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# Existence Theorems of Fixed Point for Enriched Contraction in Complex Valued Convex Metric Spaces *

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**Abstract**

The objective of this article is to establish some fixed point results for mapping $T : X \to X$ satisfying the enriched contraction in complete complex valued convex metric space. Second, we introduce a mapping $T : X \to X$ satisfying $\varphi$-enriched contraction and prove the existence of a fixed point of $T$ in complete complex valued convex metric space. The obtained results generalized and improved some results in the literature.

## 1. INTRODUCTION

Fixed point theory is a highly dynamic field within nonlinear analysis, widely recognized for its broad range of applications in disciplines like engineering, computer science, and economics. Contractive-type conditions are fundamental to this theory, playing a crucial role in its development. Due to its significance, numerous researchers have worked on extending and generalizing the Banach contraction principle, which serves as the core foundation of fixed point theory.

The notion of a metric space was first introduced by Fréchet in 1906 [7]. Mathematicians have explored the existence and uniqueness of fixed points through the Banach contraction principle. This fundamental principle has been proven in various newly developed generalized metric spaces [5].

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Fixed point theorems in metric spaces have been widely explored by numerous researchers, as seen in [12, 6] and [10]. In 1989, Bakhtin [3] introduced the concept of b-metric spaces. Following this, many researchers extended fixed point theorems from metric spaces to b-metric spaces, as demonstrated in [1, 2].

In 1970, Takahashi [11] introduced the concept of convexity structure in metric spaces to investigate fixed point problems for nonexpansive mappings. A convex metric space provides the essential framework for developing various fixed point iterative methods used to approximate fixed points of nonlinear operators. Examples of such iterative schemes include those proposed by Krasnoselskij, Mann, and Ishikawa, which rely on the linearity and convexity of the surrounding topological space.

In 2011, A. Azam, B. Fisher, and M. Khan [2] introduced the concept of complex-valued metric spaces and provided sufficient conditions for the existence of common fixed points for a pair of mappings under a contractive condition. This notion extends the classical metric space, offering a broader framework for fixed point theory.

More recently, V. Berinde V. and M. Păcurar [4], use the technique of enrichment of con-tractive type mappings and obtain general results which extend the well known Banach contraction mapping principle from metric spaces as well as other corresponding results for enriched mappings defined on Banach spaces.

In this research, we study and improve the results of V. Berinde V. and M. Păcurar [4] to complex valued convex metric space and prove the existence results of fixed point, which are presented in the next section.

## 2. PRELIMINARIES

In this section, we present some definitions and lemmas for using in section 3, and define the definition of b-metric space in the complex plane.

**Definition 2.1.** *Let* $X$ *be a nonempty set. A function* $d : X \times X \to [0, \infty)$ *is called a metric if for* $x, y, z \in X$ *the following conditions are satisfied.*

$(i)$ $d(x, y) = 0$ *if and only if* $x = y$;

$(ii)$ $d(x, y) = d(y, x)$;

$(iii)$ $d(x, z) \leq d(x, y) + d(y, z)$.
*The pair* $(X, d)$ *is called a metric space, and* $d$ *is called a metric on* $X$.

**Definition 2.2.** *[11] Let* $(X, d)$ *be a metric space. A continuous function* $W : X \times X \times [0, 1] \longrightarrow X$ *is said to be a convex structure on* $X$ *if, for all* $x, y \in X$

*and any* $\lambda \in [0,1]$,

$$d(u, W(x,y,\lambda),) \leq \lambda d(u,x) + (1-\lambda)d(u,y), \text{ for all } u \in X. \tag{1}$$

*A metric space* $(X,d)$ *endowed with a convex structure* $W$ *is called a* **Takahashi convex metric space** *and is usually denoted by* $(X,d,W)$.

**Definition 2.3.** *[4] Let* $(X,d,W)$ *be a convex metric space. A mapping* $T : X \to X$ *is said to be an* **enriched contraction** *if there exist* $c \in [0,1)$ *and* $\lambda \in [0,1)$ *such that*

$$d(W(x,Tx;\lambda), W(y,Ty;\lambda)) \leq cd(x,y), \forall x,y \in X. \tag{2}$$

*To specify the parameters* $c$ *and* $\lambda$ *involved in (2), we called* $T$ *a* $(\lambda, c)$-**enriched contraction**.

There is a completeness property in real number but on order relation is not welll-defined in complex numbers. Before giving the definition of complex valued metric spaces and complex-valued b-metric spaces, we define partial order in complex numbers (see [9]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order relation $\preccurlyeq$ on $\mathbb{C}$ as follows;

$$z_1 \preccurlyeq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

This means that we would have $z_1 \preccurlyeq z_2$ if and only if one of the following conditions holds:

$(i)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

$(ii)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

$(iii)$ $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

$(iv)$ $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

If one of the conditions $(ii), (iii)$, and $(iv)$ holds, then we write $z_1 \prec z_2$. From the above partial order relation we have the following remark.

**Remark 2.1.** *We can easily check as follows:*

$(i)$ *If* $a, b \in \mathbb{R}, 0 \leq a \leq b$ *and* $z_1 \preccurlyeq z_2$ *then* $az_1 \preccurlyeq bz_2, \forall z_1, z_2 \in \mathbb{C}$.

$(ii)$ *If* $0 \preccurlyeq z_1 \prec z_2$ *then* $|z_1| < |z_2|$.

$(iii)$ *If* $z_1 \preccurlyeq z_2$ *and* $z_2 \prec z_3$ *then* $z_1 \prec z_3$.

$(iv)$ *If* $z \in \mathbb{C}$, *for* $a, b \in \mathbb{R}$ *and* $a \leq b$, *then* $az \preccurlyeq bz$.

A metric on a metric sapce is a funcion having real value. Based on the definition of partial order on complex number, real-valued metric can be generalized into complex-valued metric as folllows.

**Definition 2.4.** *[2] Let $X$ be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex valued metric on $X$ if for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:*

$(i)$ $0 \precsim d(x, y)$ *and* $d(x, y) = 0$ *if and only if* $x = y$,

$(ii)$ $d(x, y) = d(y, x)$,

$(iii)$ $d(x, z) \precsim d(x, y) + d(y, z)$.
*Then $d$ is called a complex valued metric on $X$ and $(X, d)$ is called a complex valued metric space.*

**Definition 2.5.** *[2] Let $(X, d)$ be a complex valued metric space.*

$(i)$ *A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that*

$$B(x, r) = \{y \in Y : d(x, y) \prec r\} \subseteq A.$$

$(ii)$ *A point $x \in X$ is called limit point of a set $A$ if for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - x) \neq \emptyset$*

$(iii)$ *A subset $A \subseteq X$ is open if each element of $A$ is an interior point of $A$.*

$(iv)$ *A subset $A \subseteq X$ is closed if each limit point of $A$ is contained in $A$.*

**Definition 2.6.** *[2] Let $(X, d)$ be complex valued metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$.*

$(i)$ *The sequence $\{x_n\}$ is converges to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x) \prec r$. Thus $x$ is the limit of $(x_n)$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

$(ii)$ *The sequence $\{x_n\}$ is said to be a Cauchy sequence if for ever $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N, d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.*

$(iii)$ *If for every Cauchy sequence in$X$ is convergent, then $(X, d)$ is said to be a complete complex valued b-metric space.*

**Lemma 2.1.** *[2] Let $(X, d)$ be a complex valued metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

**Lemma 2.2.** *[2] Let $(X, d)$ be a complex valued metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

**Definition 2.7.** *Let* $(X, d)$ *be a complex valued metric space. A continuous function* $W : X \times X \times [0, 1] \longrightarrow X$ *is said to be a convex structure on* $X$ *if, for all* $x, y \in X$ *and any* $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda), ) \precsim \lambda d(u, x) + (1 - \lambda)d(u, y), \textit{ for all } u \in X. \tag{3}$$

*A metric space* $(X, d)$ *endowed with a convex structure* $W$ *is called a* **complex valued convex metric space** *and is usually denoted by* $(X, d, W)$.

Next, we prove some Lemmas for use in the main Theorems.

**Lemma 2.3.** *Let* $(X, d, W)$ *be a complex valued convex metric space. For all* $x, y \in X$ *and for all* $\lambda \in [0, 1]$, *such that*

$$\Big|d(x, y)\Big| = \Big|d(x, W(x, y, \lambda))\Big| + \Big|d(W(x, y, \lambda), y)\Big|. \tag{4}$$

*Proof.* From Definition 2.4 $(i)$, $(iii)$ and Definition 2.7, we have

$$\begin{aligned} d(x, y) &\precsim d(x, W(x, Tx; \lambda)) + d(W(x, Tx; \lambda)) \\ &\precsim \Big[\lambda d(x, x) + (1 - \lambda)d(x, y)\Big] + \Big[\lambda d(x, y) + (1 - \lambda)d(y, y)\Big] \\ &= d(x, y). \end{aligned}$$

By Remark 2.1, we have

$$\Big|d(x, y)\Big| \leq \Big|d(x, W(x, Tx; \lambda)) + d(W(x, Tx; \lambda), y)\Big| = \Big|d(x, y)\Big|,$$

it follows that

$$\Big|d(x, y)\Big| = \Big|d(x, W(x, Tx; \lambda)) + d(W(x, Tx; \lambda), y)\Big|.$$

From Definition 2.4 $(i)$, we obtain that

$$\Big|d(x, y)\Big| = \Big|d(x, W(x, Tx; \lambda))\Big| + \Big|d(W(x, Tx; \lambda), y)\Big|.$$

Which complete the proof. □

**Lemma 2.4.** *Let* $(X, d, W)$ *be a complex valued convex metric space. For all* $x, y \in X$ *and for all* $\lambda \in [0, 1]$, *such that*

$$\Big|d(x, W(x, y; \lambda))\Big| = (1 - \lambda)\Big|d(x, y)\Big| \textit{ and } \Big|d(W(x, y; \lambda), y)\Big| = \lambda\Big|d(x, y)\Big| \tag{5}$$

*Proof.* From Definition 2.7 and Definition 2.4 $(i)$, we get

$$d(x, W(x, Tx; \lambda)) \precsim \lambda d(x, x) + (1-\lambda)d(x, y) = (1-\lambda)d(x, y).$$

By Remark 2.1, we have

$$\Big|d(x, W(x, Tx; \lambda))\Big| \leq (1-\lambda)\Big|d(x, y)\Big|. \tag{6}$$

Similarly,

$$d(W(x, Tx; \lambda), y) \precsim \lambda d(x, y) + (1-\lambda)d(y, y) = \lambda d(x, y),$$

it imples that

$$\Big|d(W(x, Tx; \lambda), y)\Big| \leq \lambda\Big|d(x, y)\Big|. \tag{7}$$

From (6) and (7) we considers that, if

$$\begin{aligned}\Big|d(x, W(x, Tx; \lambda))\Big| &< (1-\lambda)\Big|d(x, y)\Big| \\ \Big|d(W(x, Tx; \lambda), y)\Big| &< \lambda\Big|d(x, y)\Big|,\end{aligned}$$

it implies that

$$\Big|d(x, W(x, Tx; \lambda))\Big| + \Big|d(W(x, Tx; \lambda), y)\Big| < \Big|d(x, y)\Big|.$$

A contradiction from Lemma 2.3. Hence, from (6) and (7), we obtain that

$$\Big|d(x, W(x, y; \lambda))\Big| = (1-\lambda)\Big|d(x, y)\Big| \text{ and } \Big|d(W(x, y; \lambda), y)\Big| = \lambda\Big|d(x, y)\Big|$$

□

## 3. ENRICHED CONTRACTION IN COMPLEX VALUED CONVEX METRIC SPACES

In this section, we consider some enriched contractions and peoved the existence theorems of fixed points, as follows;

**Definition 3.1.** *Let* $(X, d, W)$ *be a complex valued convex metric space. A mapping* $T : X \to X$ *is said to be an enriched contraction if there exists* $\alpha \in \mathbb{C}$ *with* $|\alpha| \in [0, 1)$ *and* $\lambda \in [0, 1)$ *such that*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \precsim \alpha d(x, y), \forall x, y \in X. \tag{8}$$

*We called a mapping* $T$ *a* $(\lambda, \alpha)$*-enriched contraction.*

First, we defined some mapping from a self mapping $T : X \to X$ and prove the equal of fixed point set.

**Theorem 3.1.** *Let* $(X, d, W)$ *be a complex valued convex metric space and* $T : X \to X$. *Define the mapping* $T_\lambda : X \to X$ *by*

$$T_\lambda x = W(x, Tx; \lambda), \text{ for all } x \in X. \tag{9}$$

*Then,* $Fix(T) = Fix(T_\lambda)$ *for all* $\lambda \in [0, 1)$

*Proof.* Let $\lambda = 0$. If $u \in Fix(T)$ then $Tu = u$, by Remark 2.1 and Definition 2.7, we have

$$0 \leq \Big|d(u, T_\lambda u)\Big| \leq \Big|d(u, Tu)\Big| = 0,$$

if $v \in Fix(T_\lambda)$ then $v = T_\lambda v$, by Remark 2.1 and Lemma 2.4, we have

$$0 = \Big|d(v, T_\lambda v)\Big| = \Big|d(v, W(v, Tv; \lambda))\Big| = (1 - \lambda)\Big|d(u, Tu)\Big| = \Big|d(u, Tu)\Big|.$$

Hence, $Fix(T) = Fix(T_\lambda)$.

Assume that $\lambda \in (0, 1)$ and let $u \in Fix(T)$. From Definition 2.7 and Definition 2.4 $(i)$, we have

$$\begin{aligned} d(u, T_\lambda u) &= d(u, W(u, Tu; \lambda)) \\ &\precsim \lambda d(u, u) + (1 - \lambda) d(u, Tu) \\ &= 0. \end{aligned}$$

By Remark 2.1, we get

$$\begin{aligned} \Big|d(u, T_\lambda u)\Big| &\leq 0 \\ \therefore\ d(u, T_\lambda u) &= 0. \end{aligned}$$

From Definition 2.4 $(i)$, we have $u = T_\lambda u$. Hence, $u \in Fix(T_\lambda)$.

Conversely, let $v \in Fix(T_\lambda)$, we have $v = T_\lambda v$ and $d(v, T_\lambda v) = 0$. Which impleis that

$$d(v, W(v, Tv; \lambda)) = d(v, T_\lambda v) = 0. \tag{10}$$

From Lemma 2.4, we have

$$\Big|d(v, W(v, Tv; \lambda))\Big| = (1 - \lambda)\Big|d(v, Tv)\Big|. \tag{11}$$

From (10) and (11), it follows that

$$(1 - \lambda)\Big|d(v, Tv)\Big| = 0.$$

Since, $\lambda \in [0, 1)$ it implies that $\Big|d(v, Tv)\Big| = 0$. Hence, $d(v, Tv) = 0$. By Definition 2.4 $(i)$, we have $v = Tv$. Therefore, $v \in Fix(T)$. This complete the proof. □

Next, we proved the existance theorem result of fixed point in complete complex valued convex metric spaces.

**Theorem 3.2.** *Let* $(X, d, W)$ *be a complete complex valued convex metric space and let* $T : X \to X$ *be a* $(\lambda, \alpha)$*-enriched contraction, then*

1. *The sequence* $\{x_n\}_{n=0}^{\infty}$ *obtained from the iterative process, for* $x_0 \in X$,
$$x_{n+1} = W(x_n, Tx_n; \lambda), n \geq 0, \tag{12}$$
*converges to* $u$, *for some* $u \in X$,
2. $Fix(T) = \{u\}$.

*Proof.* From Definition of $\{x_n\}$, a mapping $T_\lambda$ and Definition 3.1, we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T_\lambda x_n, T_\lambda x_{n-1}) \\ &= d(W(x_n, Tx_n; \lambda), W(x_{n-1}, Tx_{n-1}; \lambda)) \\ &\precsim \alpha d(x_n, x_{n-1}). \end{aligned}$$

By Remark 2.1, we have

$$\Big|d(x_{n+1}, x_n)\Big| \leq |\alpha| \Big|d(x_n, x_{n-1})\Big|,$$

which inductively implies

$$\Big|d(x_{n+1}, x_n)\Big| \leq |\alpha|^n \Big|d(x_1, x_0)\Big|, n \geq 1. \tag{13}$$

Since $|\alpha| < 1$ and (13), we obtain that

$$\lim_{n\to\infty} \Big|d(x_{n+1}, x_n)\Big| = 0. \tag{14}$$

Consider, for any $n, k \in \mathbb{N}$ with $k > 0$, we have

$$\begin{aligned} d(x_{n+k+1}, x_{n+1}) &= d(T_\lambda x_{n+k}, T_\lambda x_n) \\ &= d(W(x_{n+k}, Tx_{n+k}; \lambda), W(x_n, Tx_n; \lambda)) \\ &\precsim \alpha d(x_{n+k}, x_n) \\ &\precsim \alpha \Big[d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+1}) + d(x_{n+1}, x_n)\Big], \end{aligned}$$

it implies that

$$d(x_{n+k+1}, x_{n+1}) \precsim \frac{\alpha}{1-\alpha} \Big[d(x_{n+k}, x_{n+k+1}) + d(x_{n+1}, x_n)\Big]. \tag{15}$$

By Remark 2.1 and (15), we have

$$\Big|d(x_{n+k+1},x_{n+1})\Big| \leq \Big|\frac{\alpha}{1-\alpha}\Big|\Big[\Big|d(x_{n+k},x_{n+k+1})\Big| + \Big|d(x_{n+1},x_n)\Big|\Big]. \tag{16}$$

Since, $\Big|\frac{\alpha}{1-\alpha}\Big| \neq 0$, (14) and (16), we obtain that

$$\lim_{n\to\infty}\Big|d(x_{n+k+1},x_{n+1})\Big| = 0. \tag{17}$$

Hence, $\{x_n\}$ ia a Cauchy sequence. Since $X$ is a complete space there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{18}$$

Next, we show that $u$ is a fixed point of $T$. Since $T_\lambda$ is continuous, we get

$$\lim_{n\to\infty} T_\lambda x_n = T_\lambda u. \tag{19}$$

From (18) and (19), it follows that

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T_\lambda x_n = T_\lambda u.$$

Hence, $u \in Fix(T_\lambda)$. From Theorem 3.1, we get $Fix(T) = Fix(T_\lambda)$. Therefore, $u \in Fix(T)$ and from (18) it implies that $\{x_n\}$ converge to a fixed point of $T$.

Finally, we prove that $Fix(T) = \{u\}$. Let $v \in Fix(T_\lambda)$, consider

$$\begin{aligned} d(u,v) &= d(T_\lambda u, T_\lambda v) \\ &= d(W(u,Tu;\lambda), W(v,Tv;\lambda)) \\ &\precsim \alpha d(u,v). \end{aligned}$$

By Remark 2.1, we have

$$\begin{aligned} \Big|d(u,v)\Big| &\leq |\alpha|\Big|d(u,v)\Big| \\ \therefore\ (1-|\alpha|)\Big|d(u,v)\Big| &\leq 0. \end{aligned}$$

Since, $|\alpha| < 1$ it follows that $|d(u,v)| = 0$. Hence, $d(u,v) = 0$. From Definition 2.4 $(i)$, we have $u = v$. Therefore, $Fix(T) = \{u\}$. This complete the proof. □

## 4. ENRICHED $\varphi$-CONTRACTION IN COMPLEX VALUED CONVEX METRIC SPACES

In this section, we defined a self mapping $T : X \to X$ is a $\varphi$-contraction in complex valued convex metric space and prove the existence and uniqueness of a fixed point of $T$ as follows.

**Definition 4.1.** *[8] Let* $P = \Big\{z \in \mathbb{C} : Re(z) \geqslant 0$ *and* $Im(z) \geqslant 0\Big\}$. *A nondecreasing mapping* $\varphi : P \to P$ *is called a* $\varphi$*-mapping if*

1. $\varphi(0) = 0$ *and* $0 \prec \varphi(z) \prec z$ *for* $z \in P - \{0\}$*;*
2. $\varphi(z) \prec z$ *for every* $z \succ 0$;
3. $\lim_{n\to\infty} \varphi^n(z) = 0$ *for every* $z \in P - \{0\}$.

**Definition 4.2.** *Let* $(X, d, W)$ *be a complex valued convex metric space. A mapping* $T : X \to X$ *is said to be an complex valued enriched* $\varphi$*-contraction if there exists nondecreasing mapping* $\varphi : P \to P$*, such that*

$$d(W(x, Tx; \lambda), W(y, Ty; \lambda)) \precsim \varphi\Big(d(x, y)\Big), \forall x, y \in X. \tag{20}$$

*We called a mapping* $T$ *a* $(\lambda, \alpha)$*-enriched contraction.*

**Theorem 4.1.** *Let* $(X, d, W)$ *be a complete complex valued convex metric space and let* $T : X \to X$ *be a complex valued enriched* $\varphi$*-contraction defined in (20), then*

1. *The sequence* $\{x_n\}_{n=0}^{\infty}$ *obtained from the iterative process, for* $x_0 \in X$*,*

   $$x_{n+1} = W(x_n, Tx_n; \lambda), n \geq 0, \tag{21}$$

   *converges to* $u$*, for some* $u \in X$*,*
2. $Fix(T) = \{u\}$.

*Proof.* Let $x_0 \in X$ and definition of $x_n$, by Definition 4.2, we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T_\lambda x_n, T_\lambda x_{n-1}) \\ &= d\Big(W(x_n, Tx_n; \lambda), W(x_{n-1}, Tx_{n-1}; \lambda)\Big) \\ &\precsim \varphi\Big(d(x_n, x_{n-1})\Big). \end{aligned}$$

By inductively, we have

$$d(x_{n+1}, x_n) \precsim \varphi^n\Big(d(x_1, x_0)\Big). \tag{22}$$

By Definition 4.1 (3), we have $\lim\limits_{n\to\infty} \varphi^n\Big(d(x_1, x_0)\Big) = 0$, it follows that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{23}$$

We now prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Assume that $\{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence. By Lemma 2.2, there exists $r \in P$ and two subsequence $\{x_{n_k}\}, \{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\Big|d(x_{n_k}, x_{m_k})\Big| \succ r, \quad \text{for all } k \geq 0.$$

Since, $\{d(x_{n+1}, x_n)\}$ is decreasing and bounded below from $\epsilon$, it implies that

$$\lim_{k\to\infty} \Big|d(x_{n_k}, x_{m_k})\Big| = r. \tag{24}$$

We consider,

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &= d(T_\lambda x_{n_k-1}, T_\lambda x_{m_k-1}) \\ &= d\Big(W(x_{n_k-1}, Tx_{n_k-1}; \lambda), W(x_{m_k-1}, Tx_{m_k-1}; \lambda)\Big) \\ &\preccurlyeq \varphi\Big(d(x_{n_k-1}, x_{m_k-1})\Big). \end{aligned}$$

From (24), we have

$$r = \lim_{k\to\infty} d(x_{n_k}, x_{m_k}) \preccurlyeq \lim_{k\to\infty} \varphi\Big(d(x_{n_k-1}, x_{m_k-1})\Big) = \varphi(r),$$

it follows that $r \preccurlyeq \varphi(r)$, a contradiction from Definition 4.1 (2). Hence, $\{x_n\}$ is a Cauchy sequence. Since, $X$ is a complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{25}$$

Next, we show that $u$ is a fixed point of $T$. Since $T_\lambda$ is continuous, we get

$$\lim_{n\to\infty} T_\lambda x_n = T_\lambda u. \tag{26}$$

From (25) and (26), it follows that

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T_\lambda x_n = T_\lambda u.$$

Hence, $u \in Fix(T_\lambda)$. From Theorem 3.1, we get $Fix(T) = Fix(T_\lambda)$. Therefore, $u \in Fix(T)$ and from (25) it implies that $\{x_n\}$ converge to a fixed point of $T$.

Finally, we prove that $Fix(T) = \{u\}$. Let $v \in Fix(T_\lambda)$ with $u \neq v$, then $d(u, v) \succ 0$, it follows that

$$\begin{aligned} d(u, v) &= d(T_\lambda u, T_\lambda v) \\ &= d(W(u, Tu; \lambda), W(v, Tv; \lambda)) \\ &\preccurlyeq \varphi\Big(d(u, v)\Big). \end{aligned}$$

Which is a contradiction by Definition 4.1 (2). Hence, $u = v$. Therefore, $Fix(T) = \{u\}$. This complete the proof. $\square$

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