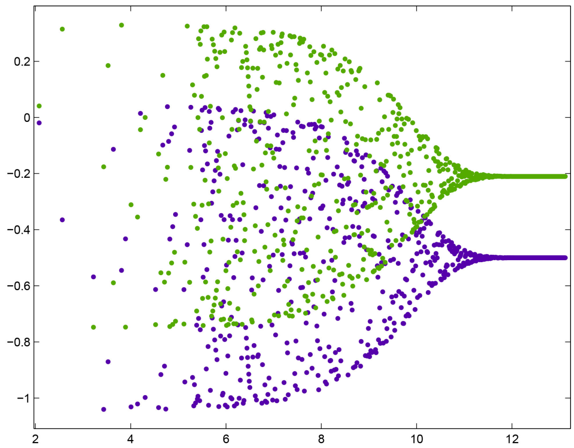

Figure 6

This exponentially weighted Dirichlet series will be denoted by $D(n, a, b)$ to distinguish it from the D series. The series converges to about (−0.5000, −0.2102).

4. AN APPLICATION TO EIGENVALUES OF HERMITIAN MATRICES

A plot of the convergents of the real parts of $D(n, a, b)$ versus the imaginary parts (that is, $C(n, a, b)$ weighted by $D(n, a, b)$) for the above eigenvalues is

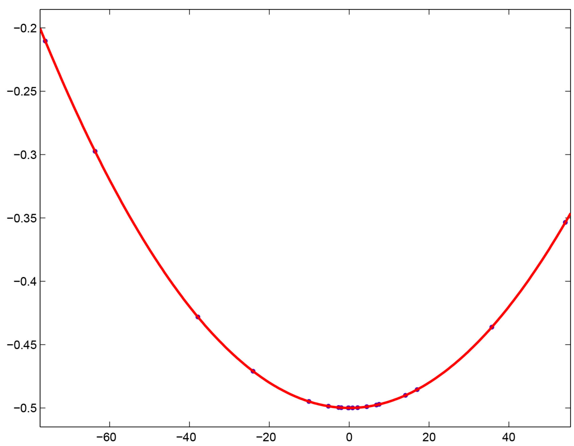

Figure 7

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of $(5.0 \cdot 10^{-5}, 5.0 \cdot 10^{-5})$, $p_2 = -5.464 \cdot 10^{-9}$ with a 95% confidence interval of $(-1.148 \cdot 10^{-8},\ 2.073 \cdot 10^{-10})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5, -0.5)$, SSE=$1.1324 \cdot 10^{-12}$, R-squared=1, and RMSE=$2.877 \cdot 10^{-7}$. The tenth eigenvalue (with a small absolute value of $2.72966475 \cdot 10^{-5}$) is omitted to avoid skewing the distribution. This eigenvalue is not omitted in the following. The expected p_1 parameter is the real part of s and the expected p_3 parameter is -0.5.

A plot of the first component of the Fourier transform of this one-sided parabola (neglecting the first point) is

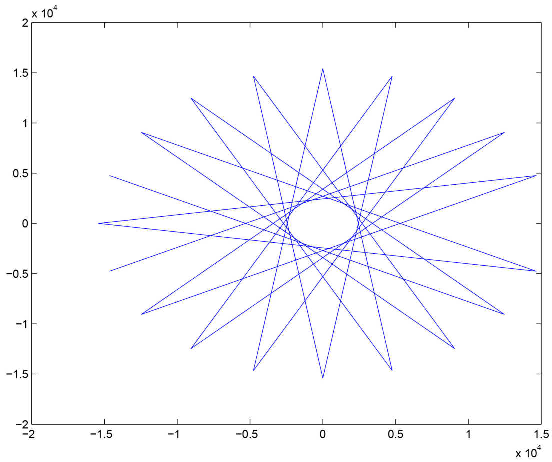

Figure 8

A plot of the second component of the Fourier transform (neglecting the first point) is

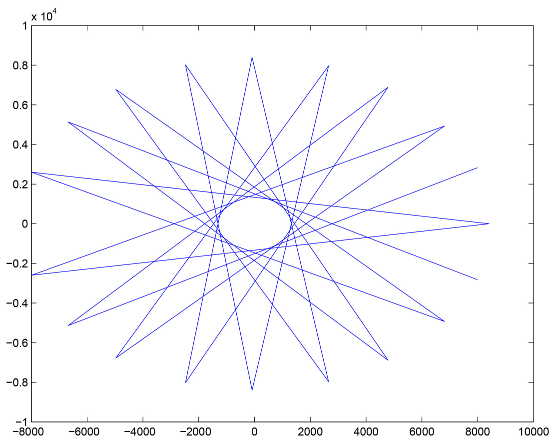

Figure 9

5. A FUNCTION DERIVED FROM $\zeta_1(s,n)$

Let $\alpha(s,n)$ denote $\zeta_1(s,n)\zeta(s,n)$ where the zeta function is defined for $\Re(s) > 1$. A plot of $\alpha(s,n)$ for the first zeta function zero and $n \leq 1000$ is

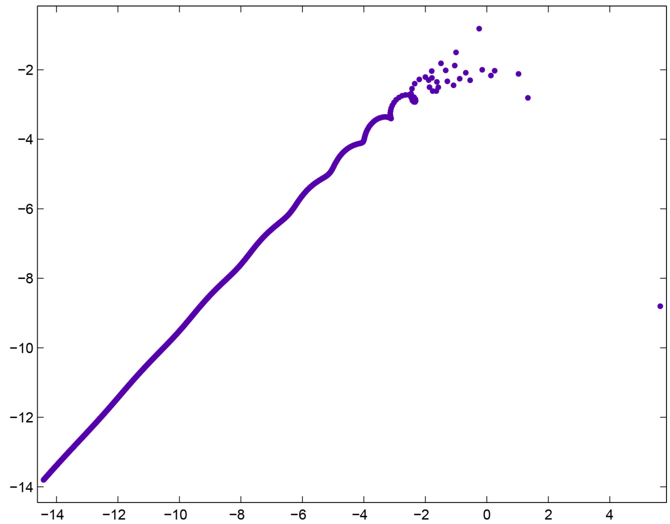

Figure 10

A plot of $\alpha(s,n)$ for $s = (0.5, 13.5)$ and $n \leq 1000$ is

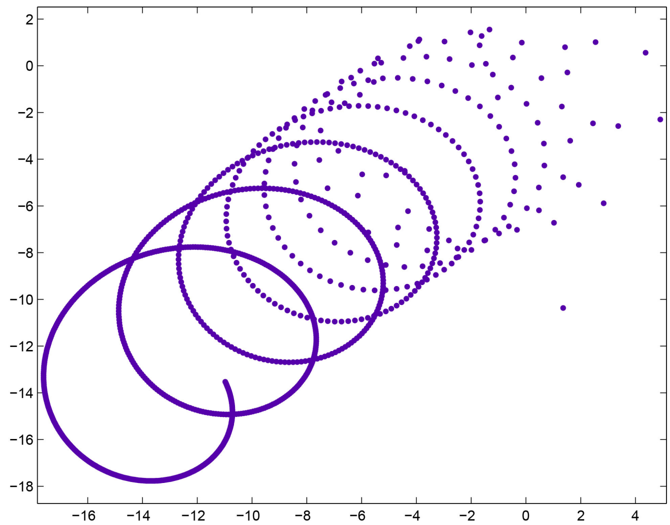

Figure 11

Such curves are typical for $\Re s$ values of 1/2 that are not zeta function zeros.

A plot of the real part of $\alpha(s, n)$ for the first hundred zeta function zeros and $n = 201$ to 300 is

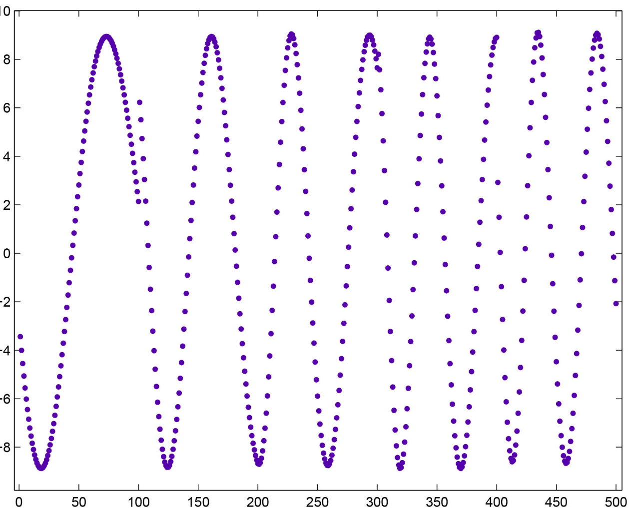

Figure 12

A plot of the real and imaginary components is

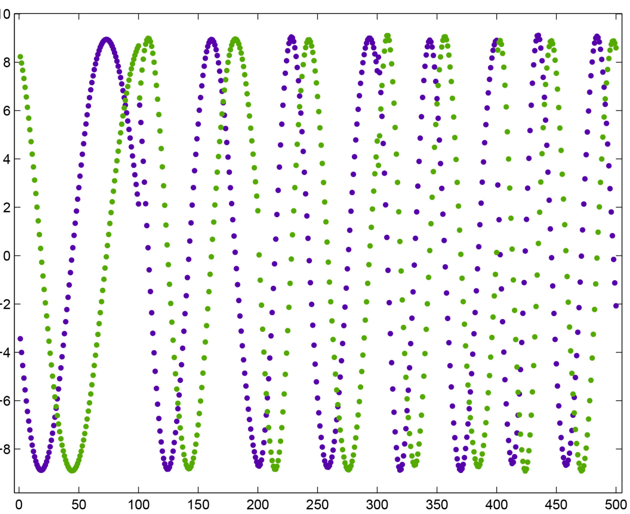

Figure 13

The curves resemble the sine and cosine functions.

A plot of the eigenvalues of the 100x100 Hermitian matrix generated from the real parts is

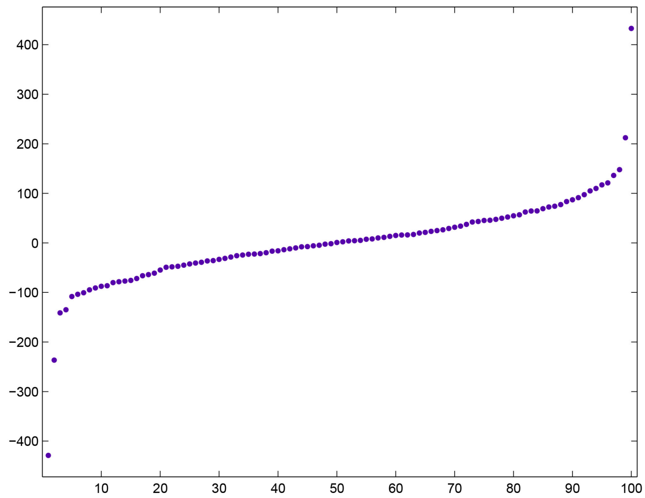

Figure 14

A normal probability plot of the eigenvalues is

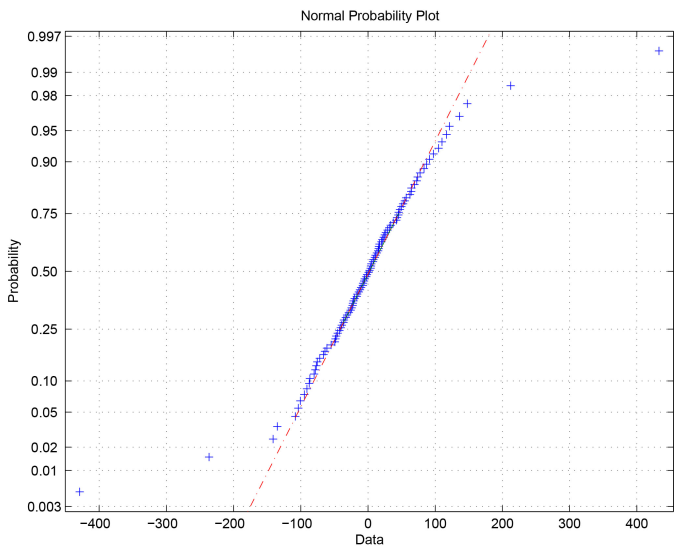

Figure 15

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts (that is, $C(n,a,b)$ weighted by $D(n,a,b)$) for these eigenvalues is

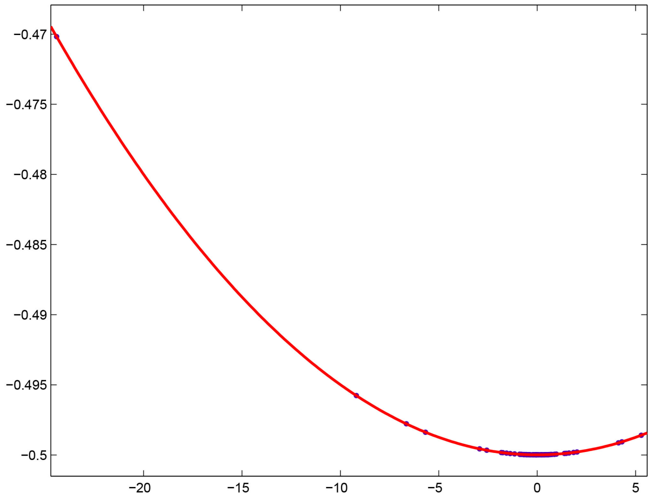

Figure 16

For a quadratic least-squares fit of the curve, $p_1 = 5.0 \cdot 10^{-5}$ with a 95% confidence interval of $(5.0 \cdot 10^{-5}, 5.0 \cdot 10^{-5})$, $p_2 = 2.806 \cdot 10^{-10}$ with a 95% confidence interval of $(-2.858 \cdot 10^{-10}, 8.469 \cdot 10^{-10})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5, -0.5)$, SSE=$1.741 \cdot 10^{-15}$, R-squared=1, and RMSE=$4.237 \cdot 10^{-9}$.

A plot of the first component of the Fourier transform (neglecting the first point) is

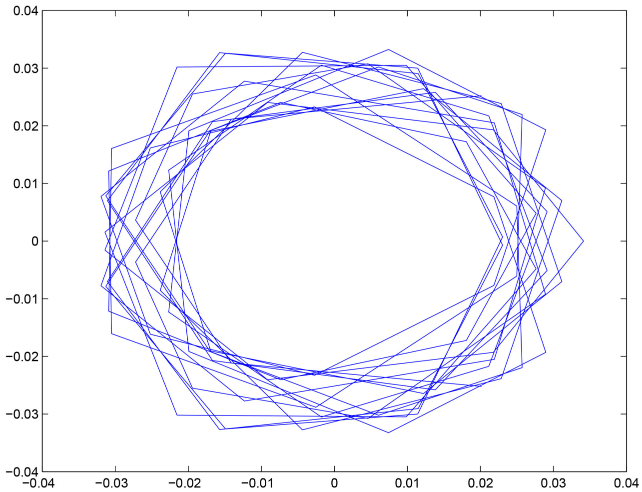

Figure 17

A plot of the second component of the Fourier transform (neglecting the first point) is

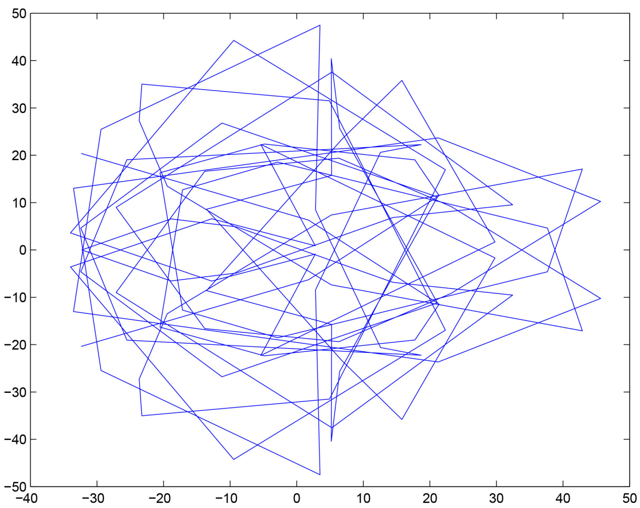

Figure 18

6.
A SUM INVOLVING THE MÖBIUS FUNCTION AND THE EIGENVALUES OF HERMITIAN MATRICES

Let w_n denote $\sum_{i|n}(e_{i+1} - e_i)\mu(i)$ where e_i denotes the eigenvalues of a Hermitian matrix. A plot of w_n, $n = 1$ to 99, for the above eigenvalues is

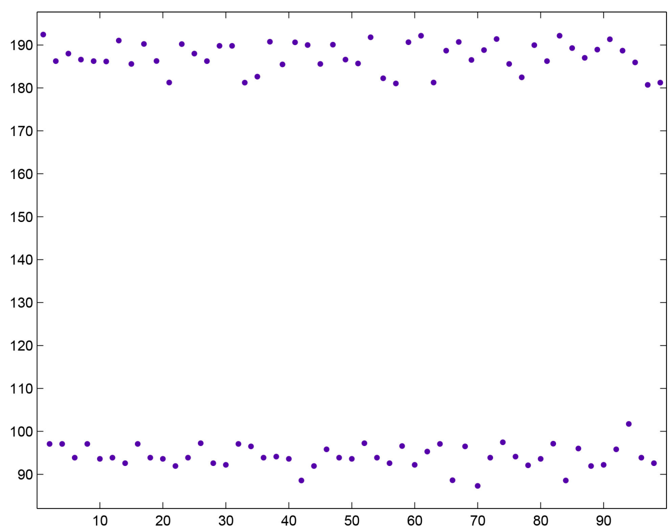

Figure 19

The upper points are for odd n values and the lower points are for even n values. The significance of the patterns in the points will be discussed later.

7. A SCALING FUNCTION INVOLVING THE GAMMA FUNCTION

Let $S(s,n)$ denote

$$\frac{\Gamma_n(-\frac{a}{2})\zeta_n(s)}{2\pi} \tag{7}$$

where the zeta function is defined for $\Re(s) > 1$. A plot of $S(s,n)$ for the first zeta function zero and $n \leq 1000$ is

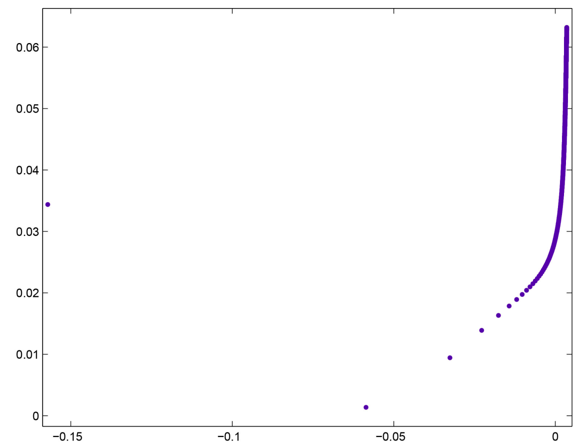

Figure 20

A plot of $S(s,n) \cdot \alpha(s,n)$ is

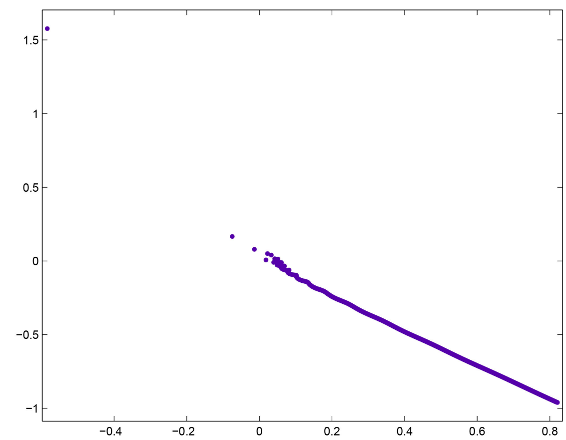

Figure 21

The multiplication changes the size and orientation of $\alpha(s,n)$.

8. A VARIANT OF $\alpha(s, n)$

Let $\alpha'(s, n)$ denote $S(s, n) \cdot \alpha(s)$. A plot of the real part of $\alpha'(s, n)$ for the first hundred zeta function zeros and $n = 201$ to 300 is

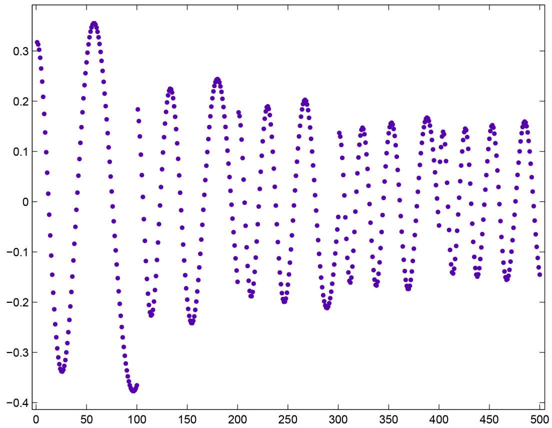

Figure 22

A plot of the real and imaginary components is

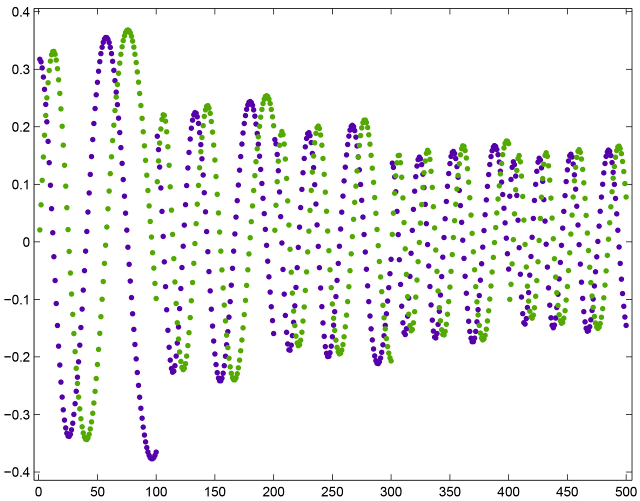

Figure 23

The curves slowly converge.

A plot of the eigenvalues of the 100x100 Hermitian matrix generated from the real parts is

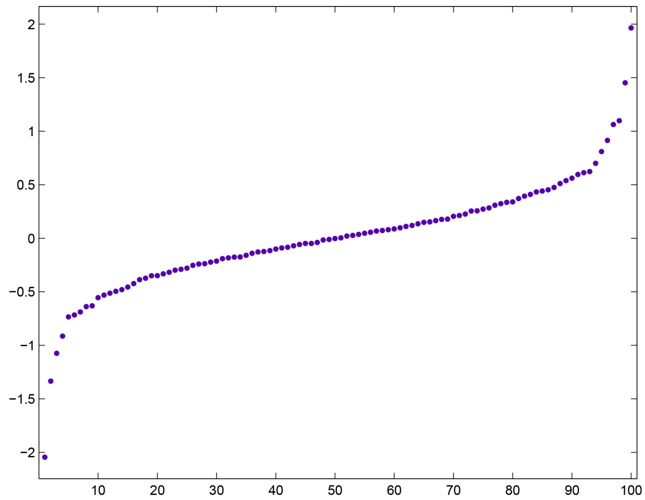

Figure 24

A normal probability plot of the eigenvalues is

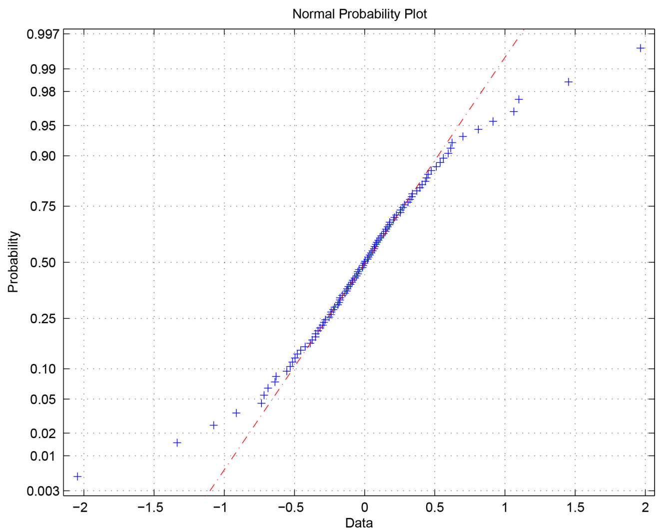

Figure 25

A plot of the convergents of the real parts of $D(n,a,b)$ versus the imaginary parts (that is, $C(n,a,b)$ weighted by $D(n,a,b)$) for these eigenvalues is

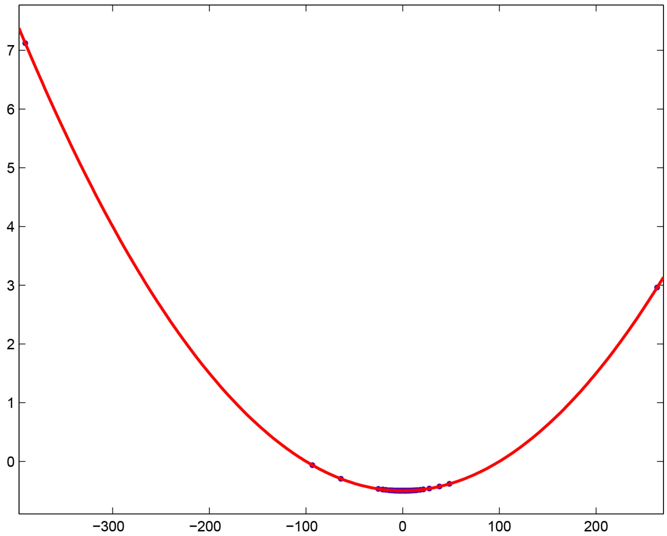

Figure 26

For a quadratic least-squares fit of the curve, $p_1 = 5.002 \cdot 10^{-5}$ with a 95% confidence interval of $(5.001 \cdot 10^{-5}, 5.002 \cdot 10^{-5})$, $p_2 = -1.281 \cdot 10^{-6}$ with a 95% confidence interval of $(-1.538 \cdot 10^{-6}, -1.226 \cdot 10^{-6})$, $p_3 = -0.5$ with a 95% confidence interval of $(-0.5, -0.5)$, SSE=$1.076 \cdot 10^{-7}$, R-squared=1, and RMSE=$3.331 \cdot 10^{-5}$.

A plot of the first component of the Fourier transform (neglecting the first point) is

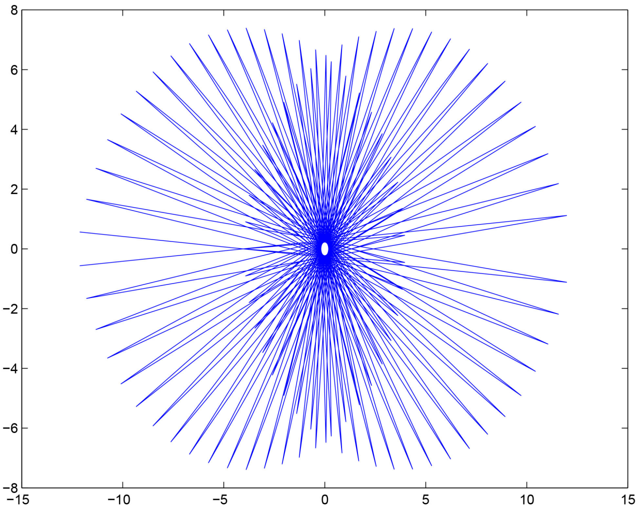

Figure 27

A plot of the real parts is

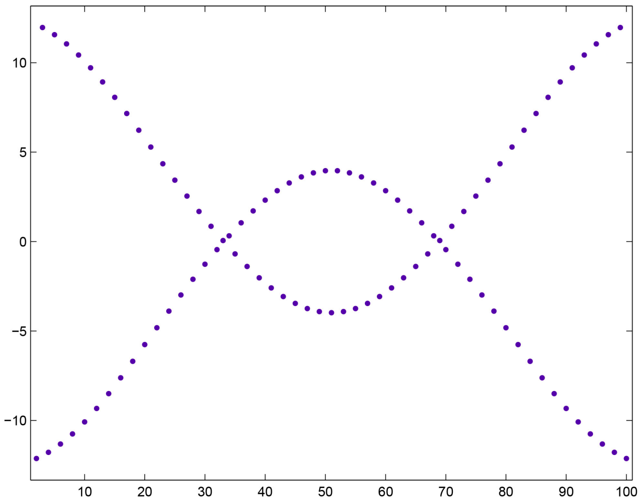

Figure 28

A plot of the second component of the Fourier transform (neglecting the first point) is

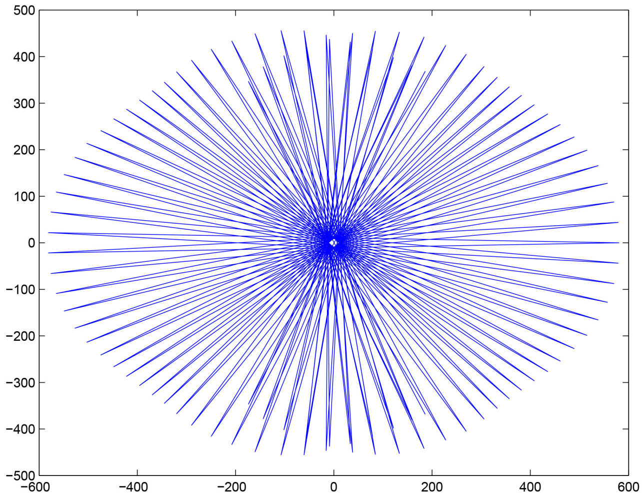

Figure 29

A plot of the real parts is

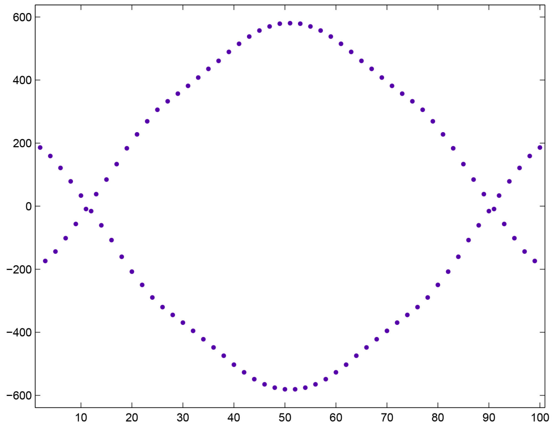

Figure 30

A plot of w_n, $n = 1$ to 99, for the above eigenvalues is

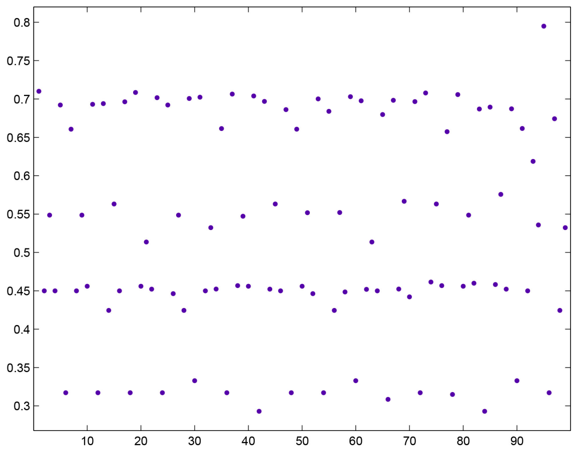

Figure 31

There are now four sets of points. A plot of the "sorted" values is

1 192.435040
91 191.340060
87 187.006791
93 188.663211

primes
11 186.140896

13 191.023013
17 190.232316
19 186.275871
23 190.208067
29 189.786826
31 189.801191
37 190.750318
41 190.621086
43 189.988155
47 190.065823
53 191.800011
59 190.633991
61 192.143256
67 190.704210
71 188.809909
73 191.375524
79 189.958922
83 192.150558
89 188.916356
97 180.713050

multiples of 5
65 188.674372
85 189.269471
95 185.945651

powers of 5
5 187.997590*
25 187.997590*

powers of 3
3 186.224570*
9 186.224570*
27 186.224570*
81 186.224570*

powers of 7
7 186.599840*
49 186.599840*

multiples of 15
15 185.571680*
45 185.571680*
75 185.571680*

3 times primes
39 185.465058
51 185.698512
57 181.030569
69 186.513597

5 times primes
35 182.623476
55 182.246805

7 times primes
77 182.462698

multiples of 21
21 181.251238*
63 181.251238*

multiples of 11
33 181.228858*
99 181.228858*

powers of 2
2 97.079030*
4 97.079030*
8 97.079030*
16 97.079030*

32 97.079030*
64 97.079030*

miscellaneous
52 97.240964
68 96.511680
76 94.113885
92 95.833606

multiples of 6
6 93.876100*
12 93.876100*
18 93.876100*
24 93.876100*
30 92.224760
36 93.876100*
42 88.567632
48 93.876100*
54 93.876100*
60 92.224760
66 88.622780
72 93.876100*
78 92.109831
84 88.567632
90 92.224760
96 93.876100*

multiples of 10
10 93.618659*
20 93.618659*
40 93.618659*
50 93.618659*
80 93.618659*

miscellaneous
70 87.299191

multiples of 14
14 92.592230*
28 92.592230*
56 92.592230*

miscellaneous
98 92.592230

multiples of 22
44 91.938790*
88 91.938790*

2 times primes
22 91.938790,
26 97.240964,
34 96.511680,
38 94.113885,
46 95.833606,
58 96.578375,
62 95.300467,
74 97.467584,
82 97.130950,
86 96.023870,
94 101.747383,

The asterisks in a group indicate the same value for the n values. Prime powers have the same value. The primes greater than 11 usually have the largest values. The largest value for a non-prime n value occurs for $n = 85$. The value is 189.269471. This is greater than the values for n equal to 71, 89, and 97. In general, the values for n close to 1 and 100 (corresponding to the tails of the eigenvalue distribution) are erratic. Primes and their powers are the basis of the ”prime” staircase where the step is determined by the Mangoldt function. The Mangoldt function, denoted by $\Lambda(n)$, equals $\log p$ if $n = p^k$ for some prime p and integer $k \geq 1$ or 0 otherwise. See Mazur and Stein [3] for an introduction to the prime staircase and the Riemann hypothesis.

See Cox [4] for applications of the eigenvalues of Hermitian matrices to the Barnes G-function and a related function. Se Cox [5] for applications of the eigenvalues of Hermitian matrices to the Hilbert-Polya conjecture. See Cox [6] for more details on the reflection formula of the gamma function. See Cox and Bhattacharjee [7] for a variant gamma function pertaining to the Riemann hypothesis. See Cox [8] for variants of the gamma function and logarithmic spirals.

9. METHODS

The following C code computes the gamma function variant. The prime look-up table contains the primes less than 1500000.

```
//
// compute Mobius function
//
#include <math.h>
#include <stdio.h>
#include "table5.h" // prime look-up table
extern char *malloc();
int mobius(unsigned int a, unsigned int *table, unsigned int tsize) {
unsigned int i,count,p;
if (a==1)
    return(1);
count=0;
for (i=0; i<tsize; i++) {
    p=table[i];
    if (p>a)
        break;
    if (a==(a/p)*p) {
        a=a/p;
        if (a==(a/p)*p)
            return(0);
        count=count+1;
        if (a==1)
            break;
        }
    }
if ((count&1)==0)
    return(1);
```

```
else
    return(-1);
}
//
// compute Euler's phi function
//
int mobius(unsigned int a, unsigned int *t, unsigned int tsize);
unsigned int nueuler(unsigned int n, unsigned int *table,
unsigned int tsize) {
unsigned int d;
int sum;
if (n==1)
    return(1);
sum=0;
for (d=1; d<=n; d++) {
    if (n==(n/d)*d)
        sum=sum+(n/d)*mobius(d, table, tsize);
    }
return((unsigned int)sum);
}
//
// 2πζ(s−1)/(Z(s−1)Π(s−1))
//
unsigned int nueuler(unsigned int a, unsigned int *table, unsigned int tsize);
unsigned int max=100000;
double s=0.50; // usually set to 0.50
//double t=13.5;
double t=14.13472514173470;
//double t=21.02203963877156;
//double t=25.01085758014569;
//double t=30.42487612585951;
//double t=32.93506158773919;
//double t=37.58617815882568;
//double t=40.91871901214750;
//double t=43.32707328091500;
//double t=48.00515088116716;
//double t=49.77383247767230;
double pi=3.14159265359;
```

```
unsigned int n=1; // select n
unsigned int xmin=0;
unsigned int out=1; // 2 for top, 3 for left, 4 for bottom, 5 for right
// usually set to 4 when the spiral is not output
unsigned int skip=0; // if set, don't do final multiplication
unsigned int tsize=114155; // size of prime look-up table
void main() {
unsigned int temp,x;
double *rsave,*isave,suma,sumb,sumr,sumi;
double temp1,temps,tempt,prods,a,b,c,d,e,f,olds,oldt,sums,sumt;
double R,I;
FILE *Outfp;
Outfp = fopen("transs1.dat","w");
if (max>1500000) {
    printf("max too large \n");
    return;
    }
rsave=(double*) malloc(16000004);
if (rsave==NULL)
    return;
isave=(double*) malloc(16000004);
if (isave==NULL)
    return;
//
// compute Π(s − 1)
//
prods=1.0;
sums=0.0;
sumt=0.0;
for (x=1; x<=max; x++) {
    if (n==0)
        temp=nueuler(x,table,tsize);
    else
        temp=x;
    prods=prods*(double)temp/((double)temp+s);
    a=s-1.0;
    if (a>=0.0)
        temp1=pow((double)(x+1),a);
```

```
    else {
        temp1=pow((double)(x+1),-a);
        temp1=1.0/temp1;
        }
    temps=temp1*(cos(t*log(x+1)));
    tempt=temp1*(sin(t*log(x+1)));
    a=prods*temps-tempt;
    b=prods*tempt+temps;
    rsave[x-1]=a;
    isave[x-1]=b;
    }
//
// divide 2π/Z(s − 1) by Π(s − 1)
//
sumr=0.0;
sumi=0.0;
for (x=1; x<=max; x++) {
    if (n==0)
        temp=nueuler(x,table,tsize);
    else
        temp=x;
    a=2.0*(s-1);
    if (a>=0.0)
        temp1=pow((double)temp,a);
    else {
        temp1=pow((double)temp,-a);
        temp1=1.0/temp1;
        }
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    a=sumr;
    b=sumi;
    c=2.0*pi;
    d=2.0*pi;
    temp1=a*a+b*b;
```

```
    e=(a*c+b*d)/temp1;
    f=-(a*d-b*c)/temp1;
    c=rsave[x-1];
    d=isave[x-1];
    temp1=c*c+d*d;
    temps=(c*e+d*f)/temp1;
    tempt=-(c*f-d*e)/temp1;
    rsave[x-1]=temps;
    isave[x-1]=tempt;
    }
//
// multiply by ζ(s − 1) (and ζ(s))
//
olds=0.0;
oldt=0.0;
sumr=0.0;
sumi=0.0;
suma=0.0;
sumb=0.0;
for (x=1; x<=max; x++) {
    if (n==0)
        temp=nueuler(x,table,tsize);
    else
        temp=x;
    a=s-1.0;
    if (a>0.0)
        temp1=pow((double)temp,a);
    else {
        temp1=pow((double)temp,-a);
        temp1=1.0/temp1;
        }
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    sumr=sumr+R/temp1;
    sumi=sumi-I/temp1;
    if (s>0.0)
        temp1=pow((double)temp,s);
```

```
    else {
        temp1=pow((double)temp,-s);
        temp1=1.0/temp1;
        }
    R=temp1*cos(t*log((double)temp));
    I=temp1*sin(t*log((double)temp));
    temp1=R*R+I*I;
    suma=suma+R/temp1;
    sumb=sumb-I/temp1;
    c=rsave[x-1];
    d=isave[x-1];
    temps=c*sumr-d*sumi;
    tempt=c*sumi+d*sumr;
    if (skip==0) {
        a=temps;
        b=tempt;
        temps=a*suma-b*sumb;
        tempt=a*sumb+b*suma;
        }
    if (x>xmin) {
        if (out==1)
            fprintf(Outfp," %.10llf %.10llf \n",temps,tempt);
        if ((out==2)&&((olds>0.0)&&(temps<0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        if ((out==3)&&((oldt>0.0)&&(tempt<0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        if ((out==4)&&((olds<0.0)&&(temps>0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        if ((out==5)&&((oldt<0.0)&&(tempt>0.0)))
            fprintf(Outfp," %.10llf \n",log((double)x));
        olds=temps;
        oldt=tempt;
        }
    }
fclose(Outfp);
return;
}
```

```
#include <math.h>
#include <stdio.h>
//
// compute C(n,a,b), 09/20/2024 (dkc)
//
unsigned int max=200001;
double a=0.0001;
double b=14.13472514173470;
//double b=21.02203963877156;
//double b=25.01085758014569;
//double b=30.42487612585951;
//double b=32.93506158773919;
//double b=37.58617815882568;
//double b=40.91871901214750;
//double b=43.32707328091500;
//double b=48.00515088116716;
//double b=49.77383247767230;
//double b=52.97032147771446;
//double b=56.44624769706339;
//double b=59.34704400260235;
//double b=60.83177852460981;
//double b=65.11254404808160;
//double b=67.07981052949417;
//double b=69.54640171117399;
//double b=72.06715767448191;
//double b=75.70469069908393;
//double b=77.14484006887480;
//double b=79.33737502024937;
//double b=84.73549298051705;
//double b=87.42527461312523;
//double b=88.80911120763446;
//double b=92.49189927055849;
//double b=94.65134404051989;
//double b=95.87063422824531;
//double b=98.83119421819369;
//double b=101.31785100573138;
unsigned int xmin=0; // usually set to 0
unsigned int out=2; // usually 1, 2 for inflection points
```

```
unsigned int out3p=1; // usually 0, 1 for differences in j values >=2
unsigned int polar=1; // set to use polar coordinates
void main() {
unsigned int x,oldx;
double sumr,sumi,R,I,temp1,oldsumr,oldsumi,temp,tempa,tempb,y,e,f,g;
double tempr,esumr,esumi;
FILE *Outfp;
Outfp = fopen(”c2nab2f.dat”,”w”);
y=1.0-a;
if (y>=0.0)
    temp1=pow((double)2,y);
else {
    temp1=pow((double)2,-y);
    temp1=1.0/temp1;
    }
e=temp1*(cos(b*log(2)));
f=temp1*(sin(b*log(2)));
e=1.0-e;
f=-f;
y=-a;
if (y>=0.0)
    temp1=pow((double)max,y);
else {
    temp1=pow((double)max,-y);
    temp1=1.0/temp1;
    }
y=2.0*temp1;
oldx=0;
sumr=0.0;
sumi=0.0;
oldsumi=0.0;
oldsumr=0.0;
esumr=0.0;
esumi=0.0;
for (x=1; x<=(max-1); x++) {
    tempr=1.0/exp((double)x*a);
    esumr=esumr+tempr*cos((double)x*b);
    esumi=esumi+tempr*sin((double)x*b);
```

```
temp1=pow((double)x,a);
R=temp1*cos(b*log((double)x));
I=temp1*sin(b*log((double)x));
temp1=R*R+I*I;
if (x!=(x/2)*2) {
     sumr=sumr+R/temp1;
     sumi=sumi-I/temp1;
     }
else {
     sumr=sumr-R/temp1;
     sumi=sumi+I/temp1;
     }
temp=cos(b*log((double)max/(double)x));
tempa=sumr*temp;
tempb=sumi*temp;
tempa=tempa*y;
tempb=tempb*y;
g=tempa*e-tempb*f;
tempb=tempa*f+tempb*e;
tempa=g;
if (x>xmin) {
     if (out==1)
          fprintf(Outfp," %.10llf, %.10llf \n",tempa,tempb);
     if ((out==2)&&((oldsumr¿0.0)&&(tempa¡0.0))) {
          if (out3p==0)
               fprintf(Outfp," %d %.10llf %.10llf \n",x,tempa,tempb);
          if ((x-oldx)>2) {
               printf(" %d %d %.10llf %.10llf \n",x,oldx,oldsumr,tempa);
               if (out3p!=0) {
                    if (polar==0)
                         fprintf(Outfp," %d %d \n",x,oldx);
                    else
                         fprintf(Outfp," %d %d %.10llf %.10llf %.10llf %.10llf
\n",x,oldx,
                              sqrt(tempa*tempa+tempb*tempb),atan2(tempb,tempa),esumr,esumi);
                    }
               }
          oldx=x;
```

```
            }
        oldsumr=tempa;
        oldsumi=tempb;
        }
    }
fclose(Outfp);
return;
}
```

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